Analytic dependence on parameters for Evans' approximated Weak KAM solutions

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Abstract

We consider a variational principle for approximated Weak KAM solutions proposed by Evans. For Hamiltonians in quasi-integrable form $h(p) + \varepsilon f(\varphi, p)$, we prove that the map which takes the parameters $(\varepsilon, P, \varrho)$ to Evans' approximated solution $u_{\varepsilon,P,\varrho}$ is real analytic. In the mechanical case, we compute a recursive system of periodic partial differential equations identifying univocally the coefficients for the power series of the perturbative parameter ε.

1 Introduction

In the classical integrability theory of Hamiltonian systems, a central role is played by the Hamilton-Jacobi method. The basic idea is to integrate the Hamilton's ODE by a change of variables $(x, p) \to (X, P)$ implicitly defined by a generating function $v(x, P)$. That is

$$\begin{cases} X = \partial_P v(x, P) \\ p = \partial_x v(x, P) \end{cases} \tag{1}$$

In particular, one looks for a function $v(x, P)$ and for an integrable Hamiltonian $\bar{H}(P)$ which solve the so-called Hamilton-Jacobi equation

$$H(x, \partial_x v(x, P)) = \bar{H}(P). \tag{2}$$

If there exists a smooth change of variable $(x, p) \to (X, P)$ which satisfies (1), then the original Hamiltonian dynamics transforms into the trivial dynamics

$$\begin{cases} \dot{X} = D_P \bar{H}(P) \\ \dot{P} = 0 \end{cases}$$

Clearly, only special Hamiltonians are integrable in the above sense: the Hamilton-Jacobi equation (2) does not in general admit smooth global solutions and, even if it does, the new variables (X, P) are not globally defined. However, most mechanical systems are quasi-integrable. That is

$$H(\varphi, p) = h(p) + \varepsilon f(\varphi, p), \tag{3}$$

where $(\varphi, p) \in \mathbb{T}^d \times \mathbb{R}^d$ are the angle-action variables, ε is a small real parameter and $d \in \mathbb{N}$, $d \geq 1$, is the fixed dimension of the ambient space.

For quasi-integrable Hamiltonians, the classical perturbation approach consists in finding a canonical transformation which pushes the perturbation to the order ε^2 and then iterating the procedure. Since for $\varepsilon = 0$ the Hamiltonian (3) is integrable, we look for a generating function in the form

$$v(\varphi, P) = P \cdot \varphi + \varepsilon u(\varphi, P) + \mathcal{O}(\varepsilon^2)$$

and possibly expand $v(\varphi, P)$ in a power series of ε. We note here that the ε-dependence of the generating function $v(x, P)$ is crucial also for numerical investigations, *e.g.* in Celestial and Quantum Mechanics. We also observe that in this context one has to deal with the resonances related to the so-called small divisors. The main strategies to handle such a problem are based on KAM and Nekhoroshev theorems (cf. [13, 2, 20, 23]) and on Newton-Nash-Moser implicit function theorem (cf. [21, 12]).

The application of such deep results leads to new intriguing questions concerning, for example, the generalization of the KAM Theory to a wider class of Hamiltonians which are not necessarily almost-integrable. The most important outcomes in this direction have been obtained by the Weak KAM Theory introduced by Mather, Mané and Fathi (see, *e.g.*, [19, 18, 10]) which exploits variational and PDE's methods to treat Tonelli Hamiltonians. In particular, by the Weak KAM Theorem one can prove that, for any P in $\mathbb{R}^d$ (and then with no non-resonance conditions) the Hamilton-Jacobi equation (2) admits global Lipschitz continuous solutions. The corresponding Hamiltonian $\bar{H}(P)$ is given by

$$\bar{H}(P) = \inf_{u \in C^1(\mathbb{T}^d)} \sup_{\varphi \in \mathbb{T}^d} H(\varphi, P + \partial_\varphi u(\varphi, P)) \,. \tag{4}$$

and is called "effective Hamiltonian". However, since Weak KAM solutions are in general not differentiable, they cannot be used as generating functions in order to conjugate the original flow to an integrable one.

In order to bypass this lack of regularity, in [7, 8] Evans introduced a sort of approximated integrability for Tonelli Hamiltonians. The main result of his approach is a sequence of smooth functions uniformly converging to a Weak KAM solution and defining, for any $P \in \mathbb{R}^d$, a dynamics on $\mathbb{T}^d$. The properties of this torus dynamics and its relations with the original Hamiltonian flow have been discussed in [8] and in [3]. More recently, Evans returned to this subject in [9].

In the present paper, we propose a functional analytic approach to investigate the variational approximated version of Weak KAM Theory introduced by Evans. For

Hamiltonians in the quasi-integrable form (3), we analyze the dependence on parameters of the sequence of Evans' approximated smooth solutions. In particular, we prove that the map which takes the perturbative parameter ε to the approximated solution is real analytic in a neighborhood of 0 (see Theorem 1 here below). As a consequence, it can be written in terms of a converging power series of ε for ε close to 0. Moreover, for mechanical Hamiltonians, we compute a recursive system of periodic partial differential equations which identifies univocally the coefficients of the power series of the parameter ε (see Section 4). We underline two possible applications of this regularity result. First, the converging power series of ε can be used in order to investigate the asymptotic behavior of the parameters involved in Evans' construction. Moreover, this series can be useful for a numerical treatment of the above sequence of smooth functions uniformly converging to a Weak KAM solution.

