Lesson 2

Non linear Equations

Youndé – 7 August 2013

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The problem

Definition 2.1 *Let f be a function of the (real or complex) variable x. The roots of the equation*

$$f(x) = 0$$

are the numbers ξ for which $f(\xi) = 0$. Each root of the equation $f(x) = 0$ is said to be a zero of the function f.

Example

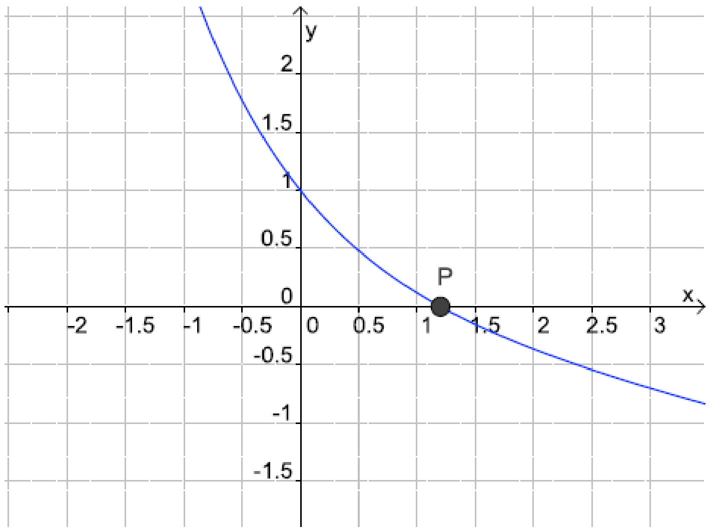
Let $f(x) = x^2-1$. Then, the zeros of the function f(x) (or, it is the same, the roots of the equation f(x)=0) are the real numbers x such that

$$f(x) = x^2 - 1 = 0$$

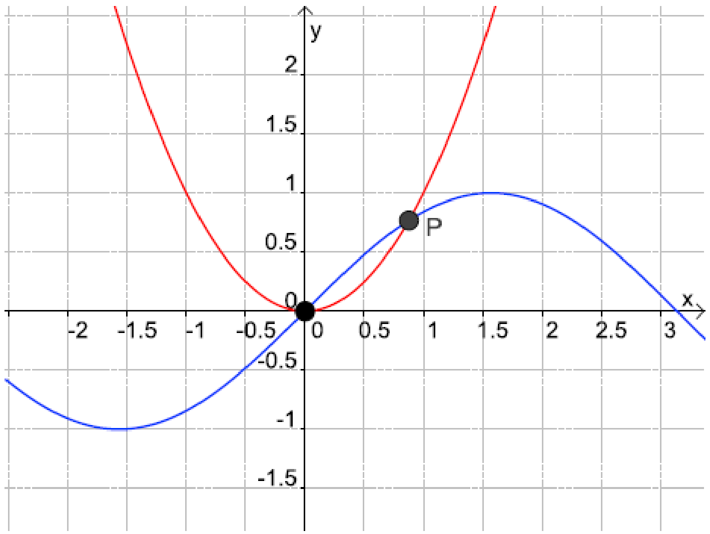
So, we have two zeros $\xi_1 = -1$ and $\xi_2 = 1$.
The number $x = 0$ is not a zero of the function f since $f(0) = 0^2 -1 = -1 \neq 0$.

Geometric interpretation

The roots of the equation $f(x) = 0$ are the intersections of the graph $y = f(x)$ with the real axis, i.e., the line $y = 0$. In the same manner, the roots of the equation $f(x) = g(x)$ are the abscissas of the intersection points of the two graphs $y = f(x)$ and $y = g(x)$.



(a) The equation $e^{-x} - 0.25 \cdot x = 0$ has only one root $\xi \in (1,\ 1.5)$ since the corresponding graph $y = e^{-x} - 0.25 \cdot x$ intersect the x axis only in one point $P = (\xi, 0)$.



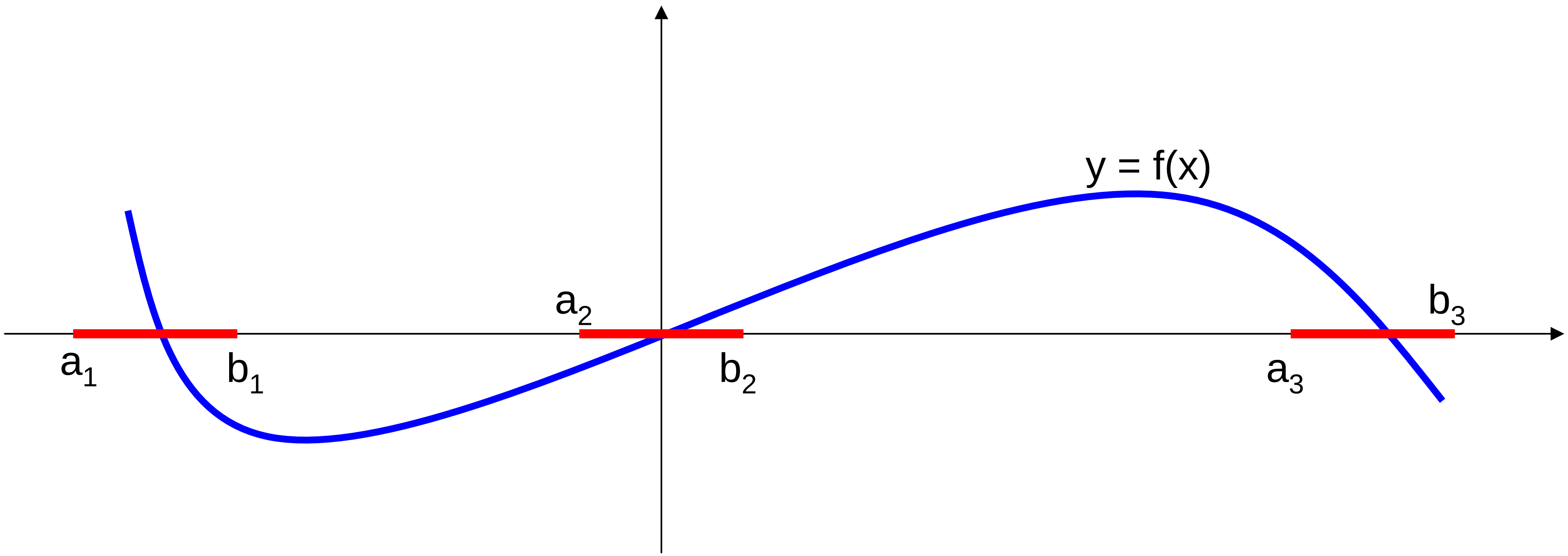
(b) The equation $\sin(x) - x^2 = 0$ has two roots: $\xi_1 = 0$ and $\xi_2 \in (0.5,\ 1)$ since the graphs $y = \sin(x)$ and $y = x^2$ have two intersection points $O = (0, 0)$ and $P = (\xi_2, f(\xi_2))$.

