Mather measures associated with a class of Bloch wave functions

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Abstract. In this paper we study the Wigner transform for a class of smooth Bloch wave functions on the flat torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$:

$$\psi_{\hbar,P}(x) = a(\hbar, P, x)e^{\frac{i}{\hbar}(P\cdot x+\hat{v}(\hbar,P,x))}.$$

On requiring that $P \in \mathbb{Z}^n$ and $\hbar = 1/N$ with $N \in \mathbb{N}$, we select amplitudes and phase functions through a variational approach in the quantum states space based on a semiclassical version of the classical effective Hamiltonian $\bar{H}(P)$ which is the central object of the weak KAM theory. Our main result is that the semiclassical limit of the Wigner transform of $\psi_{\hbar,P}$ admits subsequences converging in the weak* sense to Mather probability measures on the phase space. These measures are invariant for the classical dynamics and Action minimizing.

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1. Introduction

The semiclassical limit of the *Wigner transform* has been intensively studied in the literature, with respect to different settings and related problems. From the evolutionary viewpoint, we recall the papers where the semiclassical limits of the Wigner transform, the so-called *Wigner measures*, are studied for states evolved by the quantum dynamics and it is proved that such time-dependent limits solve the Liouville equation in the measure sense ([L-P], [Pul], [A-F-P], [A-F-G], [A-P], [A-P2], [Ge1], [G-M-M-P], [M-P-S]). From the stationary viewpoint, we recall the several papers involving the semiclassical limit of the Wigner transform of energy eigenfunctions as well as of energy quasimodes, and showing its convergence to invariant measures under the classical dynamics ([An], [C-R-R], [E-G-I], [H-M-R], [T-Z], [Z]).

In our paper we look at Wigner measures which are also Mather measures, i.e. probability measures on the phase space that are invariant under the classical dynamics and Action minimizing. These measures represent one of the central objects of *Aubry-Mather theory*, which is currently and widely studied by several authors (see e.g. [B2], [B3], [B-B], [C-G-T], [D-I-S-Y], [F], [So]).

In particular, here we study the Wigner measures associated with a class of Bloch wave functions selected by a variational approach on the quantum state space inspired from the *weak KAM theory* (see [B1], [C-I-P], [E2], [E4], [F], [F-S], and references therein).
More precisely, we are concerned with the semiclassical analysis of the Weyl quantization on the flat torus $\mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ of the classical Hamiltonian system related to $H(x,p) := \frac{1}{2}|p|^2 + V(x)$ with $V \in C^\infty(\mathbb{T}^n;\mathbb{R})$. The (semiclassical) Wigner distribution of a state ψ is here defined by

$$W_\hbar\psi(x,\xi) := (2\pi)^{-n} \int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\xi\rangle} \psi(x-z)\overline{\psi}(x+z)dz, \tag{1.1}$$

so that for the Weyl-quantization of a symbol $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ we have

$$\langle \psi, \mathrm{Op}^w_\hbar(b)\psi\rangle_{L^2(\mathbb{T}^n)} = \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} b(y,\xi)W_\hbar\psi(y,\xi)dy.$$

The aim is to look at the stationary study of the Wigner transform $W_\hbar\psi$ for a class of Bloch wave functions ψ in $C^k(\mathbb{T}^n;\mathbb{C})$, with k sufficiently large, where

$$\psi(x) = \psi_{\hbar,P}(x) = a(\hbar,P,x)e^{\frac{i}{\hbar}(P\cdot x + \hat{v}(\hbar,P,x))},\ P \in \mathbb{Z}^n,\ \hbar = 1/N, \tag{1.2}$$

for some $N \in \mathbb{N}$. We remark that taking $\hbar = 1/N$, for some $N \in \mathbb{N}$, allows to work with 2π-periodic phase functions and hence gives well-defined Bloch wave functions on the flat torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. We select these states in such a way that $W_\hbar\psi_{\hbar,P}$ admits a semiclassical limit as $\hbar \to 0^+$ (i.e. $N \to +\infty$), by possibly passing to a subsequence, which is a Mather probability measure in the phase space $\mathbb{T}^n \times \mathbb{R}^n$. In fact, we follow the original notion of semiclassical measures (see for example [G-L], [Ge]) but here with the additional feature of being Mather measures, and as a consequence also stationary solutions of the Liouville equation. The supports of these measures are invariant sets (the union is the so-called Mather set) naturally contained in weak KAM tori of the phase space, namely the graphs of weak KAM solutions of the stationary Hamilton-Jacobi equation

$$H(x, P + \nabla_x v(P,x)) = \bar{H}(P), \tag{1.3}$$

where, in the general and classical setting, $P \in \mathbb{R}^n$ and $\bar{H}(P)$ is the *effective Hamiltonian* (see [C-I-P], [E2], [F]).

Our variational approach is based on a version of the *semiclassical effective Hamiltonian*, that here we define as

$$\bar{H}_\hbar(P) := \inf_v \sup_a \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x))a(x)^2 - \frac{\hbar^2}{2}|\nabla_x a(x)|^2\, dx \tag{1.4}$$

for every fixed $P \in \mathbb{R}^n$ and the inf-sup procedure is taken with respect to all amplitudes and phase functions that belong to $C^\infty(\mathbb{T}^n;\mathbb{R})$ and fulfill

$$\int_{\mathbb{T}^n} a(x)^2 dx = 1, \quad \int_{\mathbb{T}^n} v(x) dx = 0. \tag{1.5}$$

In the asymptotic setting $\hbar \to 0^+$, formula (1.4) gives exactly the classical effective Hamiltonian, that is

$$\lim_{\hbar \to 0^+} \bar{H}_\hbar(P) = \bar{H}(P). \tag{1.6}$$

This fact has been proved by Evans in [E4], Section 3. In particular, in that paper Evans introduces the variational principle to obtain new approximations and estimates giving, in appropriate asymptotic limits, the fundamental PDE for weak KAM theory, namely the stationary Hamilton-Jacobi equation and its coupled continuity equation. Moreover, assuming the existence and smoothness of the minimizers v, he also discusses a sort of "approximate integrability" of certain phase-space evolutions related to the original Hamiltonian flow. Here we prove (1.6) with a more refined estimate (see Theorem 4.5). We finally stress that the semiclassical effective Hamiltonian $\bar{H}_\hbar(P)$ we introduce here, is different from the one introduced by Evans in [E3], Sections 6-7, in order to prove a *quantum analog to weak KAM theory*. In [E3], the selection involves particular phase functions and amplitudes related to local minima of the quantum Action functional $A_\hbar$ on ψ of the form (1.2),

$$A_\hbar[\psi] = \int_{\mathbb{T}^n} \frac{\hbar^2}{2} |\nabla_x \psi(x)|^2 - V(x)|\psi(x)|^2 \, dx.$$

Moreover, in that paper the rigorous construction of a candidate minimizer ψ is used to propose a sort of semiclassical quantization of weak KAM theory. In particular, considering the state ψ as a quasimode, Evans derives an $O(h)$-error term. Similar results have been proved by Gomes and Valls in the paper [G-V], where the authors investigate the above quantum action problem using Wigner measures on the torus, thus suggesting a *quantum version of the Aubry-Mather theory.*

Here we study the main variational features of the functional in (1.4) and we show that the supremum of the functional

$$\lambda_0[\hbar, P, v] := \sup_a \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x)) a(x)^2 - \frac{\hbar^2}{2} |\nabla_x a(x)|^2 dx \tag{1.7}$$

is in fact a maximum realized on a unique smooth amplitude $\varphi_0(\hbar, P, v)(x)$. This function is the unique positive and normalized eigen function belonging to the principal eigenvalue of the elliptic differential operator $\frac{1}{2}\hbar^2 \Delta_x + H(x, P + \nabla_x v(x))$. This works for all v as in (1.5). About the minimizers of λ_0, we prove that there exists $\hat{w} = \hat{w}(\hbar, P, \cdot) \in W^{1,2}(\mathbb{T}^n;\mathbb{R})$ realizing global minima, whence

$$\bar{H}_\hbar(P) = \lambda_0[\hbar, P, \hat{w}].$$

However, we may also look for C^k *and* $o(\hbar^\alpha)$*-approximated global minimum points.* Indeed, in the classical setting we have

$$\inf_v \lambda_0[0,P,v] = \inf_v \max_{x\in\mathbb{T}^n} H(x, P+\nabla_x v(x)) = \bar{H}(P),$$

and this value becomes a minimum when $\lambda_0[0,P,\cdot]$ is evaluated over an arbitrary $C^{0,1}$ - critical subsolution $\tilde{v}$ of the stationary Hamilton-Jacobi equation, that is

$$H(x, P+\nabla_x \tilde{v}(P,x)) \leq \bar{H}(P). \tag{1.8}$$

As a consequence, we may apply an Ekeland's variational principle (see [A-E]) and prove that for every fixed $v(P,x)$ and for k sufficiently large, there exist C^k-functions $\hat{v} = \hat{v}(\hbar, P, x)$ such that for some constants $c_i(P) > 0$, $i = 1,2,3$, for all fixed $0 < \alpha < 1$, we have $\hbar$-perturbations of the critical subsolution, namely

$$\|\hat{v}(\hbar,P,\cdot) - \tilde{v}(P,\cdot)\|_{C^{0,1}} \leq c_2(P)\hbar^{\alpha/2},$$

and approximated global minima, in the sense that

$$\bar{H}_\hbar(P) \leq \lambda_0[\hbar,P,\hat{v}] \leq \bar{H}_\hbar(P) + c_1(P)\hbar^\alpha, \quad \Big\| \frac{D\lambda_0}{Dv}[\hbar,P,\hat{v}] \Big\|_\star \leq c_3(P)\hbar^{\alpha/2}.$$

The set of all such phase functions is denoted by $\mathbf{\Gamma}_{\hbar,P}$ and consequently our selected set of smooth Bloch wave functions, *defined for* $P\in\mathbb{Z}^n$ *and* $\hbar = 1/N$, $N\in\mathbb{N}$, is the following

$$\mathbf{\Psi}_{\hbar,P} := \Big\{ \psi_{\hbar,P}(x) = \varphi_0(\hbar,P,\hat{v})(x) e^{\frac{i}{\hbar}(P\cdot x + \hat{v}(\hbar,P,x))} \quad | \quad \hat{v}\in\mathbf{\Gamma}_{\hbar,P} \Big\}. \tag{1.9}$$

The relevance of the states in $\mathbf{\Psi}_{\hbar,P}$ appears in the main results of the paper, that involve the semiclassical approximations as well as the asymptotics of their Wigner transform. In what follows, the notion of convergence is considered with respect to the weak* topology on the set of complex measures, using test functions $\phi\in C^\infty(\mathbb{T}^n\times\mathbb{R}^n;\mathbb{R})$ such that $\mathrm{supp}(\mathcal{F}_y^{-1}\phi(x,\cdot))$ is compact ($\mathcal{F}_y^{-1}$ denoting the inverse Fourier transform on $\mathbb{R}^n$ in the y variable).