2 Analytical setting and main result

We start by recalling the main lines of the approach to Weak KAM Theory proposed by Evans in [7, 8]. Instead of looking for minimizers u for the sup norm

$$I[u] = \sup_{\varphi \in \mathbb{T}^d} H(\varphi, P + \partial_\varphi u(\varphi, P))$$

as suggested by formula (4), Evans considers a positive real number ϱ and looks for minimizers u of the functional

$$I_\varrho[u] = \int_{\mathbb{T}^d} e^{\varrho H(\varphi, P + \partial_\varphi u)} d\varphi\,. \tag{5}$$

Then, for all $(P, \varrho) \in \mathbb{R}^d \times \mathbb{R}_+$ the corresponding Euler-Lagrange equation is

$$\operatorname{div}_\varphi \left(e^{\varrho H(\varphi, P+\partial_\varphi u)} \frac{\partial H}{\partial p}(\varphi, P + \partial_\varphi u) \right) = 0. \tag{6}$$

In detail:

$$\begin{aligned} &\frac{1}{\varrho} \sum_{i=1}^{d} (H_{p_i}(\varphi, P + \partial_\varphi u))_{\varphi_i} + \sum_{i,j=1}^{d} H_{p_i}(\varphi, P + \partial_\varphi u) H_{p_j}(\varphi, P + \partial_\varphi u) u''_{ij} + \\ &+ \sum_{i=1}^{d} H_{\varphi_i}(\varphi, P + \partial_\varphi u) H_{p_i}(\varphi, P + \partial_\varphi u) = 0 \end{aligned} \tag{7}$$

where $u''_{ij} = \frac{\partial^2 u}{\partial \varphi_i \partial \varphi_j}$. Under suitable convexity hypotheses on H –see $(c1)$, $(c2)$ and $(c3)$ below– and by using standard variational techniques, Evans proves the existence of minimizers u for (5) for all $(P, \varrho) \in \mathbb{R}^d \times \mathbb{R}_+$. He also shows that such minimizers are smooth and unique up to an additive constant. (So that there exists a unique minimizer with zero integral mean, *i.e.* such that $\int_{\mathbb{T}^d} u d\varphi = 0$.)

In the present paper we focus our attention on smooth real valued Hamiltonians H defined on the covering space $\mathbb{R}^d \times \mathbb{R}^d$ of $\mathbb{T}^n \times \mathbb{R}^n$ by the quasi–integrable form

$$H(\varphi, p) = h(p) + \varepsilon f(\varphi, p)$$

where the functions h and f satisfy the following conditions:

($c1$) (*periodicity in* φ) For any $p \in \mathbb{R}^d$, the mapping $\varphi \mapsto f(\varphi, p)$ is $\mathbb{T}^d$-periodic;

($c2$) (*strict convexity*) There exists a constant $\gamma > 0$ such that

$$\frac{\partial^2 h}{\partial p_i \partial p_j}(p)\xi_i \xi_j \geq \gamma |\xi|^2 \tag{8}$$

for each $p, \xi \in \mathbb{R}^d$;

($c3$) (*growth bounds*) There exists a constant $C > 0$ such that

$$|f(\varphi, p)| \leq C, \quad |D^2_{\varphi,p} f(\varphi, p)| \leq C(1 + |p|),$$

$$|D^2_{\varphi} f(\varphi, p)| \leq C(1 + |p|^2), \quad |D^2_p H(\varphi, p)| \leq C$$

for each $\varphi, p \in \mathbb{R}^d$;

($c4$) (*regularity of* f *and* h) We suppose that $f(\varphi, p)$ is a jointly real analytic function of $(\varphi, p) \in \mathbb{T}^d \times \mathbb{R}$ and that h is real analytic.

As proved by Evans [7, Thm. 5.2], conditions (c1) – (c3) imply the existence of a unique solution of equation (6) with zero integral mean. We shall denote such a solution by $u_{\varepsilon,P,\rho}$. Then we ask the following question:

what can be said on the function which takes (ε, P, ρ) to $u_{\varepsilon,P,\rho}$?

In particular,

what about the ε-dependence?

In our main Theorem 1 we prove that under conditions (c1) – (c4) the map which takes (ε, P, ρ) to $u_{\varepsilon,P,\rho}$ is real analytic. However, one may wish to relax the regularity condition in (c4) and –for example– ask a differentiability condition on f and h instead of the real analyticity prescribed in (c4) (cf. Proposition 4 below). As one can expect, a weaker regularity assumption on f and h leads to a lower regularity of the function which takes (ε, P, ρ) to the solution $u_{\varepsilon,P,\rho}$ (cf. Thm. 7 below).

The proof of Theorem 1 utilizes a functional analytic approach. We identify $u_{\varepsilon,P,\varrho}$ as the implicit solution of a functional equation $\tilde{M}(\varepsilon, P, \varrho, u) = 0$, where $\tilde{M}$ is a (non-linear) operator acting between suitable Banach spaces (see (12) and (13) below). Then we study the dependence of $u_{\varepsilon,P,\varrho}$ upon $(\varepsilon, P, \varrho)$ by means of the Implicit Function Theorem for real analytic maps (cf., *e.g.*, Deimling, Ch. 4 in [6]). We observe that methods based on the Implicit Function Theorem have been largely exploited for the study of nonlinear perturbation problems. We refer for example to the works of Stoppelli and Valent in nonlinear elasticity (see, *e.g.*, [24, 25, 26]) and to the approach of Henry for the analysis of (regular) perturbations of the domain in boundary value problems (cf. [11]). We also mention the papers written by the second named author together with Lanza de Cristoforis and Musolino where a method based on the Implicit Function Theorem is applied to the study of singular perturbations of the domain in linear and nonlinear boundary value problems (see, for example, [5, 15]).

In the present paper we will need to set the problem in the frame of Banach spaces of periodic functions with the following two properties: they have to be appropriate for the application of the standard elliptic regularity theory and, in addition, they have to be closed under the product of functions. A suitable choice is that of periodic Schauder spaces. Here below, we first introduce such spaces and then we state the main result of the paper.

