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CAUCHY PROBLEMS FOR STATIONARY HAMILTON–JACOBI EQUATIONS UNDER MILD REGULARITY ASSUMPTIONS

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Abstract. For a Hamiltonian enjoying rather weak regularity assumptions, we provide necessary and sufficient conditions for the existence of a global viscosity solution to the corresponding stationary Hamilton–Jacobi equation at a fixed level a, taking a prescribed value on a given closed subset of the ground space. The analysis also includes the case where a is the Mañé critical value. Our results are based on a metric method extending Maupertuis approach.

For general underlying spaces, compact or noncompact, we give a global version of the classical characteristic method based on the notion of a–characteristic. In the compact case, we propose an inf-sup formula producing the minimal solution of the problem, where the generalized Aubry set is involved.

1. **Introduction.** The aim of this paper is to provide necessary and sufficient conditions for the existence of *global solutions* to a given stationary Hamilton–Jacobi equation taking a prescribed value on a closed subset of the state variable space, and to provide representation formulae for them, as well.

Even if our results hold when the equation, say $H = a$, is posed in any connected boundaryless smooth manifold, compact or noncompact, we restrict ourselves, to ease exposition, in almost all the paper to the case where the ground space is the Euclidean space $\mathbb{R}^N$ or the flat torus $\mathbb{T}^N$. As the adjective global explains, we look for solutions defined on the whole space or at least in a given neighborhood of the subset where the initial condition is assigned. The word solution must be understood in the viscosity sense.

It has to be emphasized that our analysis takes place under mild assumptions on the Hamiltonian, namely continuity, convexity in the momentum argument and coercivity, and is pushed to also include the *critical* case, i.e. equations admitting almost everywhere subsolutions but no *strict* subsolutions. Recall that when the underlying space is compact, a global (viscosity) solution does exist if and only if the equation is of this type, see [25], [18].

In the critical case the constant a appearing in right hand–side of the equation is nothing but the well known Mañé critical value which have been object of large

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interest in dynamical systems theory, see [13], as well as in partial differential equations in connection with homogenization problems, starting from the seminal paper [25]. The investigation of the critical equation is one of the main object of weak KAM theory, see [17].

Our contribution is framed in a research context exploring how tools issued from dynamical systems theory can be used, after suitable adaptation, to the study of Hamilton–Jacobi equations or related problems, even when a Hamiltonian flow cannot be defined due to weak regularity of H, see [18], [14], [22], [23]. This connection is made possible through the interpretation of dynamical objects in metric terms, where the reference metric, denoted by S_a, is the path metric of a suitable length functional intrinsically related to the a–sublevels of the Hamiltonian. In other words, the distance between two given points is defined as the infimum of the intrinsic length of the curves joining them. We discuss in Section 4 the relationship between this point of view and the classical Maupertuis principle.

Perhaps the more interesting outcome of this stream of research has been the notion of generalized Aubry set whose points are detected looking at the intrinsic length of cycles passing through them. It extends, for Hamiltonians just continuous, the (projected) Aubry set of the Aubry–Mather theory. It is an essential device for the analysis of the critical case and we make large use of it throughout the paper. In the same spirit we give the definition of a–characteristic, see Definition 5.1, generalizing the classical concept of characteristic at the energy level a, or, to be more precise, of the projection of such a curve on state variable space. A new formulation of Maupertuis principle given in Theorem 4.2 clarifies this link. This also develops a notion given in [14], under the name of critical curve, and employed to study the long–time behavior of solutions of a class of time–dependent Hamilton–Jacobi equations.

We present two main results. In the first one, which holds for compact as well as non compact ground spaces and can be interpreted as a more general and global version of the characteristic method, we establish that the Cauchy problem under consideration, with continuous initial datum f assigned on a closed subset Σ, have a global solution if and only if for any $\chi_0 \in \Sigma$ there exists an a–characteristic ξ, defined in $]-\infty, 0]$, with $\xi(0) = \chi_0$ and

$$f(\chi_0) - S_a(\xi(t), \chi_0) = \sup\{f(\chi) - S_a(\xi(t), \chi) \,:\, \chi \in \Sigma\}.$$

See Theorem 5.6. A similar assertion holds true, see Corollary 2, if one looks for solutions defined in a given neighborhood of Σ, instead of on the whole space.

The second outcome is valid in compact environments, and so it only applies to the critical equation, as already pointed out. The novelty here is that Σ is not contained in the Aubry set. This case, in fact, have already been settled in [18] where the existence of a solution has been proved to be equivalent to the datum f satisfying the compatibility condition

$$f(\chi_2) - f(\chi_1) \le S(\chi_1, \chi_2) \qquad \text{for every } \chi_1, \chi_2 \in \mathcal{A},$$

where $\mathcal{A}$ denotes the Aubry set. In this event a solution is represented by the Lax formula (7), and it is actually unique if, in addition, Σ coincides with the Aubry set.

For general Σ we supply in Theorem 6.1 the following characterizing condition on f and Σ for the existence of solutions

$$\inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S(x, \bar{\chi}) - S(x, \chi) + f(\chi) \Big) \le f(\bar{\chi}) \qquad \text{for every } \bar{\chi} \in \Sigma,$$

we moreover provide a representation formula for the minimal solution u of the problem

$$u(y) = \inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S_a(x,y) - S_a(x,\chi) + f(\chi) \Big).$$

A nonuniqueness example is also exhibited at the end of Section 6. As one can expect, our existence condition reduces to the one of [18] if $\Sigma \subset \mathcal{A}$.

The above inf–sup formula prompts analogies with the symplectic framework of analytical mechanics where Lagrangian submanifolds which are geometrical solutions of the Cauchy problem are generated –in the integrable case– by a complete integral of $H = a$, through stazionarization of the auxiliary parameters, see Appendix 6 for more detail. If in the formula of a generating function for such a Lagrangian submanifold, see (30), we replace the complete integral by the family of distance functions $S_a(x,\cdot)$, with x varying in $\mathcal{A}$, viewed as a complete weak (in the viscosity sense) integral, then we obtain

$$S_a(x,y) - S_a(x,\chi) + f(\chi),$$

where $x \in \mathcal{A}$ and $\chi \in \Sigma$ play the role of extra parameters. Since no stazionarization is possible in our framework, being the functions involved not sufficiently regular, we have simply eliminated such parameters, to obtain a solution of the problem, by means of an inf–sup procedure. This kind of construction is not new in viscosity solutions theory, see [7], [12], [9] and [6].

The same idea is, in a sense, behind the notion of a–characteristic, where a minimality requirement has been taken in place of the stazionarization of the intrinsic length functional yielding, up to a change of parameter, the solutions of the Euler–Lagrange equations, according to Maupertuis principle.

The material is organized as follows. In the second section we present the assumptions and collect some well known facts about weak (sub)solutions to $H = a$. In the following one it is outlined the metric approach, including some representation formulae based on it and the definition of Aubry set. In Section 4 we revisit the Maupertuis principle in connection with the metric point of view; the case of mechanical Hamiltonian is also discussed with some detail. Sections 5, 6 contain the main results with definition of a–characteristic. Finally Appendix 7 gives a brief synopsis on symplectic environment and geometrical solutions to Hamilton–Jacobi equations.

2. Preliminaries.

As explained in the Introduction, we restrict ourselves, in order to avoid technicalities, to take the ground space M equal to the N–dimensional Euclidean space $\mathbb{R}^N$ (noncompact case) or to the flat torus $\mathbb{T}^N$ endowed with the flat Riemannian metric induced by the Euclidean metric of $\mathbb{R}^N$ and with the cotangent bundle $T^*\mathbb{T}^N$ identified to $\mathbb{T}^N \times \mathbb{R}^N$ (compact case).

We assume throughout the paper the following conditions on the Hamiltonian $H : T^*M \to \mathbb{R}$

(H1) $H(y,p)$ is continuous in both variables,
(H2) H is convex in the second argument,
(H3) $\lim_{|p| \to +\infty} H(y,p) = +\infty$ uniformly in y.

The mechanical Hamiltonians $|p|^2/2 + V(y)$ are included in this setting whenever the potential V is continuous. Notice that, due to (H2), the Lipschitz constant of $p \mapsto H(y,p)$ in some Euclidean ball is independent of y, see [14]. The previous hypotheses imply that for any $a \in \mathbb{R}$ the a–sublevels

$$Z_a(y) := \{p : \ H(y,p) \le a\}$$

are convex and compact (possibly empty). In addition, if $Z_a(y) \neq \emptyset$ for any y, then the set valued map $y \mapsto Z_a(y)$ is upper semicontinuous with respect to the Hausdorff metric and continuous at any y with $\operatorname{int} Z_a(y) \neq \emptyset$. Denoting the evaluation bracket by $\langle v, p\rangle$, $v \in T_yM$ and $p \in T_y^*M$, we set

$$\sigma_a(y,v) := \sigma_{Z_a(y)}(v) = \sup\{\langle v,p\rangle : \ p \in Z_a(y)\}, \tag{1}$$

with the usual convention that $\sigma_a(y,v) = -\infty$ for every v if $Z_a(y) = \emptyset$. If $Z_a(y)$ is nonempty valued then the function $\sigma_a(y,v)$ is convex positively homogeneous in the second argument, and, in addition, upper semicontinuous in y and continuous whenever $\operatorname{int} Z_a(y) \neq \emptyset$.

Due to the uniform coercivity condition (H3) a family of Lipschitz–continuous subsolutions is *intrinsically* related to the equation

$$H(y, Du) = a. \tag{2}$$

Actually several definitions of subsolutions detect the same family of functions. Before detailing this point, we introduce the notions of weak derivative and viscosity test function.