Our main results are the following.

Theorem 1.1. *The Wigner transform of states in* $\mathbf{\Psi}_{\hbar,P}$ *satisfies*

$$\begin{aligned}
&\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\xi) W_\hbar \psi_{\hbar,P}(x,\xi)\,dx \qquad (1.10)\\
&= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\xi)\,d\mu_{\hbar,P}(x,\xi) + \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n} \int_{\mathbb{T}^n} \phi(x,\xi)\,dr_{\hbar,P}(x,\xi).
\end{aligned}$$

The first order approximation is a probability measure $d\mu_{\hbar,P}$ *on the phase space* $\mathbb{T}^n\times\mathbb{R}^n$ *for which*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} \phi(x,\xi)\,d\mu_{\hbar,P}(x,\xi) = \int_{\mathbb{T}^n} \phi(x, P+\nabla\hat{v}(\hbar,P,x))\varphi_0(\hbar,P,\hat{v})(x)^2\,dx.$$

The remainder measure $dr_{\hbar,P}$ *fulfills the weak* limit*

$$dr_{\hbar,P} \rightharpoonup 0 \qquad \text{as } \hbar = 1/N \to 0^+. \tag{1.11}$$

Notice that, in view of (1.11), the probability measure $d\mu_{\hbar,P}$ takes an intermediate role between $W_\hbar\psi_{\hbar,P}$ and the asymptotic probability measures described in Theorem 1.2 below.

Theorem 1.2. *Every family of probability measures $\{d\mu_{\varepsilon,P}\}_{0<\varepsilon<1}$ (with $\varepsilon = 1/N$, $N \in \mathbb{N}$) has a weak* convergent subsequence*

$$d\mu_{\varepsilon(P,\alpha),P} \rightharpoonup d\tilde{\mu}_P, \qquad \varepsilon(P,\alpha) = 1/N(P,\alpha) \to 0^+ \text{ as } \alpha \to +\infty, \tag{1.12}$$

and the limit is a Mather measure $d\tilde{\mu}_P$ for which

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \phi(x,\xi) d\tilde{\mu}_P(x,\xi) = \int_{\mathbb{T}^n} \phi(x, P + \nabla\tilde{v}(P,x)) d\tilde{\sigma}_P(x) \tag{1.13}$$

where $\tilde{v}(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ solves (1.8) and $d\tilde{\sigma}_P$ is a Radon probability measure on $\mathbb{T}^n$.

About the asymptotic measures $d\tilde{\mu}_P$, we remark that

$$\int_{\mathbb{T}^n} H(x, P + \nabla_x\tilde{v}(P,x)) d\tilde{\sigma}_P(x) = \bar{H}(P), \tag{1.14}$$

$$\int_{\mathbb{T}^n} \nabla_x f(x) \cdot (P + \nabla_x\tilde{v}(P,x)) d\tilde{\sigma}_P(x) = 0, \ \forall f \in C^\infty(\mathbb{T}^n;\mathbb{R}). \tag{1.15}$$

These equations correspond respectively, in a measure sense, to the stationary Hamilton-Jacobi equation

$$H(x, P + \nabla_x v(P,x)) = \bar{H}(P) \tag{1.16}$$

and its coupled continuity equation

$$\mathrm{div}_x[(P + \nabla_x v(P,x))\sigma(P,x)] = 0. \tag{1.17}$$

The intrinsic multiplicity of the set of solutions of (1.14) and (1.15) allows for possible different semiclassical limits in (1.12). However, it is still an open question whether or not, through our technique, it is possible to get all the Mather measures in the form (1.13). Moreover, it is also an unsolved problem that of getting a measure as in (1.13) with full support in the Mather set.

We finally stress that in the case of a Hamiltonian system in the KAM setting, the dynamics on each maximal KAM torus is realized by a dense orbit conjugated to a (irrational) rotation on $\mathbb{T}^n$ with frequency vector satisfying Diophantine condition (see for example [Chi]). This implies - see Section 3 in [So] - the uniqueness of invariant measures and, as a consequence, the uniqueness of the Mather measures; moreover the Mather set coincides with the maximal KAM torus for all fixed $P \in \mathbb{R}^n$. This fact implies that the unique Mather measure has full support on the KAM torus and that it can be written in the form (1.13) where $\tilde{v}(P,x)$ is smooth and solves (1.16), meanwhile $d\tilde{\sigma}_P(x) = \sigma(P,x)dx$ where $\sigma(P,x)$ is a smooth density solving (1.17).

The content of the paper is the following. In Section 2 we give a brief overview of weak KAM theory, involving the different notions of weak solutions for stationary Hamilton-Jacobi equations. In Section 3 we recall some basic arguments of the Aubry-Mather theory, in particular the notion of

Mather measures, the definitions of Mather and Aubry sets. Section 4 is the core of the paper, namely the variational approach on the quantum state space. Here we define the notion of the semiclassical effective Hamiltonian with the corresponding study of the functional-analytical setting, followed in Subsection 4.2 by the results on the Sobolev regularity of global minimizing phase functions and the existence of approximated smooth minimizers. In Subsection 4.3 we next introduce the class of Bloch wave functions arising from this variational approach. Section 5 begins with the summary of the Weyl quantization on the torus, and continues with the proof of the results of the paper, namely Theorems 1.1 and 1.2.

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2. A brief overview of weak KAM theory

KAM theory investigates the persistence, under small perturbations, of typical invariant sets of integrable Hamiltonian systems. The main result of this theory was proved by Kolmogorov in 1954, and was followed by the fundamental works of Arnol'd and Moser in the 60's, who technically overcame the problem related to the appearance of arbitrarily small divisors due to the perturbative setting. However, the existence of smooth invariant tori for nearly integrable Hamiltonians is destroyed when the ε-perturbations of integrable Hamiltonians becomes large.

Recent alternative approaches to the study of non integrable Hamiltonian systems, in a non-perturbative setting, led to the so called *weak KAM theory*. One of the main outcomes of this theory is the existence of global weak (Lipschitz) solutions of the stationary Hamilton-Jacobi equation:

$$H(x, P + \nabla_x v(P, x)) = \bar{H}(P), \tag{2.1}$$

for general Tonelli Hamiltonians. The function $\bar{H}(P)$ is called the *effective Hamiltonian* and, as showed in [C-I-P] (see also [E2]), can be expressed by the following inf-sup formula

$$\bar{H}(P) = \inf_{v \in C^\infty(\mathbb{T}^n;\mathbb{R})} \sup_{x \in \mathbb{T}^n} H(x, P + \nabla_x v(x)). \tag{2.2}$$

For any given $P \in \mathbb{R}^n$, the corresponding weak solutions $v(P, \cdot)$ detect structures of weak integrability in the phase space, the so-called weak KAM tori:

$$\Lambda_P := \mathrm{Graph}\{P + \nabla v(P, \cdot)\} = \{(x, p) \in \mathbb{T}^n \times \mathbb{R}^n \mid p = P + \nabla_x v(P, x)\},$$

where $\nabla_x v(P, \cdot)$ is the gradient of a Lipschitz function and therefore defined almost everywhere. In our paper we directly work with Hamiltonians of mechanical type:

$$H(x, p) := \frac{1}{2}|p|^2 + V(x), \quad V \in C^\infty(\mathbb{T}^n; \mathbb{R}). \tag{2.3}$$

We recall that for a solution of the Hamilton-Jacobi equation (2.1), different notions exists: 1. *critical subsolutions*, 2. *viscosity solutions* and 3. *weak KAM solutions*. Here we give only the basics.

1. We say that a Lipschitz continuous function $u(P,\cdot):\mathbb{T}^n\to\mathbb{R}$ is a *subsolution* of the Hamilton-Jacobi equation (2.1) if almost everywhere one has

$$H(x,\nabla_x u(P,x))\leq \bar{H}(P). \tag{2.4}$$

For energy levels greater than $\bar{H}(P)$, there exist C^∞ subsolutions, see [C-I-P]. Moreover, Fathi and Siconolfi proved in [F-S] that there exist critical subsolutions with C^1 regularity. Finally, Bernard showed in [B1] the existence of $C^{1,1}$ critical subsolutions, and exhibit a mechanical type example for which C^2 critical subsolutions do not exist.

2. We say that $u:\mathbb{T}^n\to\mathbb{R}$ is a *viscosity subsolution* of $H(x,\nabla_x u)=c$ on the open set $\Omega\subset\mathbb{T}^n$ if for every C^1 test function $\phi:\Omega\to\mathbb{R}$ and every point $x_0\in\Omega$ such that $u-\phi$ has a maximum in x_0, one has $H(x,\nabla_x\phi(x))\leq c$. Conversely, it is a *viscosity supersolution* if for every C^1 function $\psi:\Omega\to\mathbb{R}$ and every point $x_0\in\Omega$ such that $u-\phi$ has a minimum in x_0, then $H(x,\nabla_x\phi(x))\geq c$. A *viscosity solution* is required to be both a subsolution and a supersolution (see for example [Ba]).

3. A constructive approach to weak KAM solutions of $H(x,\nabla_x u)=c$ involves the Lax-Oleinik semigroup of negative and positive type:

$$T_t^{\mp}u(x):=\inf_{\gamma}\left\{u(\gamma(0))\pm\int_0^t L(\gamma(s),\dot\gamma(s))ds\right\}$$

where the infimum is taken over all absolutely continuous curves $\gamma:[0,t]\to\mathbb{T}^n$ such that $\gamma(t)=x$. A function $u:\mathbb{T}^n\to\mathbb{R}$ is a *negative weak KAM solution* of (2.4) if $T_t^- u(x)=u(x)-ct$ for all $t>0$, whereas it is a *positive weak KAM solution* of (2.4) if $T_t^+u(x)=u(x)+ct$ for all $t>0$. Geometrically these two conditions mean that we are looking for functions whose gradients are invariant under the backward (resp. forward) Euler-Lagrange flow. We recall that Fathi (see [F]) proved that on the Mañé critical value weak KAM solutions exist and solve the H-J equation in the viscosity sense.