For any $m \in \mathbb{N}$ and $\beta \in [0, 1[$, we denote by $C^{m,\beta}(\mathbb{T}^d)$ the space of periodic functions from $\mathbb{R}^d$ to $\mathbb{R}$ which have continuous partial derivatives up to the order m and β-Hölder continuous derivatives of order m. As is well known, $C^{m,\beta}(\mathbb{T}^d)$ is a Banach space. In addition, we denote by $C^{m,\beta}_z(\mathbb{T}^d)$ the closed subspace of $C^{m,\beta}(\mathbb{T}^d)$ consisting of the functions with zero mean, $\int_{\mathbb{T}^d} u \, d\varphi = 0$. For the sake of brevity we write $C^m(\mathbb{T}^d)$ instead of $C^{m,0}(\mathbb{T}^d)$. Then,

$$\text{we fix once for all } \alpha \in]0, 1[$$

and we have the following Theorem 1 which is an immediate consequence of Theorem 7 below.

Theorem 1. *Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth Hamiltonian in the quasi–integrable form*

$$H(\varphi, p) = h(p) + \varepsilon f(\varphi, p) \,,$$

where the functions h and f satisfy conditions (c1) – (c4). For any $(P, \varrho) \in \mathbb{R}^d \times \mathbb{R}_+$, there exists $\varepsilon_0 > 0$ such that the map from $] - \varepsilon_0, \varepsilon_0[\to C^{2,\beta}_z(\mathbb{T}^d)$ which takes ε to the unique solution $u_{\varepsilon,P,\varrho}$ of equation (6) is real analytic.

We observe that by Theorem 7 one may also deduce that the map from $] - \varepsilon_0, \varepsilon_0[\times \mathbb{R}^d \times \mathbb{R}_+$ to $C^{2,\beta}_z(\mathbb{T}^d)$ which takes a triple $(\varepsilon, P, \varrho)$ to $u_{\varepsilon,P,\varrho}$ is real analytic.

As an immediate consequence of Theorem 1, there exists $0 < \varepsilon_1 \leq \varepsilon_0$ and a sequence $\{v_{k,P,\rho}\}_{k\in\mathbb{N}}$ in $C^{2,\alpha}_z(\mathbb{T}^d)$ such that

$$u_{\varepsilon,P,\rho} = \sum_{k=0}^{+\infty} \frac{\varepsilon^k}{k!} v_{k,P,\rho} \qquad \forall \varepsilon \in] - \varepsilon_1, \varepsilon_1[$$

where the series converges absolutely and uniformly in $C^{2,\alpha}_z(\mathbb{T}^d)$. In Section 4 we consider the mechanical case $H(\varphi,p) = |p|^2/2 + \varepsilon f(\varphi)$ and we compute a recursive system of periodic partial differential equations which identify univocally the coefficients $\{v_{k,P,\rho}\}_{k\in\mathbb{N}}$. Finally, we observe that for a numerical use of such a system, one may be interested in asymptotic approximations of $u_{\varepsilon,P,k}$ rather than having the complete series expansion. Under the hypothesis of Theorem 1 one can prove that

$$u_{\varepsilon,P,k} = \sum_{h=0}^{N} \frac{\varepsilon^h}{h!} v_{h,P,k} + O(\varepsilon^{N+1}) \qquad \text{as } \varepsilon \to 0\,,$$

for all $N \in \mathbb{N}$. However, asymptotic approximations of such a form do not require the real analyticity of the functions f and h and can be deduced under weaker regularity assumptions (cf. Theorem 7 below).

3 Proof of Theorem 1

3.1 Regularity of the operators

We start by studying the linear operator $L_{P,\varrho}$ defined by

$$L_{P,\varrho} u = \sum_{i,j=1}^{d} \left(\frac{1}{\varrho} \frac{\partial^2 h}{\partial p_i \partial p_j}(P) + \frac{\partial h}{\partial p_i}(P) \frac{\partial h}{\partial p_j}(P) \right) u''_{ij}$$

for all $u \in C^{2,\alpha}(\mathbb{T}^d)$. In view of the strict convexity hypothesis (8), we observe that

$$\sum_{i,j=1}^{d} \left(\frac{1}{\varrho} \frac{\partial^2 h}{\partial p_i \partial p_j}(P) + \frac{\partial h}{\partial p_i}(P) \frac{\partial h}{\partial p_j}(P) \right) \xi_i \xi_j \geq \frac{\gamma}{\varrho}|\xi|^2 + \left(\sum_{i=1}^{d} \frac{\partial h}{\partial p_i}(P)\xi_i \right)^2 \geq \frac{\gamma}{\varrho}|\xi|^2$$

for all $\xi \in \mathbb{R}^d$. Thus $L_{P,\varrho}$ is elliptic and we have the following

Proposition 2. *Let $(P,\varrho) \in \mathbb{R}^d \times \mathbb{R}_+$ be fixed. The following statements hold:*

- *(i) $L_{P,\varrho} u \in C^{0,\alpha}_z(\mathbb{T}^d)$ for all $u \in C^{2,\alpha}(\mathbb{T}^d)$;*
- *(ii) The map which takes u to $L_{P,\varrho} u$ is an isomorphism from $C^{2,\alpha}_z(\mathbb{T}^d)$ to $C^{0,\alpha}_z(\mathbb{T}^d)$.*

We premise an elementary lemma to the proof of Proposition 2. In the sequel, $\mathbb{Q}^d$ denotes the open domain $]0,1[^d$ with boundary $\partial\mathbb{Q}^d$, $\nu_{\mathbb{Q}^d}$ denotes the outward unit normal to $\partial\mathbb{Q}^d$, and $d\sigma$ denotes the area element on $\partial\mathbb{Q}^d$.

Lemma 3. *We have*

$$\int_{\mathbb{T}^d} \operatorname{div} v \, d\varphi = \int_{\partial\mathbb{Q}^d} \nu_{\mathbb{Q}^d} \cdot v \, d\sigma = 0$$

for all vector valued functions $v \equiv (v_1,\ldots,v_d) \in (C^1(\mathbb{T}^d))^d$.