Definition 2.1. Given a Lipschitz–continuous function w the (Clarke) generalized gradient ∂w at $y \in M$ is defined as follows:

$$\partial w(y) = \{p = \lim_i Dw(y_i) : \ y_i \text{ is a differentiability point of } w \text{ and } \lim_i y_i = y\}$$

Definition 2.2. Given two continuous functions ψ and v, we say that ψ is (strict) supertangent to v at y_0 if this point is a (strict) local maximizer of $u - \psi$.

The function ψ is instead called (strict) subtangent if maximizer is replaced by minimizer.

Definition 2.3. A Lipschitz–continuous function u is an a.e. subsolution to (2) in the sense of Clarke if

$$\partial u(y) \subset Z_a(y) \qquad \text{for } y \in M.$$

Definition 2.4. A continuous function u is a subsolution to (2) of first type if for any $y_0 \in M$ and for any ψ, C^1 supertangent to u at y_0,

$$H(y_0, D\psi(y_0)) \le a.$$

Definition 2.5. Same as the previous one with subtangent in place of supertangent.

Proposition 1. *The three previous definitions are equivalent to the notion of Lipschitz-continuous a.e. subsolution.*

See [28] for the proof. Consequently we call from now such functions simply subsolution of (2) without any further specification. Notice that the family of all subsolutions to (2), if nonempty, is equiLipschitz–continuous. Two equivalent notions of weak solutions in the viscosity sense of (2) can be given in two equivalent way by requiring equality to hold in Definition 2.4 or 2.5, loosely speaking the first choice corresponds to select among the subsolutions those enjoying some minimality properties, while in the other case maximality conditions are satisfied. We choose the second option and give

Definition 2.6. A continuous function u is called (viscosity) solution to (2) if it is a subsolution and, in addition

$$H(y_0, D\psi(y_0)) = a,$$

for any $y_0 \in M$ and for any ψ, C^1 subtangent to u at y_0.

From now on we use the term solution to (2) in the sense specified by the above definition. Analogously we define the notion of (sub)solution on a open subset of M. The introduction of this class of weak solutions is justified by the powerful properties of stability that they enjoy as well as the availability of representation formulae and comparison principles for them, see [8] for a comprehensive treatment of this topic. We just recall in view of later use, for instance in the proof of Theorem 5.6, that the local uniform limit in M of a sequence of solutions to (2) is still a solution.

We divide $\mathbb{R}$ in two classes according on whether the equation (2) possess subsolutions or not. The separation element will have some importance in what follows and will be called *critical value* of the Hamiltonian. Accordingly the constant a will be qualified as *supercritical* (resp. *subcritical*) if it is greater than (resp. less than) the critical value. To sum up: if a is supercritical then there are global *strict* subsolutions to (2), i.e. subsolutions of

$$H(x, Du) = a - \delta,$$

for some positive δ, while, for a critical, a solution is obtained as uniform limit of suitable sequences of solutions to $H = a_n$ for $a_n \to a^+$, but, of course, no global strict subsolutions can exist.

Notice, finally, that a global strict subsolution in the supercritical setup can be smoothed up by mollification still keeping such properties. Hence it is not restrictive to assume, for a supercritical, that (2) possess a smooth subsolution.

3. **Metric approach.** From now on we assume that a is critical or supercritical. Following [18] we employ in the analysis of (2) the so–called *metric method* which is based on the introduction of distance suitably related to the a–sublevels of the Hamiltonian. We will explain in the next section how this approach is related to the classical Maupertuis principle.

Given a supercritical or critical and $x, y \in M$ we set

$$S_a(x, y) = \sup\{u(y) : \ u \text{ is a subsolution of } H(y, Du) = a \text{ with } u(x) = 0\}.$$

It is easy to check that S_a satisfies the triangle inequality and vanishes whenever x and y coincide, but fails in general to be symmetric and can have any sign for $x \neq y$, nevertheless we will call it distance to ease terminology. By the very definition of S_a, any subsolution u to (2) satisfies

$$u(x) - u(y) \leq S_a(y, x) \qquad \text{for every } x, y \in M. \tag{3}$$

As a matter of facts, the converse implication also holds and the admissibility condition (3), for some u, is indeed equivalent to u being subsolution. This can proved by exploiting a more intrinsic definition of S_a which we describe below. For any (Lipschitz-continuous) curve ξ defined in some compact interval, we define the intrinsic length as follows

$$l_a(\xi) = \int_0^1 \sigma_a(\xi, \dot{\xi}) dt. \tag{4}$$

Due to positive homogeneity of σ_a, $l_a(\xi)$ is invariant for change of parameter in ξ preserving the orientation. In addition, by classical variational principles, see [10],

l_a is lower semicontinuous with respect to the uniform convergence of curves. The connection between S_a and l_a is given by the next result.

Proposition 2. *S_a is the path metric corresponding to the length functional l_a. Namely*

$$S_a(x,y) = \inf\{l_a(\xi) :\ \xi \text{ connecting } x \text{ to } y\}. \tag{5}$$

From this we derive

Corollary 1. *u is a subsolution to (2) if and only if satisfies (3).*

See [18] and [27] for some precise treatment of this issue.

In geometrical optics, through an asymptotic procedure on the light frequencies $\omega \to +\infty$, we obtain the Hamiltonian $H(y,p) = \frac{1}{2}\, n^{-1}(y)|p|^2$, where $(y,p) \in \mathbb{R}^{2N}$ and n is the refraction index. The corresponding well known *eikonal* equation is $\frac{1}{2}\, n^{-1}(y)|Du|^2 = 1$. In such a case, the intrinsic length l_1 is nothing but the usual *optical length* and related minimizing trajectories are the *optical rays*, see for example [20] and [4].

Notice that the intrinsic length of a curve can have any sign, however if ξ is a cycle then $l_a(\xi)$ is nonnegative and, in addition, strict positive when a is supercritical. This point justifies the forthcoming definition of Aubry set and will be outlined with some more detail in the next proposition. We will use the symbols l for the Euclidean length of a curve and $B(x,r)$ for the Euclidean ball centered in $x \in M$ and with radius $r > 0$.

Proposition 3. *Assume that there is a subsolution to (2) which is strict in a neighborhood of some point $x_0 \in M$. Then given $\varepsilon > 0$ we can find a positive constant $\delta = \delta(\varepsilon, x_0)$ such that any cycle ξ passing through x_0 satisfies the implication*

$$l_a(\xi) < \delta_\varepsilon \ \Rightarrow\ l(\xi) < \varepsilon.$$

Proof. By assumption there exists a subsolution u of (2) in M and two positive constants ρ, r such that

$$H(y, Du) \le a - \rho \qquad \text{a.e. in } B(x_0, r).$$

This implies

$$\sigma_a(y,v) > \langle p, v\rangle - \mu \tag{6}$$

for any $y \in B(x_0,r)$, $p \in \partial u(y)$ and some positive μ. We assume the cycle ξ to be parametrized in $[0,1]$ with $\xi(0) = \xi(1) = x_0$ and, addition $\varepsilon < r$. There is a portion of the curve, say for t in some interval $[t_1,t_2]$, with $l(\xi\big|_{[t_1,t_2]}) \ge \min\{l(\xi), r\}$ contained in $B(x_0,r)$. We derive from (6) and Corollary 1 that

$$\begin{aligned} l_a(\xi) \ &\ge\ (u(\xi(t_1)) - u(x_0)) + (u(\xi(t_2) - u(\xi(t_1)) \\ &+\ \mu(\min\{l(\xi), r\}) + (u(x_0) - u(\xi(t_2)) = \mu(\min\{l(\xi), r\}). \end{aligned}$$

If we take $\delta = \mu\varepsilon$ we have, under the condition $l_a(\xi) < \delta$, $\varepsilon > \min\{l(\xi), r\}$, and so $\varepsilon > l(\xi)$. □

The intrinsic length of cycles is involved in the next definition of Aubry set. This special subset of M will play a crucial role in the analysis of (2) in the critical case.

Definition 3.1. A point y belongs to the (projected) Aubry set $\mathcal{A}$ if there is a sequence of cycles ξ_n passing through it with

$$\inf_n l_a(\xi_n) = 0, \qquad \inf_n l(\xi_n) > 0.$$

It is clear from Proposition 3 that $\mathcal{A}$ is empty if a is supercritical, while it is non void in the critical case, at least when the underlying space is compact. We can derive, in addition, that no subsolution of (2) can be strict around a point of the Aubry set. From this point of view $\mathcal{A}$ can interpreted as the place where is concentrated the obstruction to get subsolutions under the critical level. Such an obstruction can be at infinity in the noncompact setting, which explains why, in this case, $\mathcal{A}$ is possibly empty.

In the simple case of the pendulum $H(y,p) = \frac{1}{2}|p|^2 - \cos y$, defined in $\mathbb{T}^1 \times \mathbb{R}$, the critical value is given by the maximum of the potential, which is equal to 1. This is the minimal energy level allowing motions trajectories to fill the whole ground space $\mathbb{T}^1$. Exactly at this energy level, the trajectories, precisely the so–called separatrices, admit the unstable equilibrium as α and ω–limit; as a matter of fact, this point, which is also the maximizer of the potential, makes up the Aubry set.