2.1. Some useful properties

We here give some simple but useful estimates about the effective Hamiltonian.

Lemma 2.1. *The effective Hamiltonian (*2.2*) fulfills the estimate:*

$$\max_{x\in\mathbb{T}^n}V(x)\leq\bar{H}(P)\leq\frac{1}{2}|P|^2+\max_{x\in\mathbb{T}^n}V(x). \tag{2.5}$$

Proof. The upper bound is directly computed:

$$\begin{aligned}\bar{H}(P) &= \inf_v\sup_{x\in\mathbb{T}^n}\frac{1}{2}|P+\nabla v(x)|^2+V(x)\\ &\leq \sup_{x\in\mathbb{T}^n}\frac{1}{2}|P+\nabla v(x)|^2+V(x)\Big|_{v=0}\leq\frac{1}{2}|P|^2+\max_{x\in\mathbb{T}^n}V(x).\end{aligned}$$

As for the lower bound, pick $x = x_M$ to be point of global maximum of V. Then

$$\begin{aligned} \bar{H}(P) &\geq \inf_v \left(\frac{1}{2}|P + \nabla v(x_M)|^2 + V(x_M) \right) \\ &= \inf_v \left(\frac{1}{2}|P + \nabla v(x_M)|^2 \right) + V(x_M) \geq \max_{x \in \mathbb{T}^n} V(x). \end{aligned}$$

□

In the following we provide an equivalent formulation for the effective Hamiltonian, that will be useful in the subsequent connection with the quantum setting.

Proposition 2.2. *The effective Hamiltonian (2.2) can be also computed as*

$$\bar{H}(P) = \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(x, P + \nabla v(x))\varphi(x)^2\, dx \tag{2.6}$$

where $v, \varphi \in C^\infty(\mathbb{T}^n; \mathbb{R})$ *and satisfy*

$$\int_{\mathbb{T}^n} \varphi(x)^2 dx = 1, \quad \int_{\mathbb{T}^n} v(x) dx = 0. \tag{2.7}$$

Proof. Define

$$I[f] := \sup_\varphi \int_{\mathbb{T}^n} f(x)\varphi(x)^2\, dx, \quad f \in C^\infty(\mathbb{T}^n; \mathbb{R}).$$

We easily observe that $I[f] \leq \max_{x \in \mathbb{T}^n} f(x)$. Moreover, we take a maximum point x_M of f, namely $f(x_M) = \max_{x \in \mathbb{T}^n} f(x)$ and then we define the sequence $\phi_r(x)$ as the periodic representative in $\mathbb{R}^n$ given by the periodization $r^{-n/2} \sum_{k \in \mathbb{Z}^n} \phi\left(\frac{x - x_M - 2\pi k}{r}\right)$ where $\phi \in C_0^\infty(B_\delta(0))$, $\|\phi\|_{L^2} = 1$ and where δ is so small that $B_\delta(0)$ is contained in a periodicity domain. As a consequence

$$\lim_{r \to 0^+} \int_{\mathbb{T}^n} f(x)\phi_r(x)^2 dx = f(x_M),$$

which means that the functional is simply $I[f] = \max_{x \in \mathbb{T}^n} f(x)$. Applying thus to (2.6) we have

$$\inf_v \sup_\varphi \int_{\mathbb{T}^n} H(x, P + \nabla v(x))\varphi(x)^2 dx = \inf_v \max_{x \in \mathbb{T}^n} H(x, P + \nabla v(x)) = \bar{H}(P).$$

□

3. Basics of Aubry-Mather theory

Aubry-Mather theory proves the existence of invariant and action-minimizing measures as well as invariant and action-minimizing sets in the phase space. It has been mainly developed by Aubry [A-D], Mather ([M], [M1], [M2]) and Mañé ([Ma1],[Ma2]). Although this is a wide and deep variational theory, here we review only a few results that we are going to use in the subsequent sections. For a detailed treatment we refer to Fathi [F], Sorrentino [So] and references quoted therein.

A probability measure $d\mu$ defined on $\mathbb{T}^n \times \mathbb{R}^n$ is called invariant with respect to the Lagrangian flow $\phi^t : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ related to a Lagrangian $L(x, \xi)$ if

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f(\phi^t(x,\xi)) d\mu(x,\xi) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f(x,\xi) d\mu(x,\xi), \tag{3.1}$$

for all $t \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$. A probability measure $d\mu$ is said to be closed if $\forall g \in C^\infty(\mathbb{T}^n; \mathbb{R})$ it holds

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} \nabla_x g(x) \cdot \xi \; d\mu(x,\xi) = 0, \; \int_{\mathbb{T}^n \times \mathbb{R}^n} |\xi| d\mu(x,\xi) < +\infty. \tag{3.2}$$

We say that a probability measure $d\tilde{\mu}_P$ is a Mather measure if, for any given $P \in \mathbb{R}^n$, it minimizes the Action

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,\xi) - P \cdot \xi \; d\mu(x,\xi) \tag{3.3}$$

on the set of invariant probability measures. It has been also proved that the Mather measures of a Tonelli Lagrangian are those which minimize the action in the class of all (compactly supported) closed measures (see [B2] Theorem 7). Moreover, the miminizing value of the Action is related to the effective Hamiltonian function as

$$-\bar{H}(P) = \int_{\mathbb{T}^n \times \mathbb{R}^n} L(x,\xi) - P \cdot \xi \; d\tilde{\mu}_P(x,\xi), \tag{3.4}$$

see for example [F]. The Mather set $\mathcal{M}^P$ is the closure of the union of the supports of all Mather measures. We recall that Mather proved in [M1] that $\mathcal{M}^P$ is compact and that it is a Lipschitz graph above a compact part of $\mathbb{T}^n$. From the PDE point of view (see [F]), one can consider the so-called Aubry set

$$\mathcal{A}^P := \bigcap_v \{(x, P + \nabla_x v(P,x)) \mid v(P,x) \text{ is differentiable in } x\}, \tag{3.5}$$

where the intersection is taken over all $v(P,\cdot) \in C^{0,1}(\mathbb{T}^n; \mathbb{R})$ such that the function $u(P,x) = P \cdot x + v(P,x)$ are critical subsolutions of the stationary Hamilton-Jacobi equation. This set is also invariant under the Lagrangian dynamics, and it has been proved in [C] that one always has $\mathcal{M}_P \subseteq \mathcal{A}_P$. This fact has many important geometrical and dynamical implications. In particular, here we observe that every Mather measure $d\tilde{\mu}_P$ has the property that

$$\text{supp}(d\tilde{\mu}_P) \subseteq \{(x, P + \nabla_x v(P,x)) \mid v(P,x) \text{ is differentiable in } x\}$$

where $u(P,x) = P \cdot x + v(P,x)$ is an arbitrary weak KAM solution of the stationary H-J equation.

From now on, we will restrict ourselves to the mechanical case, in other words $H(x,p) = \frac{1}{2}|p|^2 + V(x)$.

3.1. A family of Mather measures

We now introduce a family of Mather measures, that realize the semiclassical limit of the Wigner transform of our class of Bloch wave functions, as we will see in the last section.

Proposition 3.1. *Take* $P \in \mathbb{R}^n$, $v(P,\cdot) \in C^{0,1}(\mathbb{T}^n;\mathbb{R})$ *and let* $d\sigma_P$ *Radon probability measures on* $\mathbb{T}^n$ *such that* $\mathrm{supp}(d\sigma_P) \subseteq \mathrm{supp}(\nabla_x v(P,\cdot))$. *Define the probability measures* $d\tilde{\mu}_P$ *on* $\mathbb{T}^n \times \mathbb{R}^n$ *by*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} f(x,\xi)d\tilde{\mu}_P(x,\xi) := \int_{\mathbb{T}^n} f(x, P+\nabla_x v(P,x))d\sigma_P(x), \tag{3.6}$$

for all $f \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n;\mathbb{R})$. *Let us suppose that the* $d\tilde{\mu}_P$ *are closed and satisfy the property*

$$\int_{\mathbb{T}^n\times\mathbb{R}^n} H(x,\xi)d\tilde{\mu}_P(x,\xi) = \bar{H}(P). \tag{3.7}$$

Then, the $d\tilde{\mu}_P$ *are Mather measures for each* P.

Proof. We prove that these measures minimize the Action. To see this, consider

$$\begin{aligned} I(P) &:= \int_{\mathbb{T}^n\times\mathbb{R}^n} L(x,\xi) - P\cdot\xi \, d\tilde{\mu}_P(x,\xi) \\ &= \int_{\mathbb{T}^n\times\mathbb{R}^n} -H(x,\xi) + |\xi|^2 - P\cdot\xi \, d\tilde{\mu}_P(x,\xi). \end{aligned}$$

Recalling (3.7) we have

$$\begin{aligned} I(P) &= -\bar{H}(P) + \int_{\mathbb{T}^n\times\mathbb{R}^n} |\xi|^2 - P\cdot\xi \, d\tilde{\mu}_P(x,\xi) \\ &= -\bar{H}(P) + \int_{\mathbb{T}^n\times\mathbb{R}^n} |P+\nabla_x v(P,x)|^2 - P\cdot(P+\nabla_x v(P,x))\, d\sigma_P(x) \\ &= -\bar{H}(P) + \int_{\mathbb{T}^n\times\mathbb{R}^n} \nabla_x v(P,x)\cdot(P+\nabla_x v(P,x))\, d\sigma_P(x) = -\bar{H}(P). \end{aligned}$$

For the last equality, we use the density of C^∞ into $C^{0,1}$ to have a sequence $\nabla_x g_k(P,\cdot) \to \nabla_x v(P,\cdot)$ as $k \to +\infty$ in the $C^{0,1}$-topology, and then apply the closure. We have thus proved that $d\tilde{\mu}_P$ is a closed probability measure minimizing the Action, and hence it is a Mather measure. □

4. A variational approach in the quantum state space

4.1. The semiclassical effective Hamiltonian

By the same variational approach giving the classical effective Hamiltonian (see Proposition 2.2), we now provide the semiclassical version of the effective Hamiltonian.