Proof. It follows by the divergence theorem, by the periodicity of f, and by the equality $\nu_{\mathbb{Q}^d}(\varphi) = -\nu_{\mathbb{Q}^d}(\varphi')$ which holds for all points $\varphi \equiv (\varphi_1, \ldots, \varphi_{j-1}, 0, \varphi_j, \ldots, \varphi_d)$ and $\varphi' \equiv (\varphi_1, \ldots, \varphi_{j-1}, 1, \varphi_j, \ldots, \varphi_d)$ in $\partial\mathbb{Q}^d$ and for all $j \in \{1, \ldots, d\}$. □

Proof. (Proposition 2) (i) It is easily verified that $L_{P,\varrho}u \in C^{0,\alpha}(\mathbb{T}^d)$, so it remains to show that $\int_{\mathbb{T}^d} L_{P,\varrho}u\,dx = 0$. Let $A_{P,\varrho}$ denote the $d \times d$ real matrix with entries $(A_{P,\varrho})_{i,j}$ defined by

$$(A_{P,\varrho})_{i,j} \equiv \frac{1}{\varrho}\frac{\partial^2 h}{\partial p_i \partial p_j}(P) + \frac{\partial h}{\partial p_i}(P)\frac{\partial h}{\partial p_j}(P) \qquad \forall (i,j) \in \{1, \ldots, d\}^2$$

Then $L_{P,\varrho}u = \operatorname{div}(A_{P,\varrho}\nabla u)$ for all $u \in C^{2,\alpha}(\mathbb{T}^d)$. Thus $\int_{\mathbb{T}^d} L_{P,\varrho}u\,d\varphi = 0$ by the periodicity of $A_{P,\varrho}\nabla u$ and by Lemma 3.

(ii) Since $L_{P,\varrho}$ is continuous from $C^{2,\alpha}_z(\mathbb{T}^d)$ to $C^{0,\alpha}_z(\mathbb{T}^d)$ it suffices to show that it is one-to-one and onto in order to derive that it is an isomorphism by the open mapping theorem. If $L_{P,\varrho}u = 0$ then a standard energy argument shows that $\int_{\mathbb{T}^d} \nabla u \cdot A\nabla u\,d\varphi = 0$. Accordingly $\nabla u \cdot A\nabla u = 0$ on $\mathbb{T}^d$ and thus $\nabla u = 0$ by the ellipticity of $L_{P,\varrho}$. Thus u is constant and then $u = 0$ because $\int_{\mathbb{T}^d} u\,d\varphi = 0$ by the membership of u in $C^{2,\alpha}_z(\mathbb{T}^d)$. Now we have to prove that $L_{P,\varrho}$ is onto. Let $v \in C^{0,\alpha}_z(\mathbb{T}^d)$. Then we denote by $\mathcal{N}_{P,\varrho}(v)$ the periodic newtonian potential defined by

$$\mathcal{N}_{P,\varrho}(v)(\varphi) = \int_{\mathbb{T}^d} S_{L_{P,\varrho},\mathbb{T}^d}(\varphi - \vartheta)v(\vartheta)\,d\vartheta \qquad \forall \varphi \in \mathbb{T}^d\,,$$

where $S_{L_{P,\varrho},\mathbb{T}^d}$ denotes the periodic analog of a fundamental solution of $L_{P,\varrho}$ introduced in Appendix A. Then by a classical argument based on Fubini Theorem and the periodicity of $S_{L_{P,\varrho},\mathbb{T}^d}$ one verifies that

$$\int_{\mathbb{T}^d} \mathcal{N}_{P,\varrho}(v)(\varphi)\,d\varphi = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} S_{L_{P,\varrho},\mathbb{T}^d}(\varphi - \vartheta)v(\vartheta)\,d\vartheta\,d\varphi = \int_{\mathbb{T}^d} v(\vartheta)\,d\vartheta \int_{\mathbb{T}^d} S_{L_{P,\varrho},q}(\varphi)\,d\varphi = 0\,.$$

Thus, by Proposition 9 in Appendix A we have $\mathcal{N}_{P,\varrho}(v) \in C^{2,\alpha}_z(\mathbb{T}^d)$ and $L_{P,\varrho}\mathcal{N}_{P,\varrho}(v) = v$. □

We proceed by studying the (nonlinear) operator M from $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C^{2,\alpha}_z(\mathbb{T}^d)$ to $C^{0,\alpha}(\mathbb{T}^d)$ which takes $(\varepsilon, P, \varrho, u)$ to the function defined by the left hand side of (7). So that (7) is equivalent to $M(\varepsilon, P, \varrho, u) = 0$. In order to investigate the mapping properties of M and enstablish the correct regularity assumptions on the functions f and h, we exploit the following notation for the composition operators.

If F is a continuous function from $\mathbb{T}^d \times \mathbb{R}^d$ to $\mathbb{R}$, then we denote by $\mathcal{T}_F$ the (nonlinear nonautonomous) composition operator from $(C(\mathbb{T}^d))^d$ to $C(\mathbb{T}^d)$ which takes a vector valued function $v \equiv (v_1, \ldots, v_d)$ to the function $\mathcal{T}_F(v)$ defined by

$$\mathcal{T}_F(v)(\varphi) \equiv F(\varphi, v(\varphi)) \qquad \forall \varphi \in \mathbb{T}^d\,.$$

Similarly, for a continuous function G from $\mathbb{R}^d$ to $\mathbb{R}$, we denote by $\mathcal{T}_G$ the (nonlinear autonomous) composition operator from $(C(\mathbb{T}^d))^d$ to $C(\mathbb{T}^d)$ which takes a vector valued function $v \equiv (v_1, \ldots, v_d)$ to the function $\mathcal{T}_G(v)$ defined by

$$\mathcal{T}_G(v)(\varphi) \equiv G(v(\varphi)) \qquad \forall \varphi \in \mathbb{T}^d$$