We recall, see [18], that if $x \notin \mathcal{A}$ then there exists some subsolution to (2) which are strict around x. Bearing in mind that comparison principles for (2) are related to existence of strict subsolutions, it is clear that, in order to formulate such kind of results, one has to take into account the Aubry set where such kind of subsolutions do not exist. We define

$$\mathcal{E} = \{y \in M : \ \operatorname{int} Z_a(y) = \emptyset\},$$

and call it the set of equilibria. It is is clear that if $\mathcal{E}$ is nonempty then a is the critical value and, since no subsolution to (2) can be strict around a point of $\mathcal{E}$, then $\mathcal{E} \subset \mathcal{A}$.

Given a closed subset $\Sigma \subset M$ and a continuous function f satisfying the admissibility condition (3), with f in place of u, the maximal and minimal subsolution of (2) taking the value f on Σ are given by the so–called Lax formulae involving the intrinsic distance S_a, as specified in the following

Proposition 4. *Let Σ be a closed subset of M and f a continuous function defined on it. The functions*

$$u = \inf\{S_a(\chi, \cdot) + f(\chi) \, : \, \chi \in \Sigma\} \tag{7}$$

and

$$v = \sup\{-S_a(\cdot, \chi) + f(\chi) \, , \, \chi \in \Sigma\} \tag{8}$$

are respectively the maximal subsolution to (2) less than or equal to f on Σ and the minimal subsolution greater than or equal to f. If, in addition

$$f(\chi_1) - f(\chi_0) \le S_a(\chi_0, \chi_1) \qquad \text{for every } \chi_0, \chi_1 \in \Sigma. \tag{9}$$

then both u, v take the value f on Σ.

We refer to [18] for the proof. For the maximal subsolution we also have

Proposition 5. *Let Σ, f as in the previous proposition, then the function given by the formula (7) is solution to (2) in $M \setminus \Sigma$. It is moreover solution on the whole M whenever $\Sigma \subset \mathcal{A}$.*

We emphasize that, except in the special case where Σ is contained in the Aubry set, the maximal subsolution is not by any means also solution on Σ. Actually the main scope of the present paper is to find necessary and sufficient conditions for the existence of global solutions taking the value f on a general closed subset Σ.

In the special case where Σ is the boundary of an open bounded set, formula (7) provides a solution for the corresponding Dirichlet problem. To get an uniqueness result, holding also in the critical case, $\mathcal{A}$ must be taken into account. We recall that such set is empty for a supercritical. More precisely we have

Proposition 6. *Let Ω be an open bounded subset of M and f a continuous datum on $\partial\Omega \cup \{\mathcal{A} \cap \Omega\}$ satisfying (9). Then the Dirichlet problem*

$$\begin{cases} H(y, Du) &= a & \text{in } \Omega \setminus \mathcal{A} \\ u &= f & \text{on } \partial\Omega \cup \{\mathcal{A} \cap \Omega\} \end{cases}$$

admits the function (7), with $\partial\Omega \cup \{\mathcal{A} \cap \Omega\}$ in place of Σ, as unique solution.

When the underlying space is compact, global strict subsolutions and solutions cannot exist at the same time: if u, v were a solution and a strict subsolution to (2) respectively, a contradiction should be obtained looking at the minimizers of $u - v$. In fact u, which can be supposed smooth up to a regularization, is subsolution to (2) at such points but the equality condition required in Definition 2.6 is violated. On the other hand existence of solutions is guaranteed in the critical case, see [18]. Therefore we have

Proposition 7. *If M is compact then a solution to (2) in the whole space does exist if and only if a is the critical value.*

On the contrary, if M is noncompact, any critical or supercritical equations admits solution in M.

4. **Maupertuis principle.** Here we assume standard regularity conditions on the Hamiltonian, namely that H is of class C^2 in both variables, C^2–strictly convex in the second argument and superlinear. This implies, in particular, that a Lagrangian $L(x, v)$ can be defined via the Fenchel transform. Under the above hypothesis, in mechanical literature, it is also known as Legendre transform.

The scope of this section is to show that the metric point of view we have outlined in the previous section is strictly related to the classical Maupertuis principle, which, for time independent Hamiltonians, provides a variational characterization of solutions γ to Euler–Lagrange equation (Euler–Lagrange solutions for short) lying on a prescribed supercritical level of the energy, say a. This means that $H(\gamma(t), D_v L(\gamma(t), \dot\gamma(t)) \equiv a$, where $D_v L$ denotes the derivative of the Lagrangian with respect to v. Notice that in our environment the Euler–Lagrange solutions are defined in the whole $\mathbb{R}$.

We will discuss the two following equivalent formulations of Maupertuis principle.

Theorem 4.1. ***(Maupertuis I)*** *The Euler–Lagrange flow at the energy level a, with a supercritical, coincides with the geodesic flow of a Finsler metric geodesically equivalent to S_a, up to a change of parameter.*

See [13] for this statement. *Geodesically equivalent* means that the length functional of the above mentioned Finsler metric and l_a have the same extremals.

Theorem 4.2. ***(Maupertuis II)*** *Any Euler–Lagrange solution γ, defined in $\mathbb{R}$, and lying at the energy level a, with a supercritical, is characterized by the following condition: For any $t \in \mathbb{R}$ there exists $\delta > 0$ such that*

$$\int_{t-\delta}^{t+\delta} (L(\gamma, \dot\gamma) + a)\, dt = S_a(\xi(t - \delta), \xi(t + \delta)).$$

We preliminary gives a definition

Definition 4.3. We say that a curve ξ defined in some interval I has an a–*Lagrangian parametrization* if

$$\sigma_a(\xi(t), \dot{\xi}(t)) = L(\xi(t), \dot{\xi}(t)) + a \quad \text{for a.e. } t \in I.$$

It comes from the very definition of Lagrangian that any Euler–Lagrange solution lying at the energy level a has such a kind of parametrization and, conversely, if an Euler–Lagrange solution has an a–Lagrangian parametrization then it must be contained in the a–level of the energy. In addition, see [18], [14], any curve, at least for a supercritical, can be endowed, by suitably changing the parameter, of an a–Lagrangian parametrization, and if the original parameter varies in a bounded interval, the reparametrized curve is still defined in a bounded interval. We also record, for later use, the immediate inequality

$$L(x, v) + a \geq \sigma_a(x, v) \quad \text{for any } x,\, v. \tag{10}$$

In order to define the Finsler metric appearing in Theorem 4.1 we essentially exploit that a is supercritical and pick up a C^∞ strict subsolution ψ to (2). We introduce a new Hamiltonian through the formula

$$\tilde{H}(x, p) = H(x, p + D\psi(x)),$$

and define $\tilde{\sigma}_a(x, v)$ as in (1) with $\tilde{H}$ in place of H and $\tilde{l}_a$ as in (4) with $\tilde{\sigma}_a$ substituting σ_a. The reason which justifies this apparatus is that 0 is an interior point of any a–sublevel of $\tilde{H}$ so that $\tilde{\sigma}_a(x, v) > 0$ whenever $v \neq 0$ and consequently $\tilde{l}_a$ is strictly positive for any nonconstant curve. Therefore it is the length functional of a Finsler metric that we denote by $\tilde{S}_a$. It is clear that it is geodesically equivalent to S_a.

Now let η be a geodesic for S_a, i.e. an extremal for the length functional l_a or $\tilde{l}_a$. We can assume, without loss of generality, that it has an a–Lagrangian parametrization. It is a basic properties of Finsler metrics, see [5], that short geodesics are minimizers or in other terms realizes the distance S_a or $\tilde{S}_a$ of the endpoints. Let us admit that such a property holds for $\eta\big|_{[t_1,t_2]}$ for some $t_1 < t_2$, bearing in mind (10) we have

$$\begin{aligned} S_a(\eta(t_1), \eta(t_2)) &= l_a(\eta\big|_{[t_1,t_2]}) = \int_{t_1}^{t_2} (L(\eta, \dot{\eta}) + a)\, dt \\ &\leq l_a(\xi) \leq \int_{t_1}^{t_2} (L(\xi, \dot{\xi}) + a)\, dt, \end{aligned}$$

for any curve ξ defined in $[t_1, t_2]$ and with $\xi(t_1) = \eta(t_1)$, $\xi(t_2) = \eta(t_2)$. Therefore $\eta\big|_{[t_1,t_2]}$ minimizes the action functional among the curves connecting its endpoints and defined in $[t_1, t_2]$. Therefore it is an Euler–Lagrange solution and is contained in the a–level of the energy since it has a–Lagrangian parametrization. Further, since such a construction can be repeated on any sufficient small interval in the domain of definition of η, the whole curve keeps such a property.

Conversely, if γ is an Euler–Lagrange solution lying at the level a of the energy then, see [13], it is also unique among such solutions at the same level of energy connecting $\eta(t_1)$ to $\eta(t_2)$ whenever t_1 and t_2, with $t_1 < t_2$, are sufficiently close. We also know, see the forthcoming Lemma 5.4 in the next section, that there exists a curve joining $\eta(t_1)$ and $\eta(t_2)$ whose length l_a realizes $S_a(\eta(t_1), \eta(t_2))$. Reasoning as in the above step we see that such a curve is an Euler–Lagrange solution contained in the a–level of the energy, up to an a–Lagrangian reparametrization, and so must coincide with $\eta\big|_{[t_1,t_2]}$. Since sufficiently small portion of η minimizes l_a then such curve is indeed a geodesic for S_a and $\tilde{S}_a$.

The previous arguments supply a proof for both Theorems 4.1, 4.2. Notice, however, that such results does not hold any more at the critical level. For instance in [28] it is proved that in that case for Euler-Lagrange solutions connecting a point of the Aubry set, say y_0, to points outside such set, the minimality property with respect to l_a fails in any portion of the curve containing y_0.