Root separation #1

The computation of real roots of the equation $f(x) = 0$ follows two main steps

(a) **roots separation** : for each root ξ_k, we find an interval $[a_k,\ b_k]$ such that $\xi_k \in [a_k,\ b_k]$ and no one of the other roots belongs to $[a_k,\ b_k]$.

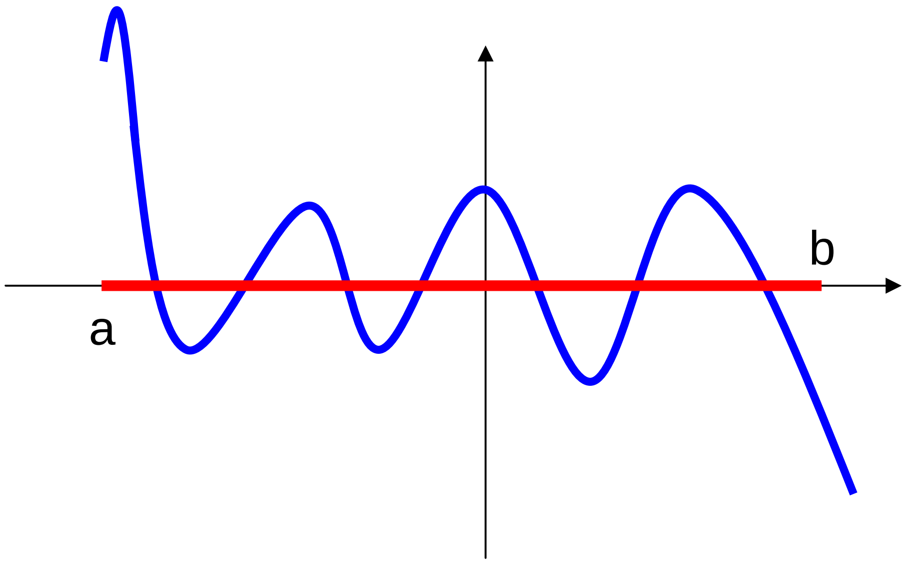
(b) **roots approximation** : we approximate some, or even all, of the roots.



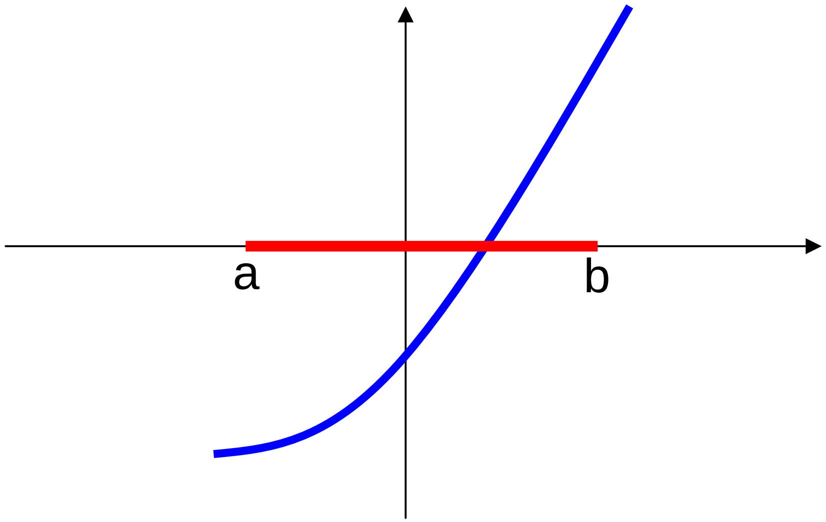
Root Separation #2

The first step may be done sketching the graph of the function f. It is also useful the following theorem.

Theorem 2.1 (zeros of a continuous function) *Let f be a continuous function (at least) in the interval $[a, b]$ with $f(a) \cdot f(b) < 0$. Then, f has almost one zero in the interval $[a, b]$. Furthermore, if the function f is strictly monotone in $[a, b]$, then the zero is unique.*



NOT monotone: we **can** have more than one root in [a, b]



STRICTLY monotone: we have exactly one root in [a, b]

Root Approximation #1

Given an interval [a, b] which contains the unique root ξ, we search for a sequence x_k such that

$$\lim_{k \to +\infty} x_k = \xi$$

Definition

We define the error e_k at step k as $e_k = x_k - \xi$.