In the sequel we shall assume the following condition:

$$\begin{aligned} &\text{The composition operators } \mathcal{T}_f,\ \mathcal{T}_h, \text{ and } \mathcal{T}_{\partial_{\varphi_j} f}, \text{ with } j \in \{1, \ldots, d\}, \text{ map} \\ &\text{functions of } (C^{1,\alpha}(\mathbb{T}^d))^d \text{ to functions of } C^{0,\alpha}(\mathbb{T}^d). \end{aligned} \tag{9}$$

In addition we shall assume either one of the following conditions (10) and (11). Here q is fixed natural number in $\mathbb{N} \setminus \{0\}$.

$$\begin{aligned} &\text{The maps } \mathcal{T}_f \text{ and } \mathcal{T}_h \text{ are of class } C^{q+2} \text{ from } (C^{1,\alpha}(\mathbb{T}^d))^d \text{ to } C^{0,\alpha}(\mathbb{T}^d) \text{ and} \\ &\text{the maps } \mathcal{T}_{\partial_{\varphi_j} f}, \text{ with } j \in \{1, \ldots, d\}, \text{ are of class } C^{q+1} \text{ from } (C^{1,\alpha}(\mathbb{T}^d))^d \text{ to } C^{0,\alpha}(\mathbb{T}^d). \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{The maps } \mathcal{T}_f \text{ and } \mathcal{T}_h \text{ are real analytic from } (C^{1,\alpha}(\mathbb{T}^d))^d \text{ to } C^{0,\alpha}(\mathbb{T}^d) \text{ and} \\ &\text{the maps } \mathcal{T}_{\partial_{\varphi_j} f}, \text{ with } j \in \{1, \ldots, d\}, \text{ are real analytic from } (C^{1,\alpha}(\mathbb{T}^d))^d \text{ to } C^{0,\alpha}(\mathbb{T}^d). \end{aligned} \tag{11}$$

We observe that condition (10) implies that $\mathcal{T}_{\partial^2_{p_i p_j} h}$, $\mathcal{T}_{\partial_{p_i} h}$, $\mathcal{T}_{\partial_{p_i} f}$, $\mathcal{T}_{\partial_{\varphi_i} f}$, $\mathcal{T}_{\partial^2_{p_i p_j} f}$, and $\mathcal{T}_{\partial^2_{\varphi_i p_i} f}$ are continuously Frechèt differentiable maps of class C^q from $(C^{1,\alpha}(\mathbb{T}^d))^d$ to $C^{0,\alpha}(\mathbb{T}^d)$ while condition (11) implies that $\mathcal{T}_{\partial^2_{p_i p_j} h}$, $\mathcal{T}_{\partial_{p_i} h}$, $\mathcal{T}_{\partial_{p_i} f}$, $\mathcal{T}_{\partial_{\varphi_i} f}$, $\mathcal{T}_{\partial^2_{p_i p_j} f}$, and $\mathcal{T}_{\partial^2_{\varphi_i p_i} f}$ are real analytic from $(C^{1,\alpha}(\mathbb{T}^d))^d$ to $C^{0,\alpha}(\mathbb{T}^d)$, see [15, Prop. 6.3]. Clearly condition (11) implies condition (10).
Finally, the next proposition gives some sufficient conditions for the validity of (9), (10), and (11). In the sequel we say that a function f belongs to $C^m(\mathbb{T}^d \times \mathbb{R}^d)$ if f belongs to $C^m(\mathbb{R}^d \times \mathbb{R}^d)$ and for every $\xi \in \mathbb{R}^d$ fixed the map which takes $x \in \mathbb{R}^d$ to $f(x, \xi)$ is periodic. Similarly, we say that f is jointly real analytic from $\mathbb{T}^d \times \mathbb{R}^d$ to $\mathbb{R}$ if it is jointly real analytic from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$ and for every $\xi \in \mathbb{R}^d$ fixed the map which takes $x \in \mathbb{R}^d$ to $f(x, \xi)$ is periodic.

Proposition 4. *The following statements hold.*

(i) *If $f \in C^{q+4}(\mathbb{T}^d \times \mathbb{R}^d)$ and $h \in C^{q+4}(\mathbb{R}^d)$, then conditions* (9) *and* (10) *are verified.*

(ii) *If f is jointly real analytic from $\mathbb{T}^d \times \mathbb{R}^d$ to $\mathbb{R}$ and h is real analytic, then conditions* (9) *and* (11) *are verified.*

Proof. Let Ω be an open neighbourhood of $\mathrm{cl}\mathbb{Q}^d$ in $\mathbb{R}^d$ and assume that Ω is of class C^1. Then the membership of f in $C^{q+4}(\mathbb{T}^d \times \mathbb{R}^d)$ imply that $f_{|\mathrm{cl}\Omega \times \mathbb{R}^d} \in C^{q+4}(\mathrm{cl}\Omega \times \mathbb{R}^d)$. Accordingly, the validity of statement (i) follows by [26, Thm. 4.4 in Chap. II]. To show that statement (ii) holds, we note that if f is real analytic then the functions from

$\mathrm{cl}\Omega \times \mathbb{R}^d$ to $\mathbb{R}$ which takes (x,ξ) to $f(x,\xi)$ and to $\partial_{x_i} f(x,\xi)$, with $i \in \{1,\dots,d\}$ are real analytic in ξ uniformly with respect to x. Then the validity of (ii) follows by [26, Thm. 5.2 in Chap. II]. $\square$

We write now the (nonlinear) operator M in terms of the operators $\mathcal{T}_{\partial^2_{p_ip_j}h}$, $\mathcal{T}_{\partial_{p_i}h}$, $\mathcal{T}_{\partial^2_{\varphi_ip_i}f}$, $\mathcal{T}_{\partial_{p_i}f}$ involving the integrable Hamiltonian h and the function f.