We proceed by discussing this topic in the case of mechanical systems. A constrained mechanical system can be described by a Riemannian metric g and a potential energy $V(y)$ on a general connected boundaryless manifold M, through the Hamiltonian

$$H(y,p) = \frac{1}{2}\langle g^{-1}(y)p, p\rangle + V(y), \tag{11}$$

with corresponding Lagrangian $L(y,v) = \frac{1}{2}\langle v, g(y)v\rangle - V(y)$. We have already reminded in Section 3, that for Hamiltonians of this type, the critical value corresponds to the maximum of the potential and the Aubry set is the set of the maximum points of V.

We notice that the a–sublevel of this Hamiltonian at a point y is the cotangent ball centered at 0 and with radius $\sqrt{2(a - V(y))}$, i.e,

$$\{p : \langle g^{-1}(y)p, p\rangle \leq 2(a - V(y))\}$$

and the radius is strictly positive for any y if a is supercritical. In other terms, 0 is in the interior of the a–sublevel, for any y, and so l_a is directly the length functional of a Finsler metric. As one can expect, according to outputs of the first part of the section, it is nothing but the Jacobi–Maupertuis metric. In fact, given a vector $v \in T_yM$, the maximum of $\langle v, p\rangle$, for p varying in the previously indicated ball, is attained at a point on the boundary of the form $\lambda\, g(y)\, v$ for some positive λ, which gives $\lambda = \sqrt{\frac{2(a-V(y))}{\langle v, g(y)v\rangle}}$. Accordingly

$$\sigma_a(y,v) = \sqrt{2(a - V(y))\langle v, g(y)v\rangle}.$$

We can therefore summarize what previously exposed as follows

Proposition 8. *The support function $\sigma_a(y,v)$ related to the mechanical Hamiltonian (11) at a supercritical level coincides with the root of the evaluation of the Jacobi-Maupertuis metric $2[a - V(y)]g(y)$ at v.*

We emphasize that the way in which we have obtained above the Jacobi-Maupertuis metric starting from the a–sublevel of the Hamiltonian is very direct and does not require any differentiability of the potential, continuity should be enough. In classical analytical mechanics the construction of Jacobi-Maupertuis metric is usually more involved and pass through Hölder's Principle (see for example [1] Theorem 3.8.5, and [3], [4]), by using the so called *iso-energetic asynchronous* variations, or equivalently, employs a nonvariational argument due to Godbillon [19] Proposition 5.11 (see also [1] Theorem 3.7.7).

5. **A generalization of the characteristics method.** We go back to the weaker hypotheses (H1)-(H2)-(H3). Exploiting the coercivity condition (H3), we can modify H, when $|p|$ is large, leaving unaffected its a–sublevel. This adjustment can be done in such a way that the emended Hamiltonian is superlinear at infinity, see for instance [18], p. 205. Since in our analysis only $\{(y,p) :\ H(y,p) \leq a\}$ matters, we

can consequently assume, without loosing generality, such a growth at infinity to hold and then define a Lagrangian $L(y,v)$ by means of the Fenchel transform.

Motivated by Theorem 4.2, we define the notion of *generalized characteristic* for S_a (a–characteristic for short).

Definition 5.1. We say that a curve ξ, defined in some interval I, is an a–characteristic if for any t_1, t_2 in I with $t_1 < t_2$

$$S_a(\xi(t_1),\xi(t_2)) = \int_{t_1}^{t_2} (L(\xi,\dot\xi)+a)\,dt \tag{12}$$

We recall that the a–Lagrangian parametrizations, appearing in the next statement, have been introduced in Definition 4.3.

Lemma 5.2. *Any a–characteristic ξ has an a–Lagrangian parametrization and is Lipschitz–continuous with Lipschitz constant depending only on a.*

Proof. We denote by I the interval of definition of ξ, and take sequences t_n and s_n with the left and the right endpoint of I, respectively, as limit. The first part of the assertion immediately comes from the chain of inequalities

$$\begin{aligned}\int_{t_n}^{s_n} (L(\xi,\dot\xi)+a)\,ds &= S_a(\xi(t_n),\xi(s_n)) \le \int_{t_n}^{s_n} \sigma_a(\xi,\dot\xi)\,ds \\ &\le \int_{t_n}^{s_n} (L(\xi,\dot\xi)+a)\,ds,\end{aligned}$$

which hold for any n and the fact that $L(y,v)+a \ge \sigma_a(y,v)$ for any y, v.
To check the Lipschitz character of ξ, let us consider a a differentiability point t_0 for which $L(\xi(t_0),\dot\xi(t_0)) + a = \sigma_a(\xi(t_0),\dot\xi(t_0))$, then $\dot\xi(t_0) \in D_p^- H(\xi(t_0),p_0)$ (D_p^- denotes the subdifferential with respect to the second variable) for some p_0 with $H(\xi(t_0),p_0) = a$.

We know by (H3) that H is coercive in p uniformly with respect to y, therefore there is $R>0$, depending on a, with $|p_0| \le R$ and so $|\dot\xi(t_0)|$ is estimated from above by the Lipschitz constant of $H(\xi(t_0),\cdot)$ on $B(p_0,R)$. Such a constant is independent of $\xi(t_0)$, as already pointed out in Section 2. This ends the proof. □

As already recalled in the previous section any curve can be endowed with an a–Lagrangian parametrization if a is supercritical. The same holds true at the critical level for curves disjoint from $\mathcal{E}$. If the curve is, in addition, at a positive distance from $\mathcal{E}$ and the original parameter varies in a bounded interval, the reparametrized curve is still defined in a bounded interval. See [18], [14].

The stability property for a–characteristics we establish next will be of crucial relevance in what follows.

Lemma 5.3. *Let ξ_n be a sequence of a–characteristics, defined on some interval I, all taking the same value at a time $t_0 \in I$, then the ξ_n locally uniformly converge to some a–characteristic ξ, up to a subsequence.*

Proof. The ξ_n are equiLipschitz–continuous by Lemma 5.2, they are moreover locally equibounded since they coincide at t_0. By applying Ascoli Theorem we thus get a local uniform limit of it in I, say ξ, up to a subsequence. It is left to prove that such a curve is indeed an a–characteristic.
Given t_1, t_2 in I with $t_1 < t_2$, we have by the continuity of S_a:

$$\lim_{n\to+\infty} S_a(\xi_n(t_1),\xi_n(t_2)) = S_a(\xi(t_1),\xi(t_2))$$

and invoke Theorem 2.9. of [14] to get

$$\lim_{n\to+\infty}\int_{t_1}^{t_2}(L(\xi_n,\dot\xi_n)+a)\,dt \geq \int_{t_1}^{t_2}(L(\xi,\dot\xi)+a)\,dt$$

so that

$$\int_{t_1}^{t_2}(L(\xi,\dot\xi)+a)\,dt \leq S_a(\xi(t_1),\xi(t_2)).$$

Being the converse inequality trivial, we obtain in the end the assertion. □

The following proposition taken from [14], see Theorems 4.8, 4.14, establish that the Aubry set is foliated by a–characteristic defined on the whole $\mathbb{R}$ and possessing a relevant property with respect to the subsolutions of (2). Of course this result is effective only in the critical case.

Proposition 9. *Given $x \in \mathcal{A}$, $t_0 \in \mathbb{R}$, there is an a–characteristic ξ defined on $\mathbb{R}$ and taking the value x at t_0 such that for any pair u, v of subsolutions to (2) $u(\xi(t)) - v(\xi(t))$ is constant.*

If, in particular, $x \in \mathcal{E}$ then any $p_0 \in Z_a(x)$ is a minimizer of $p \mapsto H(y,p)$ and so $L(y,0) = -a$. Accordingly the curve $\xi \equiv x$ is an a–characteristic satisfying the previous statement. This explains why the points of $\mathcal{E}$ deserve the name of equilibria. For a general point y_0 in M, we show here below at least the existence of global unilateral a–characteristics taking the value y_0 at 0.

Proposition 10. *For any $y_0 \in M$ there are a–characteristics defined in $[-\infty,0]$, $[0,+\infty]$ taking the value y_0 at 0.*

The proof of the proposition is related to the existence of *minimal curves* for S_a between two given points x, y, i.e. curves γ connecting them and satisfying $l_a(\gamma) = S_a(x,y)$. Notice that in this case we have

$$S_a(x,y) = S_a(x,\gamma(t)) + S_a(\gamma(t),y) \quad \text{for any } t. \tag{13}$$

In the supercritical case we have

Lemma 5.4. *Let a be supercritical. For any pair of points x, y there is an a–characteristic connecting them. Such a curve is clearly minimal for S_a.*

Proof. Let us denote by c the critical value. We consider a sequence ξ_n of curves joining x to y and with $\inf_n l_a(\xi_n) = S_a(x,y)$. Since a is supercritical, we can assume that the ξ_n have a–Lagrangian parametrizations in the intervals $[0,T_n]$, for some $T_n > 0$. The relations

$$\begin{aligned}\int_0^{T_n}(L(\xi_n,\dot\xi_n)+a)\,dt &= \int_0^{T_n}(L(\xi_n,\dot\xi_n)+c)\,dt + (a-c)\,T_n \\ &\geq l_c(\xi_n) + (a-c)\,T_n \geq S_c(x,y) + (a-c)\,T_n\end{aligned}$$

account for the fact that the T_n are bounded and so, taking into account that the curve ξ_n are equiLipschitz–continuous by Lemma 5.2, we use Ascoli Theorem to produce an uniform limit curve. We see that such a curve is an a–characteristic connecting x to y arguing as in Lemma 5.3. □

The setup is more involved if we also take into account the critical case.