Definition

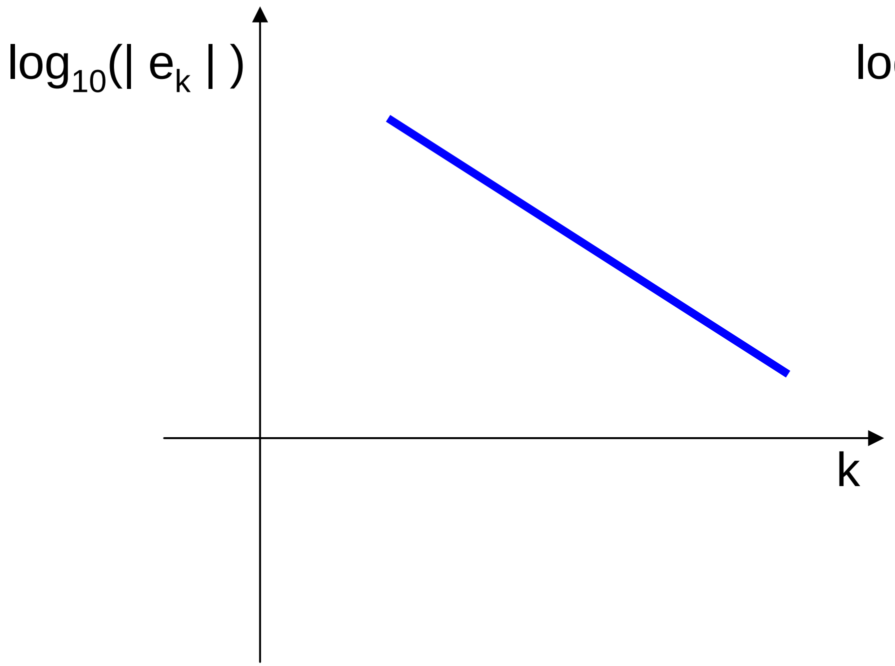
Let x_k be a sequence that converges to ξ. If there are <u>positive</u> constants c and p such that

$$\lim_{k \to +\infty} \frac{|\ e_{k+1}\ |}{|\ e_k\ |^p} = c$$

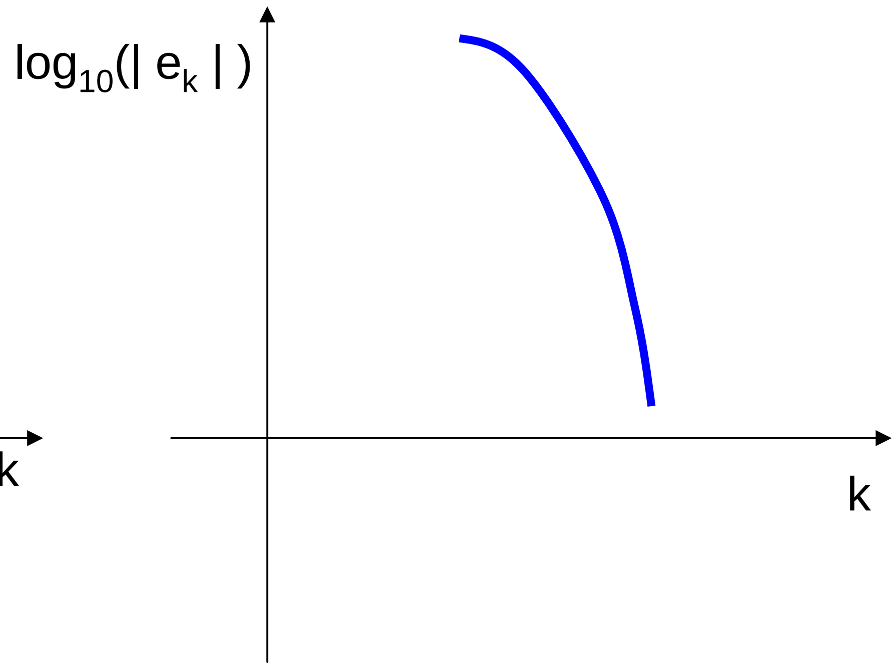
we say that the sequence x_k converges to ξ with order p and asymptotic error constant c. Moreover, the convergence process is said to be **linear** if p = 1 and **superlinear** if p>1. For the latter case, we say that it is **quadratic** if p = 2.

Root Approximation #2

If we plot the $\log_{10}(| e_k |)$ as a function of k, **it can be shown that <u>near</u> the root the behaviour of the graph** is



LINEAR CONVERGENCE



SUPERLINEAR CONVERGENCE

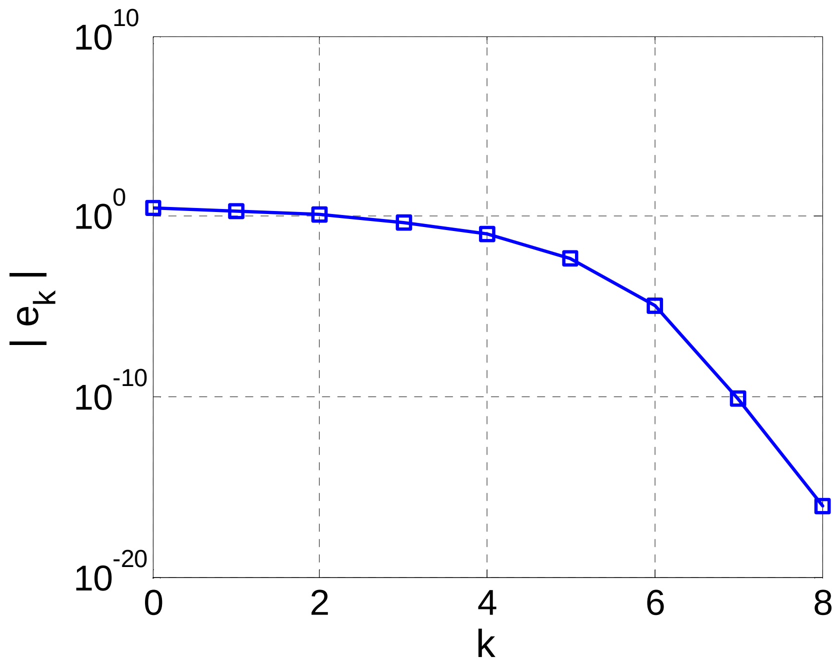
Moreover, recall that when x_k is close to the root, we may write

$$| e_{k+1} | \approx c | e_k |^p$$

Root Approximation #3

Example

Consider the computation of the root of the equation e^x-1 = 0. We have the following behaviour of the error:



k	$\| e_k \|$
0	3.0000e+000
1	2.0498e+000
2	1.1785e+000
3	4.8627e-001
4	1.0119e-001
5	4.9510e-003
6	1.2236e-005
7	7.4862e-011
8	9.2344e-017

From the plot (or from the table) we see that the convergence is superlinear.