Definition 4.1. (Semiclassical effective Hamiltonian)

$$\begin{aligned}\bar{H}_\hbar(P) &:= \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(P+\nabla_x v(x),x)\varphi(x)^2 - \frac{\hbar^2}{2}|\nabla_x\varphi(x)|^2\ dx \\ &= \inf_v \sup_\varphi \bar{H}_\hbar[P,v,\varphi] \end{aligned} \tag{4.1}$$

with $P \in \mathbb{R}^n$ and $v,\varphi \in C^\infty(\mathbb{T}^n;\mathbb{R})$ such that

$$\int_{\mathbb{T}^n} \varphi(x)^2 dx = 1, \quad \int_{\mathbb{T}^n} v(x)dx = 0.$$

We now consider the stationary equation

$$\frac{D\bar{H}_\hbar}{D\varphi}[P,v,\varphi] = 0. \tag{4.2}$$

Following Evans ([E4], section 3), for any fixed $v(x)$ we consider the elliptic second order partial differential operator

$$\mathcal{L}(\hbar,P,v) := \frac{\hbar^2}{2}\Delta_x + H(P+\nabla_x v(x),x), \tag{4.3}$$

with domain $C^\infty(\mathbb{T}^n;\mathbb{R})$. It can be easily proved that the eigenvalue problem

$$\mathcal{L}(\hbar,P,v)\varphi(x) = \lambda\varphi(x) \tag{4.4}$$

is equivalent to (4.2). Moreover, since the related spectrum is discrete and bounded from above,

$$\mathrm{Spec}(\mathcal{L}(\hbar,P,v)) = \{-\infty < ... \le \lambda_{j+1} \le \lambda_j \le ... \le \lambda_0[\hbar,P,v]\},$$

we may consider the (principal eigenvalue) functional

$$\lambda_0[\hbar,P,v] := \sup_\varphi \langle \varphi; \mathcal{L}(\hbar,P,v)\varphi\rangle_{L^2} = \sup_\varphi \bar{H}_\hbar[P,v,\varphi]$$

where $\varphi \in C^\infty(\mathbb{T}^n;\mathbb{R})$ and $\|\varphi\|_{L^2} = 1$. We look at the related unique normalized positive principal smooth eigenfunction $\varphi_0(\hbar,P,v)(\cdot) \in C^\infty(\mathbb{T}^n;\mathbb{R})$ of problem (4.4) which hence satisfies

$$\begin{aligned}\lambda_0[\hbar,P,v] &= \langle \varphi_0; \mathcal{L}(\hbar,P,v)\varphi_0\rangle_{L^2} \\ &= \int_{\mathbb{T}^n} H(x,P+\nabla_x v(x))\varphi_0(x)^2 - \frac{1}{2}\hbar^2|\nabla_x\varphi_0(x)|^2 dx.\end{aligned} \tag{4.5}$$

Notice that, by definition,

$$\inf_v \lambda_0[\hbar,P,v] = \bar{H}_\hbar(P). \tag{4.6}$$

For our purposes, we need to extend the functional $\lambda_0[\hbar,P,\cdot]$ to functions $v \in C^k(\mathbb{T}^n;\mathbb{R})$ with $k \ge k_0(n)$, where the integer $k_0(n) \ge 2$ is fixed in such a way that the eigenfunctions of the above eigenvalue problem (4.4) are at least of class $C^2(\mathbb{T}^n;\mathbb{R})$, and this is possible thanks to elliptic theory (see Gilbarg-Trudinger [G-T], Section 8.12). The variational property (4.5) then holds for all $v \in C^k(\mathbb{T}^n;\mathbb{R})$ provided $k \ge k_0(n)$.
From now on we fix $k \ge k_0(n)$.

The following result shows the continuity of $\lambda_0[\hbar,P,\cdot]$.

Theorem 4.2. *The functional $\lambda_0[\hbar, P, \cdot]$ defined on $C^k(\mathbb{T}^n;\mathbb{R})$ is continuous with respect to the topology induced by the C^k-norm.*

Proof. Take $v, w \in C^k(\mathbb{T}^n;\mathbb{R})$ and compute:

$$\begin{aligned}
&\lambda_0[\hbar, P, v]\\
&= \sup_{\varphi} \int_{\mathbb{T}^n} -\frac{\hbar^2}{2}|\nabla_x\varphi|^2 + H(x, P+\nabla_x v)\varphi^2\, dx\\
&= \sup_{\varphi} \int_{\mathbb{T}^n} -\frac{\hbar^2}{2}|\nabla_x\varphi|^2 + \frac{1}{2}|P+\nabla_x v(x)|^2\varphi^2 + V\varphi^2\, dx\\
&= \sup_{\varphi} \int_{\mathbb{T}^n} \Big\{ -\frac{\hbar^2}{2}|\nabla_x\varphi|^2 + \frac{1}{2}|P+\nabla_x w - \nabla_x w + \nabla_x v|^2\varphi^2 + V\varphi^2\Big\} dx\\
&= \sup_{\varphi} \int_{\mathbb{T}^n} -\frac{\hbar^2}{2}|\nabla_x\varphi|^2 + \frac{1}{2}|P+\nabla_x w|^2\varphi^2 + V\varphi^2\, dx\\
&\qquad + \int_{\mathbb{T}^n} \Big\{\frac{1}{2}|\nabla_x v - \nabla_x w|^2 + (\nabla_x v - \nabla_x w)\cdot(P+\nabla_x w)\Big\}\varphi^2\, dx.
\end{aligned}$$

Now see that for

$$I(v, w, \phi) := \int_{\mathbb{T}^n} \Big\{\frac{1}{2}|\nabla_x v - \nabla_x w|^2 + (\nabla_x v - \nabla_x w)\cdot(P+\nabla_x w)\Big\}\varphi^2\, dx,$$

since

$$|I(v, w, \varphi)| \leq \frac{1}{2}\|v-w\|^2_{C^1} + (|P| + \|w\|_{C^1} + \|v\|_{C^1})\|v-w\|_{C^1} =: G(v, w).$$

we get

$$\lambda_0[\hbar, P, w] - G(v, w) \leq \lambda_0[\hbar, P, v] \leq \lambda_0[\hbar, P, w] + G(v, w),$$

which concludes the proof. $\square$

Now we prove that the functional $\lambda_0[\hbar, P, \cdot]$ is Gâteaux differentiable with continuous differential.

Lemma 4.3. *One has*

$$\frac{D\lambda_0}{Dv}[\hbar, P, v](\delta v) = -\int_{\mathbb{T}^n} \nabla_x \delta v(x)\cdot(P+\nabla_x v(x))\varphi_0(\hbar, P, v)(x)^2 dx \tag{4.7}$$

where the variations $\delta v \in C^k(\mathbb{T}^n;\mathbb{R})$ are taken such that $\int_{\mathbb{T}^n}\delta v(x)dx = 0$.

Proof. We follow Evans ([E4], Th 3.1). We consider the eigenvalue equation (4.4)

$$\frac{\hbar^2}{2}\Delta_x\varphi_0(\hbar, P, v) + H(P+\nabla v, x)\varphi_0(\hbar, P, v) = \lambda_0[\hbar, P, v]\varphi_0(\hbar, P, v),$$

and take a one-parameter smooth family of functions $\{v[\tau](x) \mid 0 \leq \tau \leq 1\} \subset C^k(\mathbb{T}^n;\mathbb{R})$ with zero mean value and such that $v[0](x) = v(x)$. Let $\lambda_0(\tau) := \lambda_0(\hbar, P, v[\tau])$ be the principal eigenvalue corresponing to the potential $H(P + v[\tau], x)$. The corresponding equation is

$$\frac{\hbar^2}{2}\Delta_x\varphi_0(\hbar, P, v[\tau]) + H(P+\nabla v[\tau], x)\varphi_0(\hbar, P, v[\tau]) = \lambda_0(\tau)\varphi_0(\hbar, P, v[\tau]).$$

Since the principal eigenvalue is simple, we may differentiate with respect to τ and get, putting $\tau = 0$ and after some computations involving L^2-normalization of $\varphi_0(\hbar, P, v)$,

$$\lambda_0'(0) = -\int_{\mathbb{T}^n} \nabla_x \delta v(x) \cdot (P + \nabla_x v(x))\varphi_0(\hbar, P, v)(x)^2 dx \tag{4.8}$$

where $\nabla_x \delta v(x) = \nabla_x v'[\tau](x)|_{\tau=0}$.
In order to prove the continuity of the differential, take a sequence of variations $\{\delta v_\nu\}_{\nu\in\mathbb{N}}$ which is convergent to δv in $C^k(\mathbb{T}^n;\mathbb{R})$. Then

$$\left|\frac{D\lambda_0}{Dv}[\hbar, P, v](\delta v_\nu - \delta v)\right| \leq (|P| + \|v\|_{C^1})\|\delta v_\nu - \delta v\|_{C^k} \longrightarrow 0 \quad \text{as } \nu \to +\infty.$$

□

Here we show some useful inequalities involving $\bar{H}_\hbar(P)$ and $\bar{H}(P)$.