Lemma 5.5. *Given x_0, y_0 in M, if there are no minimal curves for S_a connecting them and not intersecting $\mathcal{E}$, we can find an a–characteristic ξ defined in some interval $]-T,0]$ ($[0,T[$ resp.) with $\xi(0)=y_0$, $\xi(0)=x_0$ resp., such that*

$$S_a(y_0,\xi(t))+S_a(\xi(t),x_0) \quad \textit{for any } t,$$

and with all limit points for $t\to -T$ ($t\to T$ resp.) belonging to $\mathcal{A}$.

Proof. We prove the statement for intervals of type $]-T,0]$, the other case can be tackled arguing similarly.

We consider a sequence ξ_n of curves between x_0 and y_0 whose intrinsic length l_a approaches $S_a(x_0,y_0)$. If the natural lengths of the ξ_n are bounded then we get, via Ascoli Theorem, an uniform limit ξ in a suitable compact interval of parametrization, say I. Exploiting the lower semicontinuity of the intrinsic length with respect to the uniform convergence and the fact that S_a is continuous in both arguments, we derive that ξ is a minimal curve for $S_a(x_0,y_0)$. If $\xi\cap\mathcal{E}\neq\emptyset$ and

$$t_0 := \max\{t\in I\,:\,\xi(t)\in\mathcal{E}\} \tag{14}$$

then by performing an a–Lagrangian reparametrization of ξ restricted to $]t_0,0]$ we obtain an a–characteristic, still denoted by ξ, on some interval $]-T,0]$, satisfying the statement.

It is therefore left to discuss the case where $\{l_a(\xi_n)\}$ is unbounded. Up to a change of parameter we can assume that all the ξ_n are defined in $]-\infty,0]$ with $\xi_n(0)=y_0$ for every n, $\xi_n\equiv x_0$ in the interval $]-\infty,-\ell(\xi_n)]$, and, in addition, $\dot\xi_n(t)=1$ for a.e. $t\in]-\ell(\xi_n),0[$. We have for any n, any $t_1<t_2$ in $]-\infty,0]$ and some infinitesimal sequence ε_n of positive numbers

$$\begin{aligned} l_a(\xi_n)-\varepsilon_n &\le S_a(x_0,y_0)\\ &\le S_a(x_0,\xi_n(t_1))+S_a(\xi_n(t_1),\xi_n(t_2))+S_a(\xi(t_2),y_0)\\ &\le l_a(\xi_n)\end{aligned}$$

which implies

$$S_a(\xi_n(t_1),\xi_n(t_2)) \ \ge\ l_a(\xi_n\big|_{[t_1,t_2]})-\varepsilon_n \tag{15}$$

$$S_a(x_0,y_0) \ \ge\ S_a(x_0,\xi_n(t_i))+S_a(\xi_n(t_i),y_0)-\varepsilon_n, \tag{16}$$

$i=1,2$. Using again Ascoli Theorem, we get a local uniform limit ξ of the ξ_n in $]-\infty,0]$, and sending n to infinity in (15), (16) we have

$$S_a(\xi(t_1),\xi(t_2)) \ =\ l_a(\xi\big|_{[t_1,t_2]}) \quad \text{for any } t_1<t_2 \text{ in }]-\infty,0] \tag{17}$$

$$S_a(x_0,y_0) \ =\ S_a(x_0,\xi(t_i))+S_a(\xi(t_i),y_0) \tag{18}$$

$i=1,2$. Let t_0 be as in (14). If it is finite then $\lim_{t\to t_0^+}\xi(t)\in\mathcal{E}\subset\mathcal{A}$ by the very definition of t_0. If, instead $t_0=-\infty$ and z_0 is an α–limit point of ξ, then we can take $\xi(t_1)$, $\xi(t_2)$, with $t_2>t_1$, both as close as we desire to z_0 and such that, in addition, $l(\xi_n\big|_{[t_1,t_2]})$ is large, for n large enough.

We construct a sequence of cycles γ_n passing through z_0 by juxtaposition of $\xi_n\big|_{[t_1,t_2]}$, and the Euclidean segments joining $\xi_n(t_2)$ to z_0 and z_0 to $\xi_n(t_1)$. For n large enough, $l_a(\xi_n\big|_{[t_1,t_2]})\sim S_a(\xi(t_1),\xi(t_2))$ since ξ_n converges locally uniformly to ξ, from this and (15) we derive that $l_a(\gamma_n)$ is infinitesimal. On the other hand the natural length of γ_n is large. This implies that $z_0\in\mathcal{A}$ by the very definition of Aubry set.

Finally, by performing an a–Lagrangian reparametrization of $\xi\big|_{]-T,0]}$, we obtain an a–characteristic in force of (17) defined in some interval $]-T,0]$. Such curve satisfies the statement. □

Proof of Proposition 10. Let x_n be a sequence of points with $|x_n| \to +\infty$. If, for any n, there exists a minimal curve ξ_n for S_a joining x_n to x_0 and $\xi_n \cap \mathcal{E} = \emptyset$, we can perform a–Lagrangian reparametrizations of the ξ_n to get a sequence of a–characteristics, still denoted by ξ_n, defined in some in interval $[-T_n, 0]$. Moreover T_n is positively divergent since the ξ_n are equiLipschitz–continuous and the initial points x_n go to infinity.

We extend the ξ_n to $]-\infty, 0]$ by setting $\xi_n \equiv x_n$ in $]-\infty, -T_n[$, and we pass to the local uniform limit of the so extended ξ_n in $]-\infty, 0]$ by using Ascoli Theorem producing a curve ξ.

Now, let us fix $T > 0$, then ξ is the uniform limit of the ξ_n, up to a subsequence, in $[-T, 0]$ and, bearing in mind that $T_n \to +\infty$, we also have that ξ_n is an a–characteristic in such an interval for n large enough. We conclude, in the light of Lemma 5.3, that ξ itself is an a–characteristic in $[-T, 0]$ and so it keeps this property in $]-\infty, 0]$, since T is arbitrary.

Hence, according to Lemma 5.5, we are reduced to the case where there exists an a–characteristic ξ in some interval $]-T, 0]$ with $\xi(0) = x_0$, and all the limit points of ξ for $t \to -T$ belonging to the Aubry set. If $T = +\infty$ the assertion is proved, otherwise we set

$$y_0 = \lim_{t \to -T} \xi(t) \in \mathcal{A}.$$

According to Proposition 9 there is a–characteristic γ defined in $\mathbb{R}$ and contained in $\mathcal{A}$ with $\gamma(-T) = y_0$. We define a new curve η in $]-\infty, 0]$ by setting

$$\begin{cases} \eta(t) &= \gamma(t) \qquad \text{for } t \in]-\infty, -T] \\ \eta(t) &= \xi(t) \qquad \text{for } t \in]-T, 0] \end{cases}$$

To prove that η is indeed an a–characteristic it is left to show that the relation (12) holds whenever $t_1 \in]-\infty, -T]$ and $t_2 \in]-T, 0]$. To this aim we exploit a crucial property of global a–geodesics contained in the Aubry set, namely the fact that on such curves the difference of any pair of subsolutions to $H = a$ is constant. Applying this principle to

$$S_a(\eta(t_1), \cdot) + S_a(\cdot, \eta(t_2))$$

and calculating it at $\xi(t_1) = \gamma(t_1)$ and at $\xi(-T) = \gamma(-T)$, we therefore get

$$S_a(\xi(t_1), \xi(t_2)) = S_a(\xi(t_1), \xi(-T) + S_a(\xi(-T), \xi(t_2)) = \int_{t_1}^{t_2} L(\eta, \dot{\eta}) + a)\, dt.$$

This ends the proof. □

We proceed by stating and proving the main result of the section.

Theorem 5.6. *Let Σ, f be a closed subset of M and a continuous function defined on it, respectively. There is a global solution of (2) taking the value f on Σ if and only if for any $\chi_0 \in \Sigma$ we can find an a–characteristic ξ defined in $]-\infty, 0]$ with $\xi(0) = \chi_0$ such that*

$$f(\chi_0) - S_a(\xi(t), \chi_0) = \sup\{f(\chi) - S_a(\xi(t), \chi) \,:\, \chi \in \Sigma\} \tag{19}$$

for any t.

Proof. We first prove the existence of a solution assuming that for any point in Σ there is a global a–characteristic satisfying (19). Writing down this condition for any such curve at $t = 0$, we see that f satisfies the compatibility condition

$$f(\chi_1) - f(\chi_2) \leq S_a(\chi_2, \chi_1) \quad \text{for any } \chi_1,\, \chi_2 \text{ in } \Sigma.$$

Therefore the function $u := \sup\{f(\chi) - S_a(\cdot, \chi) \,:\, \chi \in \Sigma\}$ is, by Proposition 4, the minimal subsolution of (2) taking the value f on Σ.

Let us now consider a curve ξ satisfying (19), for some point of Σ, and study the behavior of it at $-\infty$. The claim is that any of its α–limit points belongs to $\mathcal{A}$. If there exists an unique α–limit point, say x_0, and $x_0 \notin \mathcal{A}$, then $L(x_0, 0) + a > 0$ since $\mathcal{E} \subset \mathcal{A}$, and, by continuity of σ_a and L at $(x_0, 0)$

$$L(y, v) + a > \sigma_a(y, v),$$

when y is in a suitable neighborhood of x_0 and $|v| \leq \delta$ for some positive δ. Consequently $|\dot{\xi}(t)| > \delta$ at any large enough differentiability point t of ξ, and so the natural length of ξ is infinite.