Bisection Method #1

Let ξ be the unique zero in the interval $[a,\ b]$ of the function f which we assume continuous at least in $[a, b]$. Assume $\xi \neq a$ and $\xi \neq b$.

Starting from $I_0 := [a_0, b_0] = [a, b]$, the bisection method constructs a sequence of nested intervals $I_k = [a_k, b_k]$ containing the root:

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_k \supset I_{k+1} \cdots \qquad \text{with} \quad \xi \in I_k \ \forall\, k$$

The k-th step, $k = 0, 1, \ldots,$ of the bisection method is

Bisection Method #2

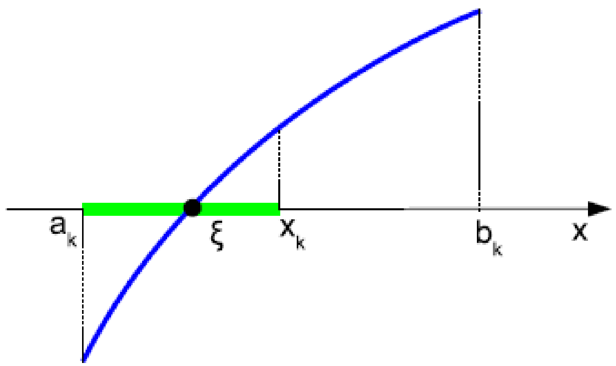
1. compute $x_k = (a_k + b_k)/2$. Note that $x_k \in I_k$.
2. compute $f(x_k)$
3. choose one of the following cases

3.1. $f(x_k) = 0$, i.e., x_k is a root of f. Since $x_k \in [a, b]$ by construction and ξ is the unique root inside $[a, b]$, then it must be $\xi = x_k$. We have find the root and the iterative process stops.

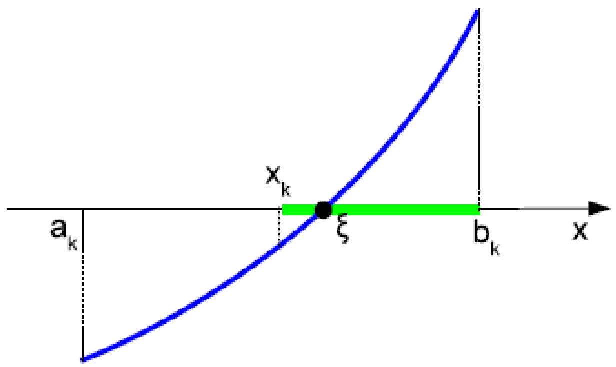
3.2. $f(a_k) \cdot f(x_k) < 0$, i.e. $f(a_k)$ and $f(x_k)$ have opposite signs. Thus $\xi \in [a_k, x_k]$. So, we set $I_{k+1} = [a_{k+1},\ b_{k+1}] = [a_k,\ x_k]$. That is, $a_{k+1} = a_k$ and $b_{k+1} = x_k$ (see Figure 2.3 on the left).

3.3. $f(a_k) \cdot f(x_k) > 0$, i.e. $f(a_k)$ and $f(x_k)$ have the same signs. Thus $\xi \in [x_k, b_k]$. So, we set $I_{k+1} = [a_{k+1},\ b_{k+1}] = [x_k,\ b_k]$. That is, $a_{k+1} = x_k$ and $b_{k+1} = b_k$ (see Figure 2.3 on the right).

Bisection Method #3



(a) $I_{k+1} = [a_k,\ x_k]$.



(b) $I_{k+1} = [x_k,\ b_k]$.

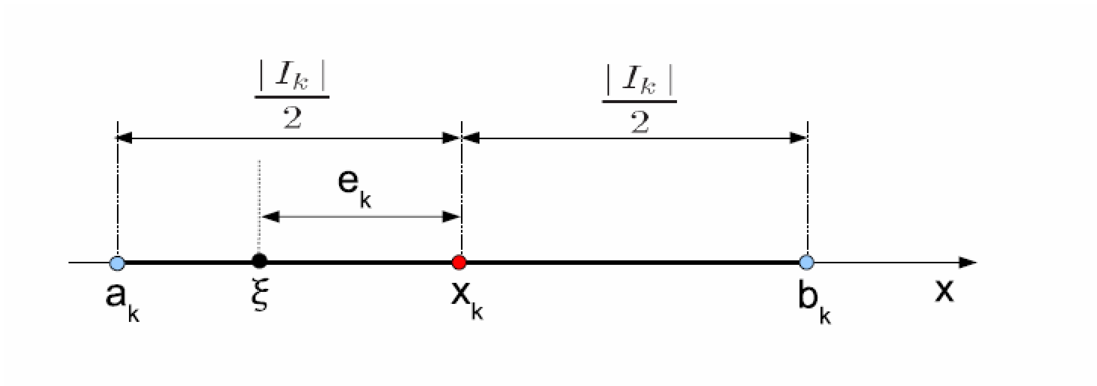
Error in the Bisection Method #1

Let us denote by $|I_k| = b_k - a_k$ the length of the interval I_k. Then, in cases 3.2 and 3.3 we have

$$|I_{k+1}| = \frac{|I_k|}{2} \stackrel{(1)}{=} \frac{|I_0|}{2^{k+1}}$$

where (1) follows from mathematical induction. So, after k-th step is complete, the error $e_k = x_k - \xi$ satisfies the inequality

$$|e_k| \leq |I_{k+1}| = \frac{|I_0|}{2^{k+1}}$$



Error in the Bisection Method #2

From the latter equation it is simple to compute the number of iterations of the bisection method that have to be performed in order to obtain $| e_k | < \varepsilon$ for some given $\varepsilon > 0$. Indeed, we have

$$| e_k | < \varepsilon \qquad \Leftrightarrow \qquad \frac{b-a}{2^{k+1}} < \varepsilon \qquad \Leftrightarrow \qquad 2^{k+1} > \frac{b-a}{\varepsilon}$$

and finally, taking the logarithm in the latter inequality, we get

$$\log\left(2^{k+1}\right) > \log\left(\frac{b-a}{\varepsilon}\right) \qquad \Leftrightarrow \qquad k > \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log(2)} - 1$$

So, to obtain $| e_k | < \varepsilon$ it is necessary to perform at least $k_{\min}$ iterations with

$$k_{\min} = \left\lceil \frac{\log\left(\frac{b-a}{\varepsilon}\right)}{\log(2)} - 1 \right\rceil . \tag{2.1}$$

where $\lceil a \rceil$ is the smallest integer greater or equal to a.