Theorem 4.4. *For all $P \in \mathbb{R}^n$ we have the inequalities*

$$\frac{1}{2}|P|^2 + \min_{x\in\mathbb{T}^n} V(x) \leq \bar{G}(P) \leq \bar{H}_\hbar(P) \leq \bar{H}(P) \leq \frac{1}{2}|P|^2 + \max_{x\in\mathbb{T}^n} V(x) \tag{4.9}$$

Here $\bar{G}(P)$ is the so-called $\mathbb{H}$-harmonic (see [P]*) value of the translated Hamiltonian $H(P+p,x)$ defined on the flat torus $\mathbb{T}^n$*

$$\bar{G}(P) := \inf_v \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x))c_0 dx = \frac{1}{2}|P|^2 + \int_{\mathbb{T}^n} V(x)c_0 dx,$$

where $c_0 := \mathrm{vol}(\mathbb{T}^n)^{-1}$. Moreover,

$$\min_{x\in\mathbb{T}^n} V(x) \leq \int_{\mathbb{T}^n} V(x)c_0 dx \leq \inf_{P\in\mathbb{Z}^n} \bar{H}_\hbar(P) \leq \max_{x\in\mathbb{T}^n} V(x). \tag{4.10}$$

Proof. We begin with

$$\begin{aligned} \bar{H}_\hbar(P) \quad &:= \quad \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(P + \nabla_x v, x)\varphi^2 - \frac{\hbar^2}{2}|\nabla_x \varphi|^2 \; dx \\ &\leq \quad \inf_v \sup_\varphi \int_{\mathbb{T}^n} H(P + \nabla_x v, x)\varphi^2 \; dx = \bar{H}(P) \end{aligned}$$

by Proposition 2.2. The upper bound for $\bar{H}(P)$ was shown in Lemma 2.1. The lower bound for $\bar{G}(P)$ is trivial, whereas to get its upper bound we compute

$$\bar{H}_\hbar(P) = \inf_v \lambda_0[\hbar, P, v] = \inf_v \sup_\varphi \langle \varphi; \mathcal{L}(\hbar, P, v)\varphi\rangle \geq \inf_v \; \langle c_0; \mathcal{L}(\hbar, P, v)c_0\rangle.$$

However $\Delta_x c_0 = 0$, so

$$\begin{aligned} \bar{H}_\hbar(P) \quad &\geq \quad \inf_v \; \langle c_0; \mathcal{L}(\hbar, P, v)c_0\rangle \\ &= \quad \inf_v \int_{\mathbb{T}^n} H(x, P + \nabla_x v(x))c_0 dx =: \bar{G}(P). \end{aligned} \tag{4.11}$$

Following Paternian ([P]), the minimizer v_0 of (4.11) exists and is a strong solution of

$$\mathrm{div}_x(\nabla_p H(P + \nabla_x v_0(x), x)) = 0,$$

that is, in this case, the equation for the harmonic functions

$$\Delta_x v_0(x) = 0, \quad x \in \mathbb{T}^n.$$

Since the only harmonic functions on the torus are the constants, and v_0 must have zero average, we conclude

$$\begin{aligned}\bar{G}(P) &= \int_{\mathbb{T}^n} H(P + \nabla_x v_0(x), x) c_0 dx = \int_{\mathbb{T}^n} H(P, x) c_0 dx \\ &= \frac{1}{2}|P|^2 + \int_{\mathbb{T}^n} V(x) c_0 dx \geq \frac{1}{2}|P|^2 + \min_{x \in \mathbb{T}^n} V(x).\end{aligned}$$

□

We now improve (4.9) as follows

Theorem 4.5. *For all $P \in \mathbb{R}^n$ we have the inequalities*

$$\bar{H}(P) - 2\hbar^\alpha \leq \bar{H}_\hbar(P) \leq \bar{H}(P) \tag{4.12}$$

for all $0 < \alpha < 1$ and $0 \leq \hbar \leq 1$.

Proof. The upper bound for $\bar{H}_\hbar(P)$ has been already proved in the previous theorem. Let us give a lower bound for the functional

$$\lambda_0[\hbar, P, v] = \sup_\varphi \int_{\mathbb{T}^n} -\frac{1}{2}\hbar^2 |\nabla_x \varphi|^2 + H(x, P + \nabla_x v)\varphi^2 dx.$$

We recall that the supremum is taken over all $\varphi \in C^\infty(\mathbb{T}^n; \mathbb{R})$ such that $\|\varphi\|_{L^2} = 1$. Now fix a phase function $v \in C^\infty(\mathbb{T}^n; \mathbb{R})$ and a point of global maximum $x_v \in \mathbb{T}^n$ of $H(P + \nabla_x v(x), x)$. Set

$$K = K(P, v) := \Big(1 + \max_{x \in \mathbb{T}^n} |\nabla_x^2 (H(x, P + \nabla_x v(x)))|\Big)^{-1/2}, \quad \varepsilon := K \hbar^{\alpha/2}. \tag{4.13}$$

Take $\gamma \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ with zero average and support contained in $B_{1/2}(0)$, $t > 0$ and define the smooth function on the torus

$$\tilde{\varphi}_{\hbar,\alpha}(x) := 1 + t\varepsilon^{-n/2} \sum_{k \in \mathbb{Z}^n} \gamma\Big(\frac{x - x_v - 2\pi k}{\varepsilon}\Big) \tag{4.14}$$

Notice that the L^2-norm on the torus of $\tilde{\varphi}_{\hbar,\alpha}$ is uniformly bounded from below by the volume of the torus. This allows us to define

$$\varphi_{\hbar,\alpha}(x) := \tilde{\varphi}_{\hbar,\alpha}(x) / \|\tilde{\varphi}_{\hbar,\alpha}\|_{L^2(\mathbb{T}^n)}$$

which therefore satisfies

$$\begin{aligned}&\frac{\hbar^2}{2} \int_{\mathbb{T}^n} |\nabla_x \varphi_{\hbar,\alpha}(x)|^2 dx \\ &= \frac{1}{\|\tilde{\varphi}_{\hbar,\alpha}\|^2_{L^2(\mathbb{T}^n)}} \frac{t^2 \hbar^2}{2\varepsilon^2} \varepsilon^{-n} \int_{[-\pi,+\pi]^n + x_v} \Big| (\nabla_x \gamma) \Big(\frac{x - x_v}{\varepsilon}\Big) \Big|^2 dx \\ &\leq \frac{1}{\mathrm{vol}(\mathbb{T}^n)^2} t^2 \hbar^{2-\alpha} K^{-2} \frac{1}{2} \|\nabla \gamma\|^2_{L^2} \leq \hbar^{2-\alpha}\end{aligned} \tag{4.15}$$

provided t is sufficiently small. So we have the lower bound

$$\begin{aligned}\lambda_0[\hbar, P, v] &\geq \int_{\mathbb{T}^n} -\frac{1}{2}\hbar^2|\nabla_x\varphi_{\hbar,\gamma}|^2 + H(x, P+\nabla_x v(x))\varphi_{\hbar,\gamma}^2 dx \\ &\geq \int_{\mathbb{T}^n} H(x, P+\nabla_x v(x))\varphi_{\hbar,\gamma}^2 dx - \hbar^{2-\alpha}. \qquad (4.16)\end{aligned}$$

We have picked $x_v \in \mathbb{T}^n$ to be a global maximum of $H(P+\nabla_x v(x), x)$. We may suppose that (a representative of) x_v belongs to the interior of the translate of a periodicity domain Q. We may hence write, for $x \in Q$,

$$H(x, P+\nabla_x v(x)) = H(x_v, P+\nabla_x v(x_v)) + \varepsilon^2 R(\varepsilon, P, x_v, x),$$

where $R \leq 0$ and $\|R(\varepsilon, P, x_v, \cdot)\|_\infty \leq \max_{x\in\mathbb{T}^n} |\nabla_x^2(H(x, P+\nabla_x v(x)))|$. We therefore have

$$\begin{aligned}&\int_{\mathbb{T}^n} H(x, P+\nabla_x v(x))\varphi_{\hbar,\gamma}(x)^2 dx \\ &= H(P+\nabla_x v(x_v), x_v) + \varepsilon^2 \int_Q R(\varepsilon, P, x_v, x)\varphi_{\hbar,\gamma}(x)^2 dx \\ &= \max_{x\in\mathbb{T}^n} H(x, P+\nabla_x v(x)) + \varepsilon^2 \int_Q R(\varepsilon, P, x_v, x)\varphi_{\hbar,\gamma}(x)^2 dx \\ &\geq \max_{x\in\mathbb{T}^n} H(x, P+\nabla_x v(x)) - \varepsilon^2\|R(\varepsilon, P, x_v, \cdot)\|_\infty \\ &\geq \Big(\inf_v \max_{x\in\mathbb{T}^n} H(x, P+\nabla_x v(x))\Big) - \varepsilon^2\|R(\varepsilon, P, x_v, \cdot)\|_\infty. \qquad (4.17)\end{aligned}$$

Because of (4.13) we have $\varepsilon^2\|R\|_\infty \leq \hbar^\alpha$, so that

$$\lambda_0[\hbar, P, v] \geq \bar{H}(P) - \hbar^\alpha - \hbar^{2-\alpha}. \qquad (4.18)$$

Observe that $0 < \alpha < 2-\alpha$ for all $0 < \alpha < 1$ and $\hbar^\alpha + \hbar^{2-\alpha} \leq 2\hbar^\alpha$ for all $\hbar \in [0,1]$. Therefore

$$\lambda_0[\hbar, P, v] \geq \bar{H}(P) - 2\hbar^\alpha, \quad \forall\, 0 < \alpha < 1.$$

Taking the infimum of $\lambda_0[\hbar, P, \cdot]$ on C^∞ gives (4.12). □

4.2. Minimizing phase functions

In this section we prove the existence of global minimizers of the functional $\lambda_0[\hbar, P, \cdot]$ as in (4.5) by using its weakly lower semicontinuity in the Sobolev space $W^{1,2}(\mathbb{T}^n;\mathbb{R})$. Afterwards, we prove for all $0 < \alpha < 1$ the existence of $o(\hbar^\alpha)$-approximated global minimizers in $C^k(\mathbb{T}^n;\mathbb{R})$ through a result based on the so-called Ekeland's variational principle.

To begin with, fix $P \in \mathbb{R}^n$, and for $\varphi \in C^\infty(\mathbb{T}^n;\mathbb{R})$, $\|\varphi\|_{L^2} = 1$, let

$$f_{\hbar,\varphi}\colon\ \mathbb{T}^n \times \mathbb{R}^n \ni (x,\xi) \longmapsto \Big(\frac{1}{2}|P+\xi|^2 + V(x)\Big)\varphi(x)^2 - \frac{\hbar^2}{2}|\nabla\varphi(x)|^2 \in \mathbb{R}.$$

If we recall Definition 4.1, it is easy to see that

$$\bar{H}[P, v, \varphi] = \int_{\mathbb{T}^n} f_{\hbar,\varphi}(x, \nabla v(x))dx, \tag{4.19}$$

$$\lambda_0[\hbar, P, v] = \sup_{\varphi} \bar{H}[P, v, \varphi], \tag{4.20}$$

where $v \in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n) := \Big\{ v \in W^{1,2}(\mathbb{T}^n; \mathbb{R}) \mid \int_{\mathbb{T}^n} v(x)dx = 0 \Big\}$.