We can therefore take two points $\xi(t_0)$, $\xi(s_0)$, with $s_0 > t_0$, on the curve which are both as close as we desire to x_0 and such that, in addition, $l(\xi\big|_{[t_0,s_0]})$ is large. Arguing as in Lemma 5.5 we construct a cycle passing through x_0 with infinitesimal intrinsic length l_a and positive natural length, which implies that x_0 belongs to the Aubry set.

If instead ξ has more that one α–limit then its natural length is automatically infinite, and we can repeat the above argument, arguing along sequences t_n , $s_n \to -\infty$, to prove the claim.

Next we consider a sequence of increasing balls B_n with $\cup_n B_n = M$, and the Dirichlet problems

$$\begin{cases} H(v_n, Dv_n) &= \quad a & \text{in } B_n \\ \quad v_n &= \quad u & \text{on } \partial B_n \cup \{\mathcal{A} \cap B_n\} \end{cases}$$

We know, thanks to Proposition 6, that such problems have an unique solution v_n, which is also maximal among the subsolutions taking the value u on $\partial B_n \cup \{\mathcal{A} \cap B_n\}$, and it is given by the Lax representation formula

$$v_n = \min\{u(y) + S(y, \cdot) \,:\, y \in \partial B_n \cup \{\mathcal{A} \cap B_n\}\}.$$

Note moreover that the v_n are Lipschitz–continuous with Lipschitz constant independent of n. Given a point $\chi_0 \in \Sigma \cap \bar{B}_n$, we still denote by ξ an unilateral global a–characteristic satisfying (19) and $\xi(0) = \chi_0$. There are two (not alternative) possibilities: either ξ goes out B_n or there is an α–limit point of it belonging to B_n. In both cases we find $x_0 \in \partial B_n \cup \{\mathcal{A} \cap B_n\}$ with

$$u(x_0) = f(\chi_0) - S_a(x_0, \chi_0).$$

Hence we have

$$f(\chi_0) = u(x_0) + S_a(x_0, \chi_0) \geq v_n(\chi_0) \geq u(\chi_0) = f(\chi_0),$$

which yields $v_n(\chi_0) = f(\chi_0)$. We can extend the v_n on the whole M keeping the same Lipschitz–constant. The sequence v_n, so extended, is equiLipschitz–continuous and locally equibounded since $v_n\big|_{\Sigma \cap \bar{B}_n} = f$, for any n, in force of what previously pointed out. We can therefore pass to the uniform limit, up to subsequences, to find, taking into account the stability properties of viscosity solutions and that $\cup_n B_n = M$, the desired global solution of $H = a$ taking the value f on Σ.

We proceed proving the converse implication. Assume v, χ_0, B_n to be a global solution of $H = a$ taking the value f on Σ, a point of Σ and an increasing sequence of balls, centered at χ_0, whose union equals the whole space. We know from Proposition 6 that v is the unique solution to

$$\left\{ \begin{array}{rcll} H(u, Dw) & = & a & \text{in } B_n \\ w & = & v & \text{on } \partial B_n \cup \{\mathcal{A} \cap B_n\} \end{array} \right.$$

We denote, for any n, by x_n a point of $\partial B_n \cup \{\mathcal{A} \cap B_n\}$ satisfying $f(\chi_0) = v(x_n) + S_a(x_n, \chi_0)$, then

$$u(x_n) = f(\chi_0) - S_a(x_n, \chi_0) = v(x_n), \tag{20}$$

otherwise there should be $\chi_1 \in \Sigma$ with

$$f(\chi_1) > v(x_n) + S_a(x_n, \chi_1),$$

which is impossible, in view of Corollary 1, since v is a solution of $H = a$ and $f(\chi_1) = v(\chi_1)$.

We first assume that, for any n, there exists a minimal curve ξ_n for S_a connecting x_n to χ_0, with $\xi_n \cap \mathcal{E} = \emptyset$, defined in the interval $[-T_n, 0]$, for some $T_n > 0$. Since $S_a(x_n, \chi_0) = S_a(x_n, \xi_n(t)) + S_a(\xi_n(t), \chi_0)$ for any n, $t \in [-T_n, 0]$ we get, exploiting (20)

$$\begin{array}{rcl} u(x_n) & = & f(\chi_0) - S_a(x_n, \xi_n(t)) - S_a(\xi_n(t), \chi_0) \\ & \le & u(\xi_n(t)) - S_a(x_n, \xi_n(t)) \le u(x_n), \end{array}$$

which, in turn, implies

$$f(\chi_0) - S_a(\xi_n(t), \chi_0) = \max\{f(\chi) - S_a(\xi_n(t), \chi) \,:\, \chi \in \Sigma\}. \tag{21}$$

Sending n to infinity and arguing as in Proposition 10 we produce an a–characteristic ξ defined in $]-\infty, 0]$. Passing to the limit in (21) we also see that ξ satisfies (19). If the above setup does not take place then, according to Lemma 5.5, there is an a–characteristic ξ defined in some interval $]-T, 0]$ with $\xi(0) = \chi_0$ and all the limit points for $t \to -T$ belonging to $\mathcal{A}$, in addition

$$S_a(x_n, \chi_0) = S_a(x_n, \xi(t)) + S_a(\xi(t), \chi_0), \tag{22}$$

for some n and any $t \in]-T, 0]$. If T is finite then we extend ξ in $]-\infty, 0]$, as in Proposition 10, in such a way that

$$S_a(x_n, \cdot) + S_a(\cdot, \chi_0)$$

is constant on $\xi\big|_{]-\infty, -T]}$. Therefore (22) holds on the whole $]-\infty, 0]$ and, arguing as above, we see that (19) is also satisfied. This relation is proved in the same way when $T = -\infty$; no extension of ξ is clearly needed in this case. □

Notice that if $\Sigma \subset \mathcal{A}$ then, taking into account Proposition 9, the previous theorem reduces to Proposition 2 in the case where the compatibility condition (9) holds for the datum f.

By obvious adaptation of the argument of Theorem 5.6, we get a local version of it.

Corollary 2. *Let f, Σ be as in the previous theorem, and denote by Ω a neighborhood of Σ. There is a global solution of (2) in Ω taking the value f on Σ if and only if for any $\chi_0 \in \Sigma$ we can find an a–characteristic ξ defined in some interval $]-T, 0]$ with $\xi(0) = \chi_0$ such that the all limit points of ξ, for $t \to -T$, belong to $\partial\Omega \cup \mathcal{A}$ and*

$$f(\chi_0) - S_a(\xi(t), \chi_0) = \sup\{f(\chi) - S_a(\xi(t), \chi) \,:\, \chi \in \Sigma\} \tag{23}$$

for any t.

6. **An inf-sup formula in the compact case.** In this section we assume the underlying space to be compact and consequently study equation (2) for a critical, taking into account Proposition 7. We will write for simplicity S instead of S_a. The aim is to provide in this setting more direct conditions for finding solutions taking the prescribed value f on Σ and to give representation formulae, as well.

As already said in the Introduction, we look for parallels with formulae giving generating functions for geometrical solutions to the Cauchy problem in the symplectic framework of analytical mechanics, see Appendix 6 for more detail. From this angle, the family of distance functions $S(x,\cdot)$, with $x \in \mathcal{A}$, plays the role of a complete weak (in the viscosity sense) integral, as specified by the following characterization of the Aubry set, see [18], [17] and [28].

Proposition 11. *The function $S(x,\cdot)$ is a solution of (2) in M if and only if $x \in \mathcal{A}$.*

Notice that, with some additional regularity assumptions on the Hamiltonian, namely for H locally Lipschitz in both variables and strictly convex in p, $S(x,\cdot)$ is also differentiable at x and

$$x \in \mathcal{A} \Longleftrightarrow H(x, D_y S(x,y)_{|_{y=x}}) = a, \tag{24}$$

see [18]. The main result of the section is the following

Theorem 6.1. *Let Σ, f be a closed subset of M and continuous function defined on it, respectively. There exists a global solution of (2) taking the value f on Σ if and only if the condition*

$$\inf_{x\in\mathcal{A}} \sup_{\chi\in\Sigma} \Big(S(x,\bar\chi) - S(x,\chi) + f(\chi) \Big) \le f(\bar\chi) \tag{25}$$

is satisfied for any $\bar\chi \in \Sigma$. In such a case,

$$u(y) = \inf_{x\in\mathcal{A}} \sup_{\chi\in\Sigma} \Big(S(x,y) - S(x,\chi) + f(\chi) \Big) \tag{26}$$

is the minimal solution to the problem.

Going back to the comparison with the symplectic framework, we point out that the family

$$y \mapsto S(x,y) - S(x,\chi) + f(\chi)$$

with $x \in \mathcal{A}$ and $\chi \in \Sigma$ is the weak analogous of the generating function (see 30) for the geometrical solution. In our case the extra parameters x and χ are eliminated through an inf–sup procedure. To further illustrate this point and the role of the Aubry set, we discuss the special case where $\Sigma = \mathcal{A}$. Under this additional assumption, Theorem 6.1 is a consequence of Propositions 2, 6, and an uniqueness property also holds, as made precise below, see [17] and [18].