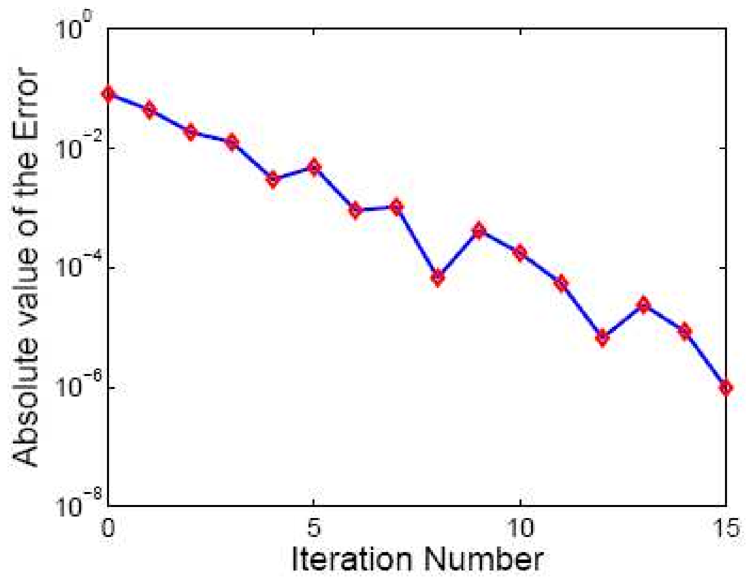
Error in the Bisection Method #3

Example 2.2 *The computation of the first positive zero of the equation*

$$x - \tan\left(\frac{x}{2}\right) = 0$$

within the tolerance $\varepsilon = 1.E - 5 = 10^{-5}$ *and with starting interval* $[a, b] = [2.0,\ 2.5]$ *requires, at least,*

$$k_{min} = \left\lceil \frac{\log\left(\frac{2.5-2.0}{10^{-5}}\right)}{\log(2)} - 1 \right\rceil = \lceil 14.61 \rceil = 15 \textit{ iterations}$$



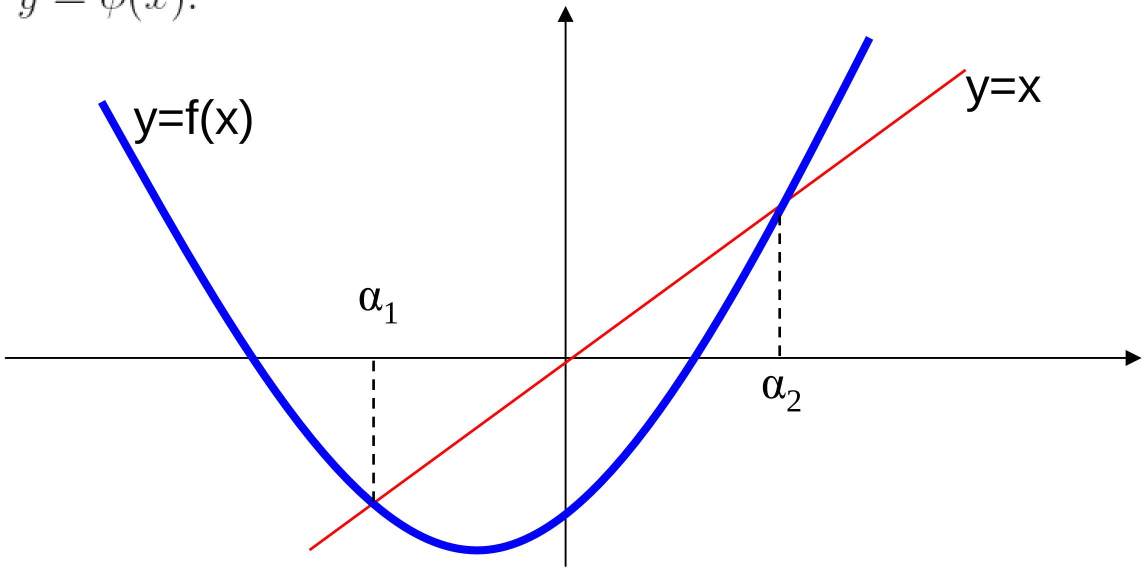
Note that the error DOES NOT decrease monotonically, i.e. we can have

$$| e_{k+1} | > | e_k |$$

Fixed Points of a Function

Definition 2.2 *The function $\phi(x)$, $x \in [a,b]$ has the fixed point $\alpha \in [a,b]$ if $\alpha = \phi(\alpha)$.*

So, fixed points of the function ϕ are, if any, the roots of the equation $x = \phi(x)$. Graphically, they are abscissas of the intersection points of the graphs $y = x$ and $y = \phi(x)$.



Function f(x)
has two fixed points

Example 2.3 *The function $\phi(x) = x^2 + 1$ does not have any fixed point since the equation $x = x^2 + 1$ has no real roots.*
The function $\phi(x) = x^2$ has two fixed points since the equation $x = x^2$ has roots $\alpha_1 = -1$ and $\alpha_2 = 0$.