$$\begin{aligned}
M(\varepsilon,P,\varrho,u) = {} & \sum_{i,j=1}^{d}\left(\frac{1}{\varrho}\mathcal{T}_{\partial^2_{p_ip_j}h}(P+\partial_\varphi u) + \mathcal{T}_{\partial_{p_i}h}(P+\partial_\varphi u)\mathcal{T}_{\partial_{p_j}h}(P+\partial_\varphi u)\right)u''_{ij} \\
& + \frac{\varepsilon}{\varrho}\sum_{i,j=1}^{d}\mathcal{T}_{\partial^2_{p_ip_j}f}(P+\partial_\varphi u) + \frac{\varepsilon}{\varrho}\sum_{i=1}^{d}\mathcal{T}_{\partial^2_{\varphi_ip_i}f}(P+\partial_\varphi u) \\
& + 2\varepsilon\sum_{i,j=1}^{d}\mathcal{T}_{\partial_{p_i}f}(P+\partial_\varphi u)\mathcal{T}_{\partial_{p_j}h}(P+\partial_\varphi u)u''_{ij} + \varepsilon\sum_{i=1}^{d}\mathcal{T}_{\partial_{\varphi_i}f}(P+\partial_\varphi u)\mathcal{T}_{\partial_{p_i}h}(P+\partial_\varphi u) \\
& + \varepsilon^2\sum_{i,j=1}^{d}\mathcal{T}_{\partial_{p_i}f}(P+\partial_\varphi u)\mathcal{T}_{\partial_{p_j}f}(P+\partial_\varphi u)u''_{ij} + \varepsilon^2\sum_{i=1}^{d}\mathcal{T}_{\partial_{\varphi_i}f}(P+\partial_\varphi u)\mathcal{T}_{\partial_{p_i}f}(P+\partial_\varphi u)
\end{aligned} \tag{12}$$

Then, by standard calculus in Banach spaces and by the continuity of the product of functions from $C^{0,\alpha}_z(\mathbb{T}^d) \times C^{0,\alpha}_z(\mathbb{T}^d)$ to $C^{0,\alpha}(\mathbb{T}^d)$, one proves the following

Proposition 5. *Let condition* (9) *hold true.*

- (*i*) *If condition* (10) *is verified for a* $q \in \mathbb{N}\setminus\{0\}$, *then the map* M *is of class* C^q *from* $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+\times C^{2,\alpha}_z(\mathbb{T}^d)$ *to* $C^{0,\alpha}(\mathbb{T}^d)$.
- (*ii*) *If in addition condition* (11) *holds true, then the map* M *is real analytic from* $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+\times C^{2,\alpha}_z(\mathbb{T}^d)$ *to* $C^{0,\alpha}(\mathbb{T}^d)$.

3.2 Applying the Implicit Function Theorem

We plan to use the Implicit Function Theorem for real analytic maps in order to study equation $M(\varepsilon,p,\varrho,u) = 0$ in a neighbourhood of a fixed point $(0,P_0,\varrho_0,0) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}_+\times C^{2,\alpha}_z(\mathbb{T}^d)$.

The partial differential of M with respect to the variable u evaluated at $(0,P_0,\varrho_0,0)$ is delivered by

$$\partial_u M(0,P_0,\varrho_0,0).\delta u = L_{P_0,\varrho_0}\delta u \qquad \forall \delta u \in C^{2,\alpha}_z(\mathbb{T}^d)$$

and L_{P_0,ϱ_0} is an isomorphism from $C^{2,\alpha}_z(\mathbb{T}^d)$ to $C^{0,\alpha}_z(\mathbb{T}^d)$ (cf. Prop. 2).
We note that, since $\int_{\mathbb{T}^d} M(\varepsilon,p,\varrho,u)\,d\varphi$ may be different from 0, the image of M is not

contained in $C_z^{0,\alpha}(\mathbb{T}^d)$. To overcome this difficulty, we introduce the auxiliary map $\tilde{M}$ defined by

$$\tilde{M}(\varepsilon, P, \varrho, u) \equiv e^{\varrho(h(P+\partial_\varphi u)+\varepsilon f(\varphi, P+\partial_\varphi u))} M(\varepsilon, P, \varrho, u)$$

for all $(\varepsilon, P, \varrho, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C_z^{2,\alpha}(\mathbb{T}^d)$, or equivalently, by using the operators $\mathcal{T}_h$ and $\mathcal{T}_f$,

$$\tilde{M}(\varepsilon, P, \varrho, u) = e^{\varrho(\mathcal{T}_h(P+\partial_\varphi u)+\varepsilon \mathcal{T}_f(P+\partial_\varphi u))} M(\varepsilon, P, \varrho, u) \tag{13}$$

Then one verifies that

$$\tilde{M}(\varepsilon, P, \varrho, u) = \frac{1}{\varrho}\mathrm{div}_\varphi \left(e^{\varrho(\mathcal{T}_h(P+\partial_\varphi u)+\varepsilon \mathcal{T}_f(P+\partial_\varphi u))} (\mathcal{T}_{\partial_{p_i} h}(P+\partial_\varphi u) + \varepsilon \mathcal{T}_{\partial_{p_i} f}(P+\partial_\varphi u))_{i\in\{1,\ldots,d\}} \right)$$

and thus, by Lemma 3, we conclude that

$$\int_{\mathbb{T}^d} \tilde{M}(\varepsilon, P, \varrho, u)\, d\varphi = 0$$

for all $(\varepsilon, P, \varrho, u) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C_z^{2,\alpha}(\mathbb{T}^d)$. Accordingly $\tilde{M}(\varepsilon, P, \varrho, u) \in C_z^{0,\alpha}(\mathbb{T}^d)$ and by using Proposition 5 one shows an analog result for the map $\tilde{M}$.

Proposition 6. *Let condition* (9) *hold true.*

(*i*) *If condition* (10) *is verified for a* $q \in \mathbb{N} \setminus \{0\}$*, then* $\tilde{M}$ *is a map of class* C^q *from* $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C_z^{2,\alpha}(\mathbb{T}^d)$ *to* $C_z^{0,\alpha}(\mathbb{T}^d)$.