Remark 4.6. It is well known that the space $W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ is a separable Hilbert space endowed with the norm induced by the inner product

$$\langle u, v \rangle_1 = \langle u, v \rangle_{L^2(\mathbb{T}^n)} + \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{T}^n)}.$$

By the Poincaré inequality, one furthermore has that an equivalent norm is given by $v \mapsto \|\nabla v\|_{L^2(\mathbb{T}^n)}$. Since $W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ is closed in the strong topology of $W^{1,2}(\mathbb{T}^n)$, it is also closed for the weak topology of $W^{1,2}(\mathbb{T}^n)$. Hence, weak limits of elements of $W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ belong to $W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$. Recall, furthermore, that a sequence $\{v_k\}_k \subset W^{1,2}(\mathbb{T}^n)$ converges weakly to v as $k \to \infty$ iff $v_k \overset{w-L^2(\mathbb{T}^n)}{\longrightarrow} v$ and $\nabla v_k \overset{w-L^2(\mathbb{T}^n)}{\longrightarrow} \nabla v$ as $k \to \infty$.

A straightforward computation shows the following lemma.

Lemma 4.7. *For any given $v \in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ we have*

$$\int_{\mathbb{T}^n} |\nabla v(x)|^2 dx \leq C_1 \lambda_0[\hbar, P, v] + C_2,$$

for positive constants C_1, C_2 depending only on $|P|$, $\mathrm{vol}(\mathbb{T}^n)$ and V.

Next, to apply classical results from the calculus of variations, we have to check that the functions $f_{\hbar,\varphi}$ are convex with respect to ξ uniformly in $x \in \mathbb{T}^n$, that is that, whatever $x \in \mathbb{T}^n$, whatever $t \in [0,1]$ and $\xi_1, \xi_2 \in \mathbb{R}^n$, we must have

$$f_{\hbar,\varphi}(x, t\xi_1 + (1-t)\xi_2) \leq t f_{\hbar,\varphi}(x, \xi_1) + (1-t) f_{\hbar,\varphi}(x, \xi_2).$$

But this amounts to the convexity of $|P + \xi|^2$, and we are done. Moreover, since

$$x \mapsto \frac{\partial f_{\hbar,\varphi}}{\partial \xi_j}(x, \nabla v(x)) \in L^2(\mathbb{T}^n), \quad \forall v \in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n),$$

we have the following lemma.

Lemma 4.8. *For every $\hbar \in [0,1]$ and $\varphi \in C^\infty(\mathbb{T}^n; \mathbb{R})$, $\|\varphi\|_{L^2} = 1$, the functional $\bar{H}[P, \cdot, \varphi] \colon W^{1,2}_{\mathrm{av}}(\mathbb{T}^n) \longrightarrow \mathbb{R}$ is (sequentially) weakly lower-semicontinuous.*

Proof. Suppose that $v_k \longrightarrow v$ weakly in $W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ as $k \to \infty$. Then, since f_φ is ξ-convex, that is

$$f_{\hbar,\varphi}(x, \xi) \geq f_{\hbar,\varphi}(x, \xi_0) + \langle \nabla_\xi f_{\hbar,\varphi}(x, \xi_0), \xi - \xi_0 \rangle,$$

we get

$$\int_{\mathbb{T}^n} f_{\hbar,\varphi}(x,\nabla v_k(x))dx$$
$$\geq \int_{\mathbb{T}^n} f_{\hbar,\varphi}(x,\nabla v(x))dx + \int_{\mathbb{T}^n} \langle \underbrace{\nabla_\xi f_{\hbar,\varphi}(x,\nabla v(x))}_{\in L^2(\mathbb{T}^n)}, \underbrace{\nabla v_k(x) - \nabla v(x)}_{\overset{w}{\to} 0 \text{ in } L^2(\mathbb{T}^n) \text{ as } k\to\infty} \rangle dx,$$

whence

$$\liminf_{k\to\infty} \bar{H}[P, v_k, \varphi] \geq \bar{H}[P, v, \varphi].$$

□

Corollary 4.9. *Since the supremum of lower semicontinuous functions is lower semicontinuous, we have that the functional* $\lambda_0[\hbar, P, \cdot]\colon W^{1,2}_{\mathrm{av}}(\mathbb{T}^n) \to \mathbb{R}$ *is also (sequentially) weakly lower-semicontinuous.*

Finally, we are in a position to prove the following theorem.

Theorem 4.10. *Let* $\bar{H}_\hbar(P) := \inf_{v\in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)} \lambda_0[\hbar, P, v]$. *Then there exists* $\hat{w} \in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ *such that* $\bar{H}_\hbar(P) = \lambda_0[\hbar, P, \hat{w}]$.

Proof. Let $\{v_k\}_k \subset W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ be a minimizing sequence. Since $\lambda_0(\hbar, P, v_k)$ is bounded, so is $\|\nabla v_k\|_{L^2(\mathbb{T}^n)}$. Hence $\{v_k\}_k \subset W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$ is bounded, and hence there exists a subsequence (that we keep denoting by $\{v_k\}_k$) which is weakly convergent to some $\hat{w} \in W^{1,2}_{\mathrm{av}}(\mathbb{T}^n)$. By the previous results,

$$\bar{H}_\hbar(P) = \liminf_{k\to\infty} \lambda_0[\hbar, P, v_k] \geq \lambda_0[\hbar, P, \hat{w}] \geq \bar{H}_\hbar(P),$$

which proves the theorem. □

Next, in order to provide a notion for *minimizing phase functions* approximate, we need first a result based on Ekeland's variational principle (see [A-E]).

Theorem 4.11. *Let* X *be a Banach space and* $F : \mathsf{X} \to \mathbb{R}$ *a lower semicontinuous functional, bounded from below and Gâteaux differentiable with continuous differential. Then, for all* $\varepsilon > 0$ *and any given* $z \in \mathsf{X}$ *such that*

$$\inf_{v\in\mathsf{X}} F(v) \leq F(z) \leq \inf_{v\in\mathsf{X}} F(v) + \varepsilon, \tag{4.21}$$

there exists $\hat{v} \in \mathsf{X}$ *satisfying:*

(a) $F(\hat{v}) \leq F(z)$,

(b) $\|\hat{v} - z\|_{\mathsf{X}} \leq \sqrt{\varepsilon}$,

(c) $\|DF(\hat{v})/Dv\|_{\mathsf{X}^*} \leq \sqrt{\varepsilon}$.

Before applying this result, we have to provide the suitable setting. Consider the Banach space $\mathsf{X} = C^k(\mathbb{T}^n;\mathbb{R})$ with $k \geq k_0(n)$ fixed as in Subsection 4.1, endowed with the usual C^k-norm.
The functional $\lambda_0[\hbar, P, \cdot] : C^k(\mathbb{T}^n;\mathbb{R}) \to \mathbb{R}$ is continuous and Gâteaux differentiable, with continuous differential, thanks to Theorem 4.2 and Lemma

4.3. Now, for any fixed $C^{0,1}$-critical subsolution $\widetilde{v}(P,\cdot)$ (see (2.4)), select $z \in C^k(\mathbb{T}^n;\mathbb{R})$ such that for all $0<\alpha<1$

$$\|z-\widetilde{v}\|_{C^{0,1}} \leq \hbar^{\alpha}, \quad \|z\|_{C^2} \leq d(P)\hbar^{-\alpha}, \tag{4.22}$$

for some $d(P)>0$. This fact is always possible thanks to the density of $C^k(\mathbb{T}^n;\mathbb{R})$ in $C^{0,1}(\mathbb{T}^n;\mathbb{R})$. All such functions $z=z(\hbar,P,x)$ fulfill condition (4.21) with a suitable ε depending on $\alpha,\hbar,P$. In fact,

$$\lambda_0(\hbar,P,z) \leq \lambda_0(0,P,z) = \max_{x\in\mathbb{T}^n} H(P+\nabla_x z(\hbar,P,x),x). \tag{4.23}$$

However,

$$|P+\nabla_x z|^2 \leq |P+\nabla_x\widetilde{v}|^2 + |\nabla_x z-\nabla_x\widetilde{v}|^2 + 2|\nabla_x\widetilde{v}|\,|\nabla_x z-\nabla_x\widetilde{v}|. \tag{4.24}$$

By using (4.23), (4.24) and the first upper bound in (4.22), we get

$$\begin{aligned}\lambda_0(\hbar,P,z) &\leq \max_{x\in\mathbb{T}^n}\Big[H(P+\nabla_x\widetilde{v}(P,x),x)\Big] + \frac{1}{2}\hbar^{2\alpha} + \|\widetilde{v}(P,\cdot)\|_{C^{0,1}}\hbar^{\alpha}\\ &\leq \bar{H}(P) + \frac{1}{2}\hbar^{2\alpha} + \|\widetilde{v}(P,\cdot)\|_{C^{0,1}}\hbar^{\alpha}.\end{aligned} \tag{4.25}$$

Now, by Theorem 4.5,

$$\lambda_0(\hbar,P,z) \leq \bar{H}_{\hbar}(P) + 2\hbar^{\alpha} + \frac{1}{2}\hbar^{2\alpha} + \|\widetilde{v}(P,\cdot)\|_{C^{0,1}}\hbar^{\alpha},$$

where $0<\alpha<1$ and $0\leq\hbar\leq 1$. We finally observe that $\bar{H}_{\hbar}(P)$ is the global infimum value of $\lambda_0(\hbar,P,\cdot)$ on C^k and that we can choose the parameters

$$c(P) := 3\max\{2, \|\widetilde{v}(P,\cdot)\|_{C^{0,1}}\}, \quad \varepsilon := c(P)\hbar^{\alpha}.$$

In view of the above considerations, we can now apply Theorem 4.11 and get the existence of some $\hat{v}(\hbar,P,\cdot)\in C^k(\mathbb{T}^n;\mathbb{R})$ satisfying the following properties

(a) $\bar{H}_{\hbar}(P) \leq \lambda_0(\hbar,P,\hat{v}) \leq \bar{H}_{\hbar}(P) + c(P)\hbar^{\alpha}$,
(b) $\|\hat{v}(\hbar,P,\cdot)-z(\hbar,P,\cdot)\|_{C^k} \leq c(P)^{1/2}\hbar^{\alpha/2}$,
(c) $\|D\lambda_0(\hbar,P,\hat{v})/Dv\|_* \leq c(P)^{1/2}\hbar^{\alpha/2}$ (where $\|\cdot\|_*$ denotes the norm in the dual space).

We remark that such a function fulfills also

$$\|\hat{v}(\hbar,P,\cdot)-\tilde{v}(P,\cdot)\|_{C^{0,1}} \leq c(P)^{1/2}\hbar^{\alpha/2} + \hbar^{\alpha}, \tag{4.26}$$

and this suggests the definition of a set of smooth approximate minimizing phase functions in connection with each selected critical subsolution of the stationary H-J equation.