Fixed Point Iterations #1

To introduce the fixed point method, the first step is to rewrite the equation $f(x) = 0$ in the form $x = \phi(x)$ for some function ϕ. The function ϕ is not unique. For example, consider the equation $x^2 - 1 = 0$. We can rewrite it as

$$\text{(a) } x = x^2 + x - 1 =: \phi(x), \qquad \text{(b) } x = \frac{1}{x} =: \phi(x), \qquad \text{(c) } x = \frac{-x^2 + 4x + 1}{4} =: \phi(x)$$

and in many other manners.

Next, let $\alpha \in [a, b]$ be the unique fixed point in the interval $[a, b]$ of $x = \phi(x)$. Given an initial estimate $x_0 \in [a, b]$ of the fixed point α, we consider the following iterative scheme for the computation of α:

$$\begin{cases} x_0 & \text{given initial estimate of } \alpha \\ x_{k+1} = & \phi(x_k), \quad k = 0, 1, 2, \ldots \end{cases}$$

The following theorem provides whether the previous iterations x_k converges to the fixed point α of $x = \phi(x)$.

Fixed Point Iterations #2

Theorem 2.3 (Convergence of the iterations) *Let ϕ be a continuous function on $[a,\, b]$, differentiable in $(a,\, b)$ with*

(i) $\phi(\,[a,\, b]\,) \subseteq [a,\, b]$;

(ii) $|\,\phi'(x)\,| \leq K < 1 \qquad \forall\, x \in\, (a, b)$

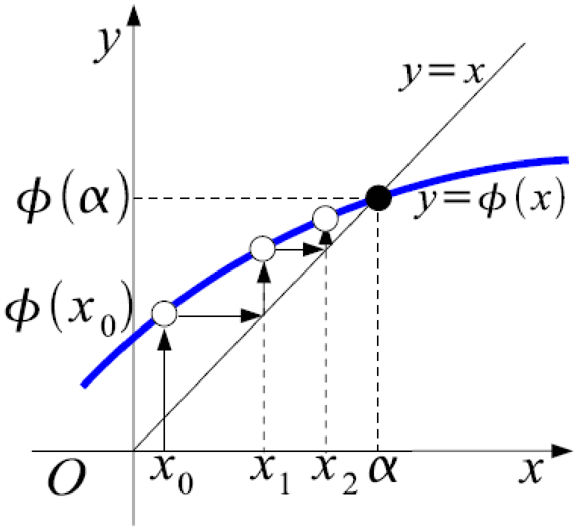
Then, the sequence

$$x_{k+1} = \phi(\,x_k\,), \quad k = 0, 1, 2, \ldots$$

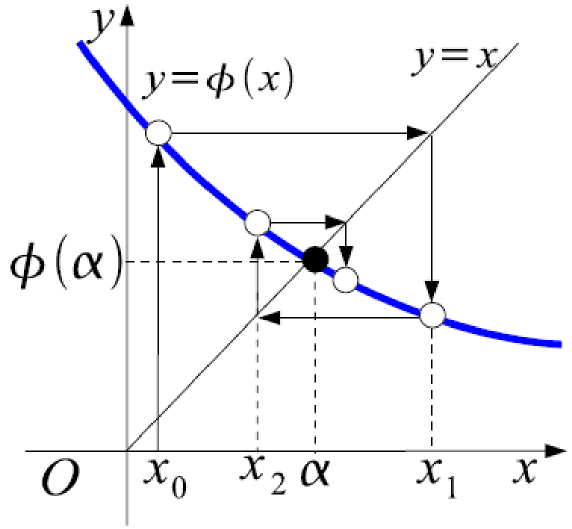
converges to the unique fixed point $\alpha \in [a,\, b]$ for any choice of $x_0 \in [a,\, b]$.

Theorem 2.4 (Ostrowski) *Let ϕ be a differentiable function in $[a,\, b]$ with fixed point $\alpha \in [a,\, b]$. If $|\,\phi'(\alpha)\,| < 1$, then exists $\delta > 0$ such that the fixed point iterations $x_{k+1} = \phi(x_k)$ converge to α for each x_0 with $|x_0 - \alpha| < \delta$.*

Fixed Point Iterations #3

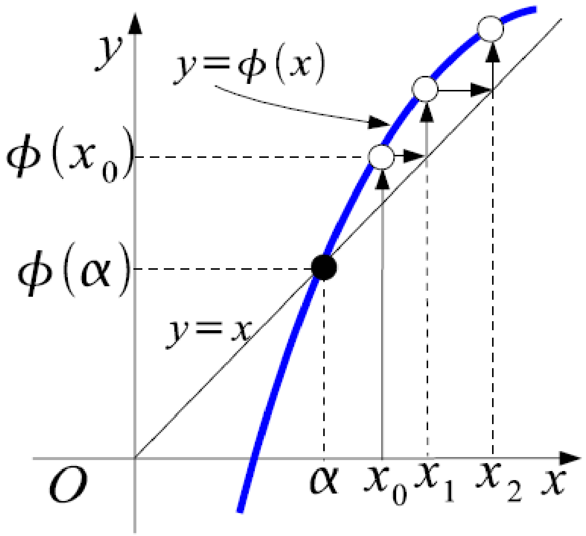


(a) $0 < \phi'(\alpha) < 1$: the iterations converge to α in a monotone fashion (increasing or decreasing accordingly to the position of x_0 with respect to α).



(b) $-1 < \phi'(\alpha) < 0$: the iterations converge to α with values alternately above and below α.

Fixed Point Iterations #4



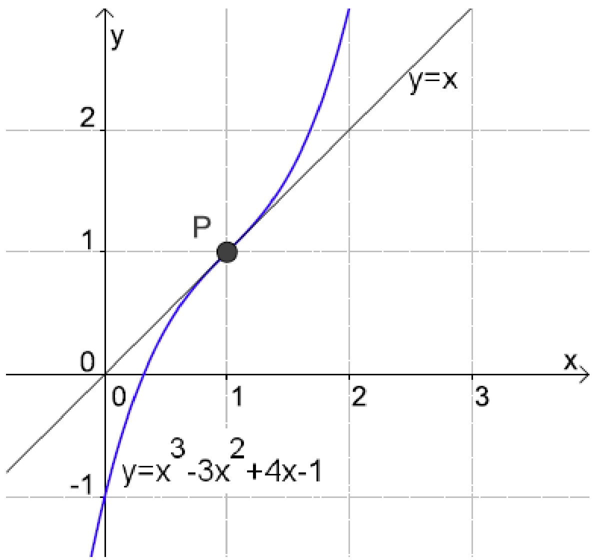
(a) $\phi'(\alpha) > 1$: the fixed point iterations diverge from α.