Theorem 6.2. *Let g a continuous datum on $\mathcal{A}$ satisfying the compatibility condition*

$$g(x_1) - g(x_0) \le S(x_0,x_1) \qquad \textit{for every } x_0, x_1 \in \mathcal{A} \tag{27}$$

then the function $u(y) := \inf_{x\in\mathcal{A}} g(x) + S(x,y)$ is the unique solution of (2) taking the value g on $\mathcal{A}$.

The novelty in Theorem 6.1 is therefore that Σ is a general closed subset of M. We note that condition (25) implies the admissibility (9) of the initial datum f. In fact, for every $\chi_0, \chi_1 \in \Sigma$ we have

$$\inf_{x\in\mathcal{A}} \Big(f(\chi_1) - S(x,\chi_1) + S(x,\chi_0)\Big) \leq \inf_{x\in\mathcal{A}} \sup_{\chi\in\Sigma} \Big(f(\chi) - S(x,\chi) + S(x,\chi_0)\Big),$$

and using condition (25) we obtain

$$f(\chi_1) - f(\chi_0) \leq \sup_{x\in\mathcal{A}} S(x,\chi_1) - S(x,\chi_0) \leq S(\chi_0,\chi_1).$$

Moreover the two assumptions are actually equivalent in the case where $\Sigma = \mathcal{A}$.

Corollary 3. *If $\Sigma = \mathcal{A}$ and g is a continuous function defined on it, condition (25) with g in place of f is equivalent to (27).*

We establish a preliminary lemma.

Lemma 6.3. *Let Σ be a closed subset non disjoint from $\mathcal{A}$ and $\bar{\chi} \in \Sigma \cap \mathcal{A}$ then*

$$\inf_{x\in\mathcal{A}} \sup_{\chi\in\Sigma} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big) \leq \sup_{\chi\in\Sigma} \inf_{x\in\mathcal{A}} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big).$$

Proof. Given any $x \in \mathcal{A}$, we denote by χ_x a maximizer in Σ of

$$f(\cdot) - S(x,\cdot) + S(x,\bar{\chi}).$$

We have

$$\inf_{x\in\mathcal{A}} \sup_{\chi\in\Sigma} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big) = \inf_{x\in\mathcal{A}} \Big(f(\chi_x) - S(x,\chi_x) + S(x,\bar{\chi})\Big) \leq$$

$$\leq f(\chi_{\bar{\chi}}) - S(\bar{\chi},\chi_{\bar{\chi}}) = \sup_{\chi\in\Sigma} f(\chi) - S(\bar{\chi},\chi).$$

Due to the inequality $S(x,\bar{\chi}) - S(x,\chi) \geq -S(\bar{\chi},\chi)$, the last term of the previous formula can be rewritten as

$$\sup_{\chi\in\Sigma} f(\chi) - S(\bar{\chi},\chi) = \sup_{\chi\in\Sigma} \inf_{x\in\mathcal{A}} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big).$$

This ends the proof. □

Proof of Corollary 3. We have already noted that condition (25) implies the admissibility of the datum on $\mathcal{A}$. Conversely, assuming the admissibility of f, we obtain by the previous lemma

$$\inf_{x\in\mathcal{A}} \sup_{\chi\in\mathcal{A}} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big) \leq \sup_{\chi\in\mathcal{A}} \inf_{x\in\mathcal{A}} \Big(f(\chi) - S(x,\chi) + S(x,\bar{\chi})\Big) =$$

$$= \sup_{\chi\in\mathcal{A}} f(\chi) - S(\bar{\chi},\chi) = f(\bar{\chi}).$$

□

As a consequence of Theorem 6.2, the search for global solutions taking the value f on Σ is thus equivalent to the determination of a continuous admissible function g on $\mathcal{A}$ such that its unique extension out of $\mathcal{A}$, say u, which is global solution of (2), i.e. u defined as in the statement of Proposition 6.2, coincides with f on Σ. Loosely speaking, this can be viewed as a sort of *rebound problem* in which the Aubry set is explicitly involved.

Proof of the theorem 6.1. We first claim that function (26) is a global solution to (2). In fact, from Proposition 4 we derive that

$$\sup_{\chi \in \Sigma} -S(\cdot, \chi) + f(\chi)$$

is the minimal subsolution to (2) coinciding with f on Σ. Hence, by Corollary 1, it gives an admissible function when restricted on $\mathcal{A}$ and, according to Proposition 6.2, the function (26) is indeed the unique extension of such a trace outside $\mathcal{A}$ providing a global solution. Moreover, as a consequence of condition (25), it takes the value f on Σ.

To prove the necessity of condition (25), let us suppose by contradiction that (25) does not hold but it exists a global solution, say w, to our problem. Let $\bar{\chi} \in \Sigma$ be such that

$$\inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S(x, \bar{\chi}) - S(x, \chi) + f(\chi) \Big) > f(\bar{\chi}). \tag{28}$$

The function w is a subsolution to (2), and so, in force of Corollary 1, $w|_{\mathcal{A}}$ is admissible, and we obtain by Corollary 4

$$w(\cdot) \geq \sup_{\chi \in \Sigma} -S(\cdot, \chi) + f(\chi) \qquad \text{on } M,$$

accordingly

$$\inf_{x \in \mathcal{A}} w(x) + S(x, \cdot) \geq \inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S(x, \cdot) - S(x, \chi) + f(\chi) \Big).$$

Now, since w is a global solution to (2) then by Proposition 6.2

$$\inf_{x \in \mathcal{A}} w(x) + S(x, \cdot) = w(\cdot) \qquad \text{on } M$$

so that

$$w(\cdot) \geq \inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S(x, \cdot) - S(x, \chi) + f(\chi) \Big).$$

This implies, in particular

$$w(\bar{\chi}) \geq \inf_{x \in \mathcal{A}} \sup_{\chi \in \Sigma} \Big(S(x, \bar{\chi}) - S(x, \chi) + f(\chi) \Big) > f(\bar{\chi}),$$

which contradicts (28). Finally the minimality property of the solution (26) is a straightforward consequence of the minimality of $\sup_{\chi \in \Sigma} f(\chi) - S(x, \chi)$ among the subsolutions taking the value f on Σ. □

In general the Cauchy problem under consideration is not uniquely solvable when the datum f is assigned on a general closed subset Σ, in contrast to what happens if $\Sigma = \mathcal{A}$, as illustrated in Theorem 6.2.

The solution u provided by the previous theorem has trace $\sup_{\chi \in \mathcal{A}} \Big(f(\chi) - S(x, \chi) \Big)$ on the Aubry set, and it cannot be excluded that other traces on $\mathcal{A}$ greater than it take the value f on Σ, when extended through the Lax formula provided in Theorem 6.2. In this way we should have different solutions. This possibility is ruled out if the following holds true: for any $x_0 \in \mathcal{A}$ there is $\chi_0 \in \Sigma$ such that

$$u(\chi_0) = f(\chi_0) = \sup_{\chi \in \Sigma} \Big(S(x_0, \chi_0) - S(x_0, \chi) + f(\chi) \Big).$$

This is indeed an uniqueness condition. In the general case it is easy to give examples of nonuniqueness.

Let us, for instance, consider the mechanical Hamiltonian $\frac{1}{2}|p|^2 + V(y)$ on $\mathbb{T}^1$ and assume the potential V to attain its maximum value at 0 and to possess two maximizers, say y_1, y_2. In this case 0 is the critical value and $\mathcal{A} = \{y_1, y_2\}$, moreover the associated intrinsic distance $S_0(x, y)$ is strictly positive whenever $x \neq y$. We take Σ so close to y_1, in the Euclidean sense, that

$$\max\{S_0(y_1, \chi) \,:\, \chi \in \Sigma\} < \min\{S_0(y_2, \chi) \,:\, \chi \in \Sigma\}.$$

We define $f = S_0(y_1, \cdot)$ on Σ. It is clear that, in force of Theorem 6.1, the condition (25) is satisfied since $S_0(y_1, \cdot)$ is a solution of the Cauchy problem. We have

$$\sup_{\chi \in \mathcal{A}} \Big(f(\chi) - S_0(y_2, \chi) \Big) = \sup_{\chi \in \mathcal{A}} \Big(S_0(y_1, \chi) - S_0(y_2, \chi) \Big) < 0 < S_0(y_1, y_2).$$

Since the solution u provided by Theorem 6.1 takes as value at y_2 the quantity in the left hand–side of the previous formula, we actually see that u is different from $S_0(y_1, \cdot)$.

Appendix: Symplectic and Lagrangian manifolds, Maslov-Hörmander parameterizations. We first review some topics from the theory of generating functions for canonical transformations in the framework of symplectic geometry.

Adopting standard notations, as in [2], we denote by ϑ the 1-form of Liouville on the cotangent bundle T^*M, which admits the local representation $\vartheta = p\,dy = \sum_{i=1}^N p_i\,dy^i$. Its exterior differential is the 2-form $\omega = d\vartheta$, which is said *symplectic*, locally $\omega = dp \wedge dy = \sum_{i=1}^N dp_i \wedge dy^i$. The cotangent bundle T^*M, equipped with ω, is the prototype of symplectic manifold, namely a (even dimensional) manifold endowed with a closed and nondegenerate 2-form.