Definition 4.12. We define $\mathbf{\Gamma}_{\hbar,P}$ as the set of functions $v:(0,1]\times\mathbb{Z}^n\times\mathbb{T}^n\to\mathbb{R}$, $(\hbar,P,x)\mapsto v(\hbar,P,x)$ with $\hat{v}(\hbar,P,\cdot)\in C^k(\mathbb{T}^n;\mathbb{R})$ such that for some $\tilde{v}(P,\cdot)\in C^{0,1}(\mathbb{T}^n;\mathbb{R})$, critical subsolution of the stationary H-J equation, and constants $0<c_i(P)<+\infty$, $i=1,2,3$, $\forall\ 0<\alpha<1$, one has

(i) $\bar{H}_{\hbar}(P) \leq \lambda_0(\hbar,P,\hat{v}) \leq \bar{H}_{\hbar}(P) + c_1(P)\hbar^{\alpha}$,
(ii) $\|\hat{v}(\hbar,P,\cdot)-\tilde{v}(P,\cdot)\|_{C^{0,1}} \leq c_2(P)\hbar^{\alpha/2}$,
(iii) $\|D\lambda_0(\hbar,P,\hat{v})/Dv\|_* \leq c_3(P)\hbar^{\alpha/2}$.

Remark 4.13. Note that, as a consequence of property (ii), all the phase functions in $\mathbf{\Gamma}_{\hbar,P}$ are C^k subsolutions of the $\hbar$-perturbed stationary Hamilton-Jacobi equation

$$H(x, P + \nabla_x \hat{v}(\hbar, P, x)) \leq \bar{H}(P) + e(P)\hbar^{\alpha/2}, \tag{4.27}$$

for $e(P) := \frac{1}{2}c_2(P)^2 + (|P| + \|\tilde{v}(P,\cdot)\|_{C^{0,1}})c_2(P)$. Moreover, by the second bound in (4.22) and the above property (b) involving the functions z and $\hat{v}$, it follows that $\|\hat{v}(\hbar, P, \cdot)\|_{C^2} \leq (d(P) + c(P)^{1/2})\hbar^{-\alpha}$.

4.3. A class of Bloch wave functions

In view of the previous definition of approximate minimizing phase functions $\mathbf{\Gamma}_{\hbar,P}$, we introduce the following class of quantum states *for* $P \in \mathbb{Z}^n$ *and* $\hbar = 1/N$ *with* $N \in \mathbb{N}$.

Definition 4.14.

$$\mathbf{\Psi}_{\hbar,P} := \left\{ \psi_{\hbar,P}(x) = \varphi_0(\hbar, P, \hat{v})(x) e^{\frac{i}{\hbar}(P\cdot x + \hat{v}(\hbar,P,x))} \quad | \quad \hat{v} \in \mathbf{\Gamma}_{\hbar,P} \right\}$$

where $\varphi_0(\hbar, P, \hat{v})(x)$ is the eigenfunction introduced in Section 4.1.

As a conseguence of the above construction, we have the next proposition.

Proposition 4.15. *For all* $\hat{v} \in \mathbf{\Gamma}_{\hbar,P}$ *and all* $0 < \alpha < 1$,

$$\begin{aligned} \bar{H}(P) - 2\hbar^\alpha \quad &\leq \quad \int_{\mathbb{T}^n} H(P + \nabla_x \hat{v}, x)\varphi_0^2 dx \leq \bar{H}(P) + e(P)\hbar^{\alpha/2}, && (4.28) \\ 0 \quad &\leq \quad \int_{\mathbb{T}^n} \frac{\hbar^2}{2}|\nabla_x \varphi_0|^2 dx \leq e(P)\hbar^{\alpha/2} + 2\hbar^\alpha. && (4.29) \end{aligned}$$

Proof. The first inequality (i) in Definition 4.12 reads

$$\bar{H}_\hbar(P) \leq \int_{\mathbb{T}^n} H(P + \nabla_x \hat{v}, x)\varphi_0^2 - \frac{\hbar^2}{2}|\nabla_x \varphi_0|^2 dx. \tag{4.30}$$

By recalling Theorem 4.5, the lower bound in (4.28) directly follows. The upper bound in (4.28) is easily proved by the L^2-normalization of φ_0 and (4.27). Finally, (4.30) and (4.28) immediately give (4.29).

Remark 4.16. The mean value of the energy operator $\hat{H} := -\frac{\hbar^2}{2}\Delta + V(x)$ on the states $\psi_{\hbar,P}$ is given by

$$\langle \psi_{\hbar,P}, \hat{H}\psi_{\hbar,P}\rangle_{L^2} = \int_{\mathbb{T}^n} H(P + \nabla_x \hat{v}, x)\varphi_0^2 + \frac{\hbar^2}{2}|\nabla_x \varphi_0|^2 dx.$$

In view of (4.28) and (4.29), one easily gets an upper and a lower bound for the energy, and this shows that the mean value is an $\hbar^{\alpha/2}$-perturbation around the value of the classical effective Hamiltonian $\bar{H}(P)$:

Corollary 4.17. *For the mean value* $\langle \psi_{\hbar,P}, \hat{H}\psi_{\hbar,P}\rangle_{L^2}$, *we have*

$$\bar{H}(P) - 2\hbar^\alpha \leq \langle \psi_{\hbar,P}, \hat{H}\psi_{\hbar,P}\rangle_{L^2} \leq \bar{H}(P) + 2e(P)\hbar^{\alpha/2} + 2\hbar^\alpha. \tag{4.31}$$

5. Approximations and asymptotics of Wigner measures

5.1. The Weyl quantization on the torus

We begin by recalling that for the so-called *phase-space symmetry* operator

$$T_{y,\eta}\psi(x) := e^{2i\langle x-y,\eta\rangle}\psi(2y-x)$$

in $\mathbb{R}^n$ one has that $T_{y,\eta} = (\pi^n\delta_{y,\eta})^{\mathrm{w}}(x,D)$. It is well known that this formula characterizes the Weyl quantization in $\mathbb{R}^n$, in that

$$a(x,\xi) = \int_{\mathbb{R}^{2n}} a(y,\eta)\delta_{y,\eta}dyd\eta,\ a^{\mathrm{w}}(x,D)\psi(x) := \frac{1}{\pi^n}\int_{\mathbb{R}^{2n}} a(y,\eta)T_{y,\eta}\psi(x)dyd\eta.$$

On the other hand, a weak formulation of the Weyl quantization involves the so-called Wigner trasform $W\psi$ of the state ψ, namely

$$\langle\psi, a^{\mathrm{w}}(x,D)\psi\rangle = \int_{\mathbb{R}^{2n}} a(y,\eta)W\psi(y,\eta)dyd\eta.$$

Starting from this characterization, we next introduce the Weyl quantization on the torus $\mathbb{T}^n$. We shall consider here the *symbol class* $S^m(\mathbb{T}^n\times\mathbb{R}^n)$, $m\in\mathbb{R}$, consisting of those functions $b(x,\xi)$ which are smooth in the x,ξ-variables, 2π-periodic in x, and satisfying: For all $\alpha,\beta\in\mathbb{Z}^n_+$ there is $C_{\alpha\beta}$ such that

$$|\partial_\xi^\alpha\partial_x^\beta b(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}. \tag{5.1}$$

We define the Weyl quantization $\mathrm{Op}^w(b) : C^\infty(\mathbb{T}^n) \to \mathcal{D}'(\mathbb{T}^n)$ of a symbol $b\in S^m(\mathbb{T}^n\times\mathbb{R}^n)$ by

$$\begin{aligned} \mathrm{Op}^w(b)\psi(x) &:= (2\pi)^{-n}\sum_{\xi\in\frac{1}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(y,\xi)T_{y,\xi}\psi(x)dy \qquad (5.2)\\ &= (2\pi)^{-n}\sum_{\xi\in\frac{1}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(y,\xi)e^{2i\langle x-y,\xi\rangle}\psi(2y-x)dy. \end{aligned}$$

Definition 5.1. The *semiclassical Weyl quantization* is given by

$$\mathrm{Op}^w_\hbar(b)\psi(x) := (2\pi)^{-n}\sum_{\xi\in\frac{1}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(y,\hbar\xi)T_{y,\xi}\psi(x)dy.$$

The *(semiclassical) Wigner transform* of $\psi\in C^\infty(\mathbb{T}^n;\mathbb{C})$ is consequently defined as

$$\mathbb{T}^n\times\frac{\hbar}{2}\mathbb{Z}^n \ni (x,\xi)\mapsto W_\hbar\psi(x,\xi) := (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2i\hbar^{-1}\langle z,\xi\rangle}\psi(x-z)\overline{\psi}(x+z)dz.$$

Remark 5.2. The reason why the summation on half-integers is present in the above definitions is that for states $\psi(x) = \sum_{\alpha\in\mathbb{Z}^n}\psi_\alpha e^{i\langle x,\alpha\rangle}\in C^\infty(\mathbb{T}^n;\mathbb{C})$, the Wigner transform has to satisfy the properties

- $\displaystyle\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n} W_\hbar\psi(x,\xi) = |\psi(x)|^2, \quad \forall x\in\mathbb{T}^n,$

- $$(2\pi)^{-n}\int_{\mathbb{T}^n} W_\hbar\psi(x,\xi)dx = \begin{cases} |\psi_\alpha|^2 & \text{when } \xi=\hbar\alpha, \quad \text{with } \alpha\in\mathbb{Z}^n, \\ 0 & \text{otherwise.}\end{cases}$$

Notice also that the integration on the torus is well-defined, for the function $z \longmapsto e^{2i\hbar^{-1}\langle z,\xi\rangle}\psi(x-z)\overline{\psi}(x+z)$ is 2π-periodic in each variable, since $\xi \in \frac{\hbar}{2}\mathbb{Z}^n$. Furthermore, we may also write, by using the change of variables $z \mapsto z-x$ (that preserves 2π-periodicity)

$$W_\hbar\psi(x,\xi) = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2i\hbar^{-1}\langle z-x,\xi\rangle}\psi(2x-z)\overline{\psi}(z)dz.$$

Using the definition of Wigner transform and the above periodicity property, one immediately has

$$\begin{aligned}
&\langle\psi, \mathrm{Op}_\hbar^w(b)\psi\rangle_{L^2(\mathbb{T}^n)} \\
&= \int_{\mathbb{T}^n}\Big[(2\pi)^{-n}\sum_{\xi\in\frac{1}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} e^{2i\langle x-y,\xi\rangle}b(y,\hbar\xi)\psi(2y-x)dy\Big]\overline{\psi}(x)dx \\
&= (2\pi)^{-n}\sum_{\xi\in\frac{1}{2}\mathbb{Z}^n}\iint_{\mathbb{T}^n\times\mathbb{T}^n} e^{2i\langle x-y,\xi\rangle}b(y,\hbar\xi)\psi(2y-x)\overline{\psi}(x)dy\,dx \\
&= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(y,\xi)\Big[(2\pi)^{-n}\int_{\mathbb{T}^n} e^{2i\hbar^{-1}\langle x-y,\xi\rangle}\psi(2y-x)\overline{\psi}(x)dx\Big]dy \\
&= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{\mathbb{T}^n} b(y,\xi)W_\hbar\psi(y,\xi)dy. \qquad (5.3)
\end{aligned}$$

Remark 5.3. We address the reader to the paper by Ruzhansky and Turunen [R-T] for the standard quantization on the torus globally seen as a Lie group (that is, without passing through local coordinates as in the standard approach). Moreover, let us observe that the above setting for the Wigner transform on the phase space $\mathbb{T}^n_x \times \mathbb{R}^n_\xi$ is equivalent to the Wigner transform introduced in the paper of Graffi and Paul [G-P], where the setting is the dual phase space $\mathbb{Z}^n_q \times \mathbb{R}^n_p$. Indeed, the link can be made through the Fourier transform.