Fixed Point Iterations #5

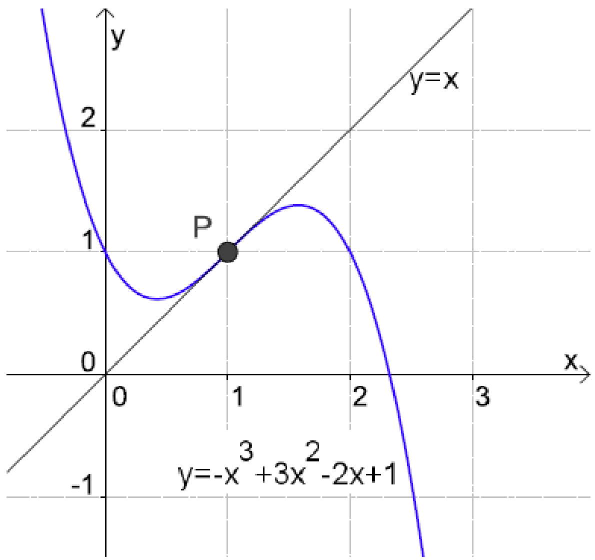
Example 2.4 *If* $|\phi'(\alpha)| = 1$ *the fixed point iteration* $x_{k+1} = \phi(x_k)$ *may, or may not, converge to the fixed point. The functions*

$$(a)\ \phi(x) = x^3 - 3x^2 + 4x - 1 \qquad (b)\ \phi(x) = -x^3 + 3x^2 - 2x + 1$$

have both the fixed point $\alpha = 1$ *with* $|\phi'(\alpha)| = 1$.



(a) Iterations x_k diverge from α for each $x_0 \neq \alpha$ chosen near α.



(b) Iterations x_k converge to α for each x_0 chosen near α.

Error Behaviour of Fixed Point Iterations

Theorem 2.5 *Let $\phi \in C^p(\ (\alpha-\delta,\alpha+\delta)\ )$ for suitable $\delta > 0$ and integer $p \geq 1$ of the fixed point α of ϕ. If*

$$\phi'(\alpha) = \phi''(\alpha) = \cdots = \phi^{(p-1)}(\alpha) = 0 \qquad \textit{and} \qquad \phi^{(p)}(\alpha) \neq 0$$

then the fixed point iterations $x_{k+1} = \phi(x_k)$ has order of convergence p and

$$\lim_{k\to+\infty} \frac{|\ e_{k+1}\ |}{|\ e_k\ |^p} = \frac{\phi^{(p)}(\alpha)}{p!}.$$

Proof. Using Taylor expansion we get

$$\begin{aligned} e_{k+1} &= x_{k+1} - \alpha = \phi(x_k) - \alpha \\ &= \sum_{j=0}^{p-1} \frac{\phi^{(j)}(\alpha)\cdot(x_k-\alpha)^j}{j!} + \frac{\phi^{(p)}(\xi_k)\cdot(x_k-\alpha)^p}{p!} \\ &\overset{(1)}{=} \frac{\phi^{(p)}(\xi_k)\cdot(\ e_k\ )^p}{p!} \end{aligned}$$

where ξ_k is a suitable point between α and x_k and (1) follows from $\phi^{(j)}(\alpha) = 0$, $j = 0, 1, \ldots, p-1$. Providing that x_k converges to the fixed point α, we also have that $\xi_k \to \alpha$ which completes the proof due to the continuity of $\phi^{(p)}$. $\square$

Stopping Criteria

It is common to terminate the convergent fixed point iterations

$$\begin{cases} x_0 & \text{given initial estimate of the fixed point } \alpha \\ x_{k+1} = & \phi(x_k), \quad k = 0, 1, 2, \ldots \end{cases}$$

when $|\, x_{k+1} - x_k \,| < \varepsilon$ for some given tolerance $\varepsilon > 0$.
Let us see how good is this stopping criteria. We have

$$x_{k+1} - \alpha = \phi(x_k) - \phi(\alpha) = \phi'(\xi_k)(x_k - \alpha)$$

for some ξ_k in the interval of endpoints α and x_k. Since it is

$$x_k - \alpha = (x_{k+1} - \alpha) - (x_{k+1} - x_k) \quad \Rightarrow \quad x_{k+1} - \alpha = x_k - \alpha + x_{k+1} - x_k$$

and denoting the error at the k-ih iteration by $e_k = x_k - \alpha$ we obtain

$$x_k - \alpha + x_{k+1} - x_k = \phi'(\xi_k)(x_k - \alpha) \quad \Rightarrow \quad e_k + x_{k+1} - x_k = \phi'(\xi_k)\, e_k$$

and finally, assuming that $\phi'(x) \neq 0$ near α and taking the absolute values,

$$\boxed{|\, e_k \,| = \frac{1}{|\, 1 - \phi'(\xi_k) \,|} \cdot |\, x_{k+1} - x_k \,|} \tag{2.2}$$

So, if $\phi'(\alpha) \approx 0$ (and, thus, $\phi'(x) \approx 0$ near α by continuity) the difference between two consecutive iterates is a reliable estimator of the error. Note that this is the case if $\phi'(\alpha) = 0$. If, otherwise, $\phi'(\alpha) \approx 1$, eq. (2.2) is not useful to estimate the error.

Exercises

Exercise 1 (1 minute, 2 points)
How many fixed points has the function $f(x) = x^2-x$?