The notion of Lagrangian submanifold of T^*M can be interpreted as a multivalued generalization of the graph of the differential of a real valued smooth function, say g, on M. It is easy to see that $\omega|_{\mathrm{Graph}(dg)} = \sum_{ij} D^2_{y^i y^j} g \ dy^i \wedge dy^j \equiv 0$, $\dim \mathrm{Graph}(dg) = N = \dim M$ and $\mathrm{Graph}(dg)$ is globally transverse to the fibers of the projection $\pi_M : T^*M \longrightarrow M$. Generalizing this setting, we say that $\Lambda \subset T^*M$ is a *Lagrangian submanifold* if

$$\omega|_\Lambda = 0 \quad \text{ and } \quad \dim \Lambda = N \ = \frac{1}{2}\dim(T^*M).$$

Notice that the transversality property of graphs has been relaxed. On the other hand Maslov [26] and Hörmander [21] theory shows that any Lagrangian manifold is, at least locally with respect to the base manifold M, the graph of the differential of some smooth function, belonging to a family indexed by a parameter ζ, on the critical set with respect to ζ. Such a family is called *generating function* of Λ and is altogether denoted by $s(y, \zeta)$, where y is, as usual, the state variable and the auxiliary parameters ζ vary in $\mathbb{R}^k$, for some $k \in \mathbb{N}$.

Theorem 6.4. ***(Maslov-Hörmander)*** *A submanifold Λ of T^*M is Lagrangian if and only if for any $\lambda \in \Lambda$ there exist an open neighborhood $U \subset M$ of $\pi_M(\lambda)$ and a smooth generating function $s : U \times \mathbb{R}^k \longrightarrow \mathbb{R}$ such that*

(i) $\Lambda|_{T^*U} = \big\{(y, p) \in T^*M : \ p = D_y s(y, \zeta), \ 0 = D_\zeta s(y, \zeta), \ y \in U, \ \zeta \in \mathbb{R}^k\big\}$,
(ii) 0 *is a regular value of the map* $(y, \zeta) \longmapsto D_\zeta s(y, \zeta)$.

We denote by $H : T^*M \to \mathbb{R}$, X_H and ϕ^t_H a Hamiltonian function, its vector field ($i_{X_H}\omega = -dH$) and the related flow, respectively. We recall that the Hamiltonian flow sends Lagrangian submanifolds into Lagrangian submanifolds and for any Lagrangian submanifold Λ contained in a regular fiber of a Hamiltonian, say $\Lambda \subset H^{-1}(a)$, $X_H(y,p) \in T_{(y,p)}\Lambda$. In addition, for any fixed $t \in \mathbb{R}$, ϕ_H^t is a canonical transformation, where a *canonical transformation* (or *symplectic diffeomorphism*) F is a diffeomorphism of T^*M into itself preserving the symplectic structure, that is, with pull–back $F^*\omega$ equal to ω.

A symplectic structure $\bar{\omega}$ can be given to $T^*M \times T^*M$ in such a way that the graph of any canonical transformation $(y,p) = F(\bar{y},\bar{p})$ is a Lagrangian submanifold. We define $\bar{\omega}$ through twofold pull-back of the standard symplectic 2-form on T^*M,

$$\bar{\omega} := pr_2^*\omega - pr_1^*\omega = dp_2 \wedge dy_2 - dp_1 \wedge dy_1,$$

where pr_1, pr_2 are the standard projections

$$T^*M \quad \overset{pr_1}{\longleftarrow} \quad T^*M \times T^*M \quad \overset{pr_2}{\longrightarrow} \quad T^*M.$$

Since $T^*M \times T^*M \cong T^*(M \times M)$, it can be deduced from a suitable resetting of Maslov-Hörmander Theorem the existence (at least locally) of generating functions $s : (M \times M) \times \mathbb{R}^k \to \mathbb{R}$, $s = s(y,\bar{y};\zeta)$, such that

$$\Lambda = \operatorname{Graph} F = \Big\{(\bar{y},\bar{p},y,p) \in T^*M \times T^*M : \ p = D_y s,\ \bar{p} = -D_{\bar{y}} s,\ 0 = D_\zeta s \Big\}.$$

Canonical transformations have a clear group structure; given two canonical transformations F and G with generating functions $s_F(y,\bar{y};\zeta)$ and $s_G(y,\bar{y};\eta)$, a generating functions $s_{G\circ F}$ for $G \circ F$ is given by means of the following *composition rule*

$$s_{G\circ F}(y,\bar{y};z,\zeta,\eta) := s_G(y,z;\eta) + s_F(z,\bar{y};\zeta).$$

In the above formula the set of variables z,ξ,η for $s_{G\circ F}$ are –all– auxiliary parameters to stationarize: it is easy to see that $D_z s_{G\circ F} = 0$ means that any image momentum of F is precisely a source momentum of G:

$$D_z s_{G\circ F} = 0 : \quad \underbrace{D_z s_F(z,\bar{y};\zeta)}_{\text{image impulse of } F} = \underbrace{-D_z s_G(y,z;\eta)}_{\text{source impulse of } G}.$$

The framework of symplectic geometry and the notion of Lagrangian submanifold allows to tackle stationary Hamilton-Jacobi equations with a fully geometrical approach. Starting from the first attempts and proposals by Maslov [26] and Vinogradov [29], this theory can be rephrased as follows.
Given a regular value $a \in \mathbb{R}$ of H, instead of looking for global classical solutions of (2), $H(y, Du(y)) = a$, which is in general hopeless, we search for a Lagrangian submanifold $\Lambda \in T^*M$ satisfying

$$\Lambda \subset H^{-1}(a),$$

and call it *geometrical solution*. Notice that for some $\lambda_0 \in \Lambda$, the set

$$\{\lambda \in \Lambda : \pi_M(\lambda) = \pi_M(\lambda_0)\}$$

is possibly multivalued, just like a Riemann surface in complex analysis. It is clear that if $\Lambda = \operatorname{Graph}(Du)$, for some C^1 function u, then such function is a classical solution of the Hamilton–Jacobi equation.

Whenever the Hamiltonian system is *Liouville integrable* –see [2]– there exists a family of classical solutions for (2), said *complete integral*, that is a smooth function $s_0 : A \times M \to \mathbb{R}$, where A is an open subset of $\mathbb{R}^{N-1}$ and the parameter ζ varies in A , satisfying

$$H(y, D_y s_0(y,\zeta)) = a \quad \text{for all } \zeta \in A,$$

and the rank of the Hessian of s_0, made up by the second mixed derivatives with respect to y and ζ, is maximum, i.e. equal to $N-1$. In other words, the function s_0 generates a Lagrangian *foliation* of $H^{-1}(a)$, i.e. $H^{-1}(a) = \cup_{\zeta \in A} \Lambda_\zeta$, where

$$\Lambda_\zeta := \{(y,p) \in T^*M : \, p = D_y s_0(y,\zeta)\} = \mathrm{Graph}(D_y s_0(\cdot,\zeta))$$

and, in addition

$$\zeta \neq \bar{\zeta} \;\Rightarrow\; \Lambda_\zeta \cap \Lambda_{\bar{\zeta}} = \emptyset.$$

We define the *characteristic relation set* $\Lambda^{(a)}$ by

$$\Lambda^{(a)} := \left\{(\bar{y},\bar{p},y,p) \, : \, (y,p) = \phi_H^t(\bar{y},\bar{p}) \text{ for some } t \in \mathbb{R}, \; H(\phi_H^t(\bar{y},\bar{p})) = a\right\}.$$

It turns out being Lagrangian in $(T^*M \times T^*M, \bar{\omega})$, and, substantially, represents free time canonical relations based on the flow ϕ_H^t: it is noteworthy pointing out that, in a sense, a similar idea is behind the very definition of the a–characteristics of Section 5. If $\hat{s}(y,\bar{y};\xi)$ is a general (possibly global) generating function for $\Lambda^{(a)}$ then it can be proved by direct computation that under the standard additional hypothesis of noncharacteristic initial data –see e.g. [24], Definition 2.5 in Appendix 7– the function

$$\mathbf{s}(y;\bar{y},\xi) = \hat{s}(y,\bar{y};\xi) + f(\bar{y}),$$

where now $\bar{y} \in \Sigma$ as well as ξ are thought as parameters, generates the Lagrangian submanifold in T^*M geometric solution of

$$\begin{cases} H(y,Du) = a \\ u|_\Sigma = f \end{cases} \tag{29}$$

In the Liouville integrable case, a generating function for $\Lambda^{(a)}$ is given by

$$\hat{s}(y,\bar{y};\zeta) := s_0(y,\zeta) - s_0(\bar{y},\zeta),$$

see [11]. As a consequence, in such a special integrable case,

$$\mathbf{s}(y;\bar{y},\zeta) := s_0(y,\zeta) - s_0(\bar{y},\zeta) + f(\bar{y}). \tag{30}$$

This procedure yields classical solutions only when the equations of stazionarization on the auxiliary parameters

$$0 = D_{\bar{y}}\mathbf{s} \qquad 0 = D_\zeta \mathbf{s}$$

explicitly give $\bar{y}$ and ζ in function of the state variable y, $\bar{y} = \bar{y}(y)$, $\zeta = \zeta(y)$; if this is the case $y \mapsto \mathbf{s}(y;\bar{y}(y),\zeta(y))$ is actually –at least locally– a classical solution.

Going back to Section 6, we have, in a sense, interpreted the intrinsic distance S_a as a weak complete integral, bearing in mind the property (24). More precisely, we have replaced the complete integral $s_0(y,\zeta)$, $\zeta \in A$, appearing in (30) by the distance function $S_a(x,y)$, $x \in \mathcal{A}$, to obtain the weak generating function

$$S(x,y) - S(x,\chi) + f(\chi) \tag{31}$$

$y \in M$, $x \in \mathcal{A}$ and $\chi \in \Sigma$. Notice that the idea of utilizing generating functions in order to construct viscosity solutions has been already exploited, see for example [7], [12], [9].

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