(*ii*) *If in addition condition* (11) *holds true, then* $\tilde{M}$ *is real analytic from* $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C_z^{2,\alpha}(\mathbb{T}^d)$ *to* $C_z^{0,\alpha}(\mathbb{T}^d)$.

Finally, a straightforward calculation shows that the partial differential of $\tilde{M}$ with respect to the variable u evaluates at $(0, p_0, \varrho_0, 0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \times C_z^{2,\alpha}(\mathbb{T}^d)$ is delivered by

$$\partial_u \tilde{M}(0, P_0, \varrho_0, 0).\delta u = e^{\varrho_0 h(P_0)} L_{P_0,\varrho_0} \delta u \qquad \forall \delta u \in C_z^{2,\alpha}(\mathbb{T}^d)$$

Then, by Proposition 2, $\partial_u \tilde{M}(0, P_0, \varrho_0, 0)$ is an isomorphism from $C_z^{2,\alpha}(\mathbb{T}^d)$ to $C_z^{0,\alpha}(\mathbb{T}^d)$ and by the Implicit Function Theorem, see [6, Ch. 4], one deduces the following

Theorem 7. *Let* $(P_0, \varrho_0) \in \mathbb{R}^d \times \mathbb{R}_+$*. Let condition* (9) *hold true.*

(*i*) *Assume that condition* (10) *is verified for a* $q \in \mathbb{N} \setminus \{0\}$*. Then there exist a neighborhood* $\mathcal{U}$ *of* $(0, P_0, \varrho_0)$ *in* $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$*, a neighborhood* $\mathcal{V}$ *of* 0 *in* $C_z^{2,\alpha}(\mathbb{T}^d)$ *and a map* U *of class* C^q *from* $\mathcal{U}$ *to* $\mathcal{V}$ *such that the set of zeros of* $\tilde{M}$ *in* $\mathcal{U} \times \mathcal{V}$ *coincides with the graph of* U.

(*ii*) *If in addition condition* (11) *is verified, then* U *is real analytic.*

In particular we have $U(0, P_0, \varrho_0) = 0$ and

$$\tilde{M}(\varepsilon, P, \varrho, U(\varepsilon, P, \varrho)) = 0 \qquad \forall(\varepsilon, P, \varrho) \in \mathcal{U}\,.$$

So that

$$M(\varepsilon, P, \varrho, U(\varepsilon, P, \varrho)) = 0 \qquad \forall(\varepsilon, P, \varrho) \in \mathcal{U}$$

(cf. equality (13)). Thus $U(\varepsilon, P, \varrho)$ coincides with the unique solution $u_{\varepsilon,P,\varrho}$ of

$$M(\varepsilon, P, \varrho, u_{\varepsilon,P,\varrho}) = 0$$

found by Evans under conditions $(c1)$–$(c3)$ in the Introduction (see also Thm. 5.2 in [7]). Accordingly, we have

$$u_{\varepsilon,P,\varrho} = U(\varepsilon, P, \varrho) \qquad \forall(\varepsilon, P, \varrho) \in \mathcal{U}\,. \tag{14}$$

Finally, since hypothesis $(c4)$ for $H(\varphi, p) = h(p) + \varepsilon f(\varphi, p)$ imply conditions (9) and (11) (cfr. Proposition 4), Theorem 1 immediately follows.

4 Mechanical case

This section is devoted to the mechanical case:

$$H(\varphi, p) = |p|^2/2 + \varepsilon f(\varphi)$$

Let us fix $P \in \mathbb{R}^d$ and $k \in \mathbb{N} \setminus \{0\}$. We focus our attention on the dependence of $u_{\varepsilon,P,k}$ upon the perturbative parameter ε. As an immediate consequence of Theorem 1 and of equality (14), there exist $\varepsilon_1 > 0$ and a sequence $\{v_{h,P,k}\}_{h\in\mathbb{N}}$ in $C^{2,\alpha}_z(\mathbb{T}^d)$ such that

$$u_{\varepsilon,P,k} = \sum_{h=0}^{+\infty} \frac{\varepsilon^h}{h!} v_{h,P,k} \qquad \forall \varepsilon \in]-\varepsilon_1, \varepsilon_1[$$

where the series converges uniformly in $C^{2,\alpha}_z(\mathbb{T}^d)$.

We now show how to compute a sequence of recursive equations which determine the $v_{h,P,k}$'s. Starting by equality $\tilde{M}(\varepsilon, P, k, u_{\varepsilon,P,\varrho}) = 0$ (see formula (13)) and using the general Leibniz rule, we have

$$\begin{aligned} \partial^h_\varepsilon(\tilde{M}(\varepsilon, P, k, u_{\varepsilon,P,\varrho})) &= e^{k\left(\frac{|P+\partial_\varphi u_{\varepsilon,P,\varrho}|^2}{2} + \varepsilon g\right)} \partial^h_\varepsilon(M(\varepsilon, P, k, u_{\varepsilon,P,\varrho})) \\ &+ \sum_{l=0}^{h-1} \binom{h}{j} \partial^{h-j}_\varepsilon \big(e^{k\left(\frac{|P+\partial_\varphi u_{\varepsilon,P,\varrho}|^2}{2} + \varepsilon g\right)}\big) \partial^j_\varepsilon(M(\varepsilon, P, k, u_{\varepsilon,P,\varrho})) \qquad \forall \varepsilon \in]-\varepsilon_1, \varepsilon_1[\end{aligned} \tag{15}$$

for all $h \in \mathbb{N}$, $h \geq 1$.
We now take the limit as $\varepsilon \to 0$ in equality (15) and apply a standard induction argument on h, verifing that equation $\lim_{\varepsilon\to 0} \partial^h_\varepsilon(\tilde{M}(\varepsilon, P, k, u_{\varepsilon,P,\varrho})) = 0$ is equivalent to