Answer. The fixed points are the solution of the equation

$$x = f(x)$$
$$x = x^2-x$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

which gives $x_1 = 0$ and $x_2 = 2$

Indeed, for example,

$$2 = x_2 = f(x_2) = f(2) = 2^2-2$$

Exercises

Exercise 2 (5 minutes, 10 points)
Let $f(x) = x^{1/2}$. (a) Compute the fixed points of f. (b) Is the fixed point iterations $x_{k+1} = f(x_k)$ convergent for $x_0 = 2$? Is the sequence x_k monotone? (c) Can we choose a starting point x_0 such that the sequence x_k converges to $x_1 = 0$?

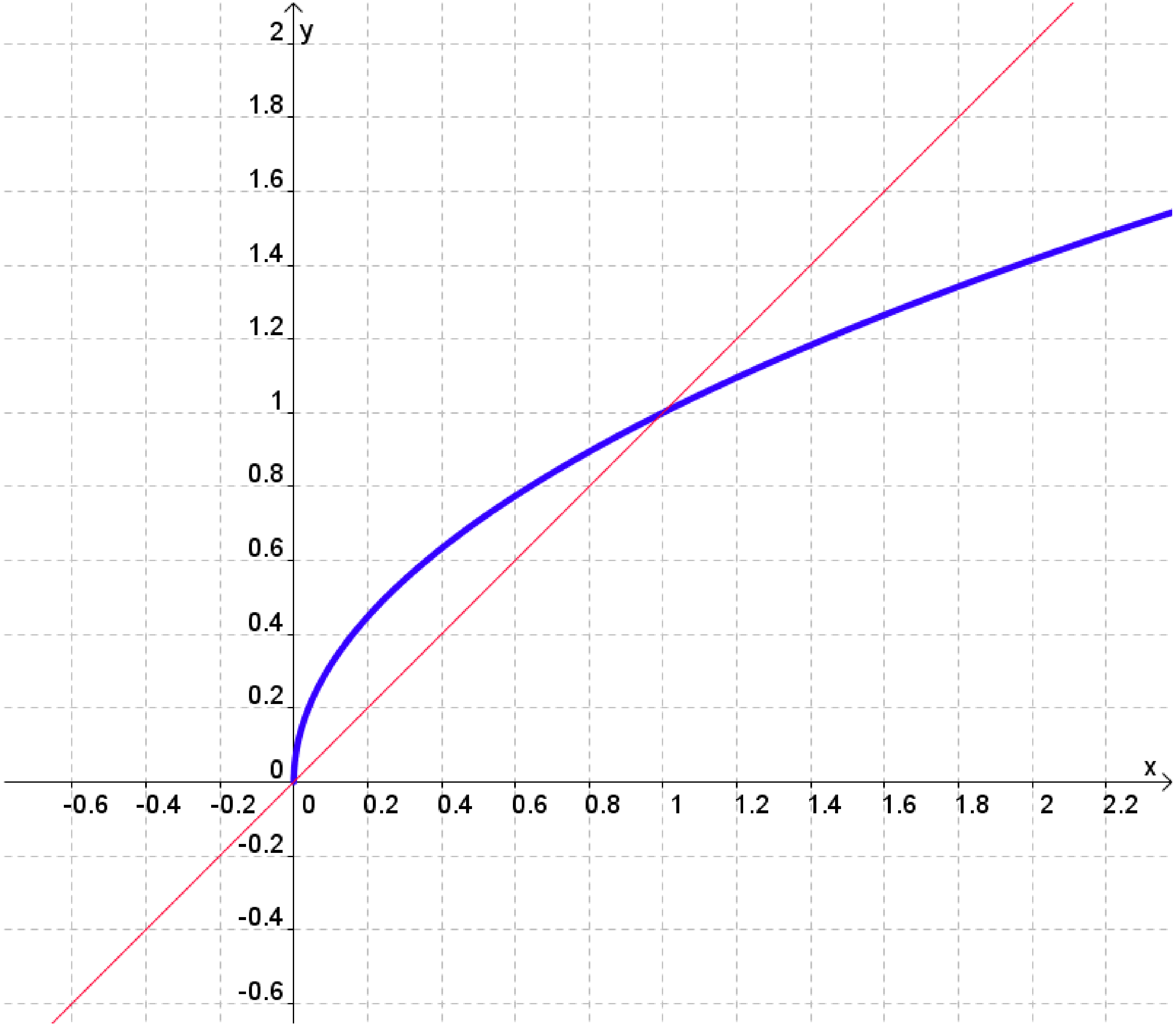
Answer. First of all note that we must have $x \geq 0$.
(a) The fixed points are solution of

$$x = f(x) \quad \text{or} \quad x = x^{1/2} \quad \text{or} \quad x^2 = x \quad \text{which gives} \quad x_1 = 0 \text{ and } x_2 = 1.$$

The graph of f(x) gives us the informations needed to answer (b) and (c).

(b) Yes, the fixed point iterations converges to the fixed point $x_2 = 1$. Moreover, the sequence x_k in monotonically decreasing.
(c) Yes, we can but the only possibility is to choose $x_0 = 0$. The correspondig fixed point iterations are $x_k = 0$ for all k.

Exercises



y
x
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2 2.2

Exercises

Exercise 3 (1 minute, 2 points)

Give an example of function f(x) which is NOT strictly monotone in [a, b] and has only one root in [a, b].

Answer. We can take $f(x) = | x |$ and $[a, b] = [-1, 1]$. The root is $x = 0$.

Exercise 4 (2 minutes, 2 points)

A problem has input $x = 1$. The correpsonding output is $y = 10$.
When $x = 1+10^{-3}$, the corresponding output becomes $y = 100$. Mark which of the following is true.

- ☐ The prolem is well conditioned.
- ☐ The condition number is K = 9000.
- ☐ It is impossible to estimate the condition number.
- ☐ The prolem is ill conditioned.

Answer. The problem is ill conditioned since a small change in the input gives a wide variation in the output. We also have

$$K = \frac{|(100-10)/10|}{|(1+10^{-3}-1) / 1|} = \frac{9}{10^{-3}} = 9000$$