$$\lim_{\varepsilon\to 0} \partial^h_\varepsilon(M(\varepsilon, P, k, u_{\varepsilon,P,\varrho})) = 0$$

for all $h \in \mathbb{N}$, $h \geq 1$. Then, by a straightforward calculation, we obtain that the equations for $v_{0,P,k}$, $v_{1,P,k}$, and $v_{2,P,k}$ are as follows:

$$
\begin{aligned}
& v_{0,P,k} = 0\,, \\
& L_{P,\varrho} v_{1,P,k} = -P \cdot \partial_\varphi g\,, \\
& L_{P,\varrho} v_{2,P,k} = -2(\partial_\varphi v_{1,P,k}) \cdot \partial_\varphi g - 4 \sum_{i,j=1}^{d} P_i\, \partial_{\varphi_i} v_{1,P,k}\, \partial^2_{\varphi_i \varphi_j} v_{1,P,k}
\end{aligned}
$$

while the (recursive) equations for the $v_{h,P,k}$'s with $h \geq 3$ are delivered by

$$
\begin{aligned}
L_{P,\varrho} v_{h,P,k} = & -h!(\partial_\varphi v_{h-1,P,k}) \cdot \partial_\varphi g - 2 \sum_{i,j=1}^{d} P_i \sum_{l=1}^{h-1} \binom{h}{l} \partial_{\varphi_i} v_{h-l,P,k}\, \partial^2_{\varphi_i \varphi_j} v_{l,P,k} \\
& - \sum_{i,j=1}^{d} \sum_{l_1=1}^{h-1} \binom{h}{l_1} \sum_{l_2=1}^{h-1-l_1} \binom{h-l_1}{l_2} \partial_{\varphi_i} v_{l_1,P,k}\, \partial_{\varphi_j} v_{l_2,P,k}\, \partial^2_{\varphi_i \varphi_j} v_{h-l_1-l_2,P,k}
\end{aligned}
$$

A Appendix

For fixed $(P, \varrho) \in \mathbb{R}^d \times \mathbb{R}_+$, we consider the partial differential operator on $\mathbb{R}^d$ defined by

$$
L_{P,\varrho} \equiv \sum_{i,j=1}^{d} \left(\frac{1}{\varrho} \frac{\partial^2 h}{\partial p_i \partial p_j}(P) + \frac{\partial h}{\partial p_i}(P) \frac{\partial h}{\partial p_j}(P) \right) \partial_{x_i} \partial_{x_j}\,.
$$

and the polynomial function

$$
\Xi_{p,\varrho}(\xi) \equiv \sum_{i,j=1}^{d} \left(\frac{1}{\varrho} \frac{\partial^2 h}{\partial p_i \partial p_j}(p) + \frac{\partial h}{\partial p_i}(p) \frac{\partial h}{\partial p_j}(p) \right) \xi_i \xi_j \qquad \forall \xi \in \mathbb{R}^d
$$

(so that $L_{p,\varrho} = \Xi_{p,\varrho}(\partial_{x_1}, \ldots, \partial_{x_d})$). As is well known, there exists a periodic tempered distribution $S_{P,\varrho,\mathbb{T}^d}$ on $\mathbb{R}^d$ such that

$$
L_{P,\varrho}\, S_{P,\varrho,\mathbb{T}^d} = \sum_{z \in \mathbb{Z}^d} \delta_z - 1\,,
$$

where δ_z denotes the Dirac measure with mass in z (cf. *e.g.* [1, page 53] and [16]). The distribution $S_{P,\varrho,\mathbb{T}^d}$ is determined up to an additive constant, and we can take

$$
S_{P,\varrho,\mathbb{T}^d}(x) = - \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{4\pi^2\, \Xi_{P,\varrho}(z)} e^{2\pi i z \cdot x}\,,
$$

in the sense of distributions in $\mathbb{R}^d$ (cf. *e.g.*, [16, Thm. 3.1]). In addition, we have the following result (for a proof we refer to [16, Thm. 3.5]).

Proposition 8. *The following statements hold.*

(i) $S_{P,\varrho,\mathbb{T}^d}$ *is real analytic in* $\mathbb{R}^d \setminus \mathbb{Z}^d$.

(ii) If $S_{P,\varrho}$ *is a fundamental solution of* $L_{P,\varrho}$ *then the difference* $(S_{P,\varrho,\mathbb{T}^d} - S_{P,\varrho})$ *is real analytic in* $(\mathbb{R}^d \setminus \mathbb{Z}^d) \cup \{0\}$.

(iii) $S_{P,\varrho,\mathbb{T}^d}$ *belongs to* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

For all functions $f \in C^{0,\alpha}(\mathbb{T}^d)$, we now denote by $\mathcal{N}_{P,\varrho}(f)$ the periodic newtonian potential defined by

$$\mathcal{N}_{P,\varrho}(f)(\varphi) = \int_{\mathbb{T}^d} S_{L_{P,\varrho},\mathbb{T}^d}(\varphi - \vartheta) f(\vartheta)\, d\vartheta \qquad \forall \varphi \in \mathbb{T}^d\,.$$

Then, by Proposition 8, by the properties of the fundamental solutions of elliptic constant coefficient operators (cf. [14, Ch. III] and [4, Thm. 5.2]) and by arguing as in [17, proof of Lem. 3.1] (see also [22, Thm. 2.1]) one verifies the validity of the following

Proposition 9. *If* $f \in C^{0,\alpha}(\mathbb{T}^d)$, *then* $\mathcal{N}_{p,\varrho}(f) \in C^{2,\alpha}(\mathbb{T}^d)$ *and*

$$L_{P,\varrho} \mathcal{N}_{P,\varrho}(f) = f - \int_{\mathbb{T}^d} f(\varphi)\, d\varphi\,.$$

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