Now we show some simple properties that will be useful in what follows.

Lemma 5.4. *Take* $f \in \mathcal{S}(\mathbb{R}^n;\mathbb{R})$ *such that* $\mathrm{supp}(\mathcal{F}^{-1}f) \subset \{|y| < R\}$, *where* $\mathcal{F}^{-1}$ *is the inverse Fourier transform. Let* $Q_n := [-\pi,\pi]^n$ *and* $w \in \mathbb{R}^n$. *The* *following properties hold:*

$$(a)\qquad \varepsilon^n \sum_{\alpha\in\varepsilon\mathbb{Z}^n} f(\alpha)e^{i\alpha\cdot y} = (2\pi)^n(\mathcal{F}^{-1}f)(y) + F(\varepsilon,y);$$
$$\forall y\in\varepsilon^{-1}Q_n;\ \forall N\in\mathbb{Z}_+\ \exists C_{f,N} \text{ s.t. } \|F(\varepsilon,\cdot)\|_\infty \le C_{f,N}\varepsilon^N.$$

$$(b)\qquad (2\pi)^{-n}\int_{\varepsilon^{-1}Q_n}\Big[\varepsilon^n\sum_{\alpha\in\varepsilon\mathbb{Z}^n} f(\alpha)e^{i\alpha\cdot y}\Big]e^{-iw\cdot y}dy = f(w);$$
$$\forall\varepsilon \text{ such that } \varepsilon^{-1}Q_n\supset \operatorname{supp}(\mathcal{F}^{-1}f)\ .$$

Proof. We use a result showed by Duistermaat-Kolk based on Poisson summation formula (see [D-K], pg. 242) easily extended to $\mathbb{R}^n$, to have

$$\begin{aligned}
\varepsilon^n\sum_{m\in\mathbb{Z}^n} f(m\varepsilon)e^{im\cdot\varepsilon y} &= \varepsilon^n\left(\frac{2\pi}{\varepsilon}\right)^n\sum_{k\in\mathbb{Z}^n}(\mathcal{F}^{-1}f)(y-\frac{2\pi}{\varepsilon}k)\\
&= (2\pi)^n(\mathcal{F}^{-1}f)(y) + (2\pi)^n\sum_{k\in\mathbb{Z}^n\setminus\{0\}}(\mathcal{F}^{-1}f)(y-\frac{2\pi}{\varepsilon}k).
\end{aligned}$$

Since $|y|\le\varepsilon^{-1}\pi$ and the requirement on the support is $|y-2\pi\varepsilon^{-1}k|<R$, the error term $F(\varepsilon,y)$ is given by the finite sum

$$(2\pi)^n\sum_{0<|k|\le R+1}(\mathcal{F}^{-1}f)(y-2\pi\varepsilon^{-1}k) =: F(\varepsilon,y) \tag{5.4}$$

which is a C^∞ function compactly supported in $\bigcup_{1\le|k|\le R+1}\{|y-2\pi\varepsilon^{-1}k|<R\}$. Therefore, for any given $N\in\mathbb{Z}_+$ we have a constant $c_{f,N}$ such that

$$|(\mathcal{F}^{-1}f)(y-2\pi\varepsilon^{-1}k)| \le \frac{c_{f,N}}{1+|y-2\pi\varepsilon^{-1}k|^N} = \frac{c_{f,N}\varepsilon^N}{\varepsilon^N+|\varepsilon y-2\pi k|^N} \le \frac{c_{f,N}\varepsilon^N}{\pi}$$

because $0<\pi<|\varepsilon y-2\pi k|^N$ for all $y\in\varepsilon^{-1}Q_n$ and $k\in\mathbb{Z}^n\setminus\{0\}$. The sum

$$\sum_{0<|k|\le R+1}\frac{c_{f,N}\varepsilon^N}{\pi} =: C_{f,N}\varepsilon^N$$

gives the L^∞ upper bound for the remainder in (a).
We next show (b). It is easily seen that

$$\begin{aligned}
&(2\pi)^{-n}\int_{\varepsilon^{-1}Q_n}\Big[\varepsilon^n\sum_{\alpha\in\varepsilon\mathbb{Z}^n} f(\alpha)e^{i\alpha\cdot y}\Big]e^{-iw\cdot y}dy\\
&= \sum_{m\in\mathbb{Z}^n} f(m\varepsilon)(2\pi)^{-n}\int_{\varepsilon^{-1}Q_n}\varepsilon^n e^{i(\varepsilon m-w)\cdot y}dy\\
&= \sum_{m\in\mathbb{Z}^n} f(m\varepsilon)(2\pi)^{-n}\int_{Q_n} e^{i(m-w\varepsilon^{-1})\cdot y}dy\\
&= \sum_{m\in\mathbb{Z}^n} f(m\varepsilon)\prod_{j=1}^n\frac{\sin(\pi\varepsilon^{-1}w_j-m_j\pi)}{\pi\varepsilon^{-1}w_j-m_j\pi} = f(w),
\end{aligned}\tag{5.5}$$

where in the last equality we used the Sampling Theorem (see [D-K], pg. 302) extended to the multivariable case. □

5.2. Main Results

In this final section we prove the main results of the paper, namely Theorems 1.1 and 1.2. Before doing this, we recall the Wigner transform on the torus

$$W_\hbar\psi(x,\xi) = (2\pi)^{-n}\int_{\mathbb{T}^n} e^{2\frac{i}{\hbar}\langle z,\xi\rangle}\psi(x-z)\overline{\psi}(x+z)dz.$$

Here we are interested in states $\psi_{\hbar,P}(x) = a(\hbar,P,x)e^{\frac{i}{\hbar}(P\cdot x+\hat{v}(\hbar,P,x))} \in \mathbf{\Psi}_{\hbar,P}$ as in Definition 4.14, so that the Wigner transform takes the form

$$W_\hbar\psi_{\hbar,P}(x,\xi) = (2\pi)^{-n}\int_{\mathbb{T}^n} a\left(\hbar,P,x+z\right)a\left(\hbar,P,x-z\right)e^{2\frac{i}{\hbar}S(\hbar,P,x,z,\xi)}dz,$$

with phase function given by

$$S(\hbar,P,x,z,\xi) = (\xi-P)\cdot z - \frac{1}{2}\Big[\hat{v}\left(\hbar,P,x+z\right) - \hat{v}\left(\hbar,P,x-z\right)\Big],$$

and amplitude

$$a(\hbar,P,x) = \varphi_0(\hbar,P,\hat{v})(x).$$

where $\hat{v}\in\Gamma_{\hbar,P}$ as in Definition 4.12.

In the first result we study the semiclassical approximation of $W_\hbar\psi_{\hbar,P}$ with respect to the weak* topology on the set of complex measures, using test functions $\phi\in C^\infty(\mathbb{T}^n\times\mathbb{R}^n;\mathbb{R})$ such that $\mathrm{supp}(\mathcal{F}_y^{-1}\phi(x,\cdot))$ is compact. We will see that the main term in the approximation is a probability measure $d\mu_{\hbar,P}$ in the phase-space $\mathbb{T}^n\times\mathbb{R}^n$, given by

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\xi)d\mu_{\hbar,P}(x,\xi) := \int_{\mathbb{T}^n}\phi(x,P+\nabla_x\hat{v}(\hbar,P,x))\varphi_0(\hbar,P,\hat{v})(x)^2dx. \tag{5.6}$$

Proof of Theorem 1.1 To begin with, for $x,z\in Q_n := [-\pi,\pi]^n$ let us write

$$\begin{aligned} a(\hbar,P,x\pm z) &= a(\hbar,P,x) \pm \int_0^1 z\cdot\nabla a(\hbar,P,x\pm tz)dt,\\ a\left(\hbar,P,x+z\right)a\left(\hbar,P,x-z\right) &= a\left(\hbar,P,x\right)^2 + \rho\left(\hbar,P,x,z\right). \end{aligned}$$

As a consequence,

$$\begin{aligned} \rho\left(\hbar,P,x,z\right) &:= a(\hbar,P,x)z\cdot\int_0^1 \nabla a\left(\hbar,P,x+tz\right) - \nabla a\left(\hbar,P,x-tz\right)dt\\ &- \int_0^1 z\cdot\nabla a\left(\hbar,P,x+tz\right)dt\int_0^1 z\cdot\nabla a\left(\hbar,P,x-t'z\right)dt'. \end{aligned}$$

In the same way,

$$\begin{aligned} v(\hbar,P,x\pm z) &= v(\hbar,P,x) \pm \int_0^1 z\cdot\nabla v(\hbar,P,x\pm tz)dt, && (5.7)\\ \nabla v(\hbar,P,x\pm tz) &= \nabla v(\hbar,P,x) \pm t\int_0^1 \nabla^2 v(\hbar,P,x\pm stz)zds. \end{aligned}$$

Hence,

$$\begin{aligned} S(\hbar,P,x,z) &= \Big(\xi - P - \nabla_x \hat{v}(\hbar,P,x)\Big)\cdot z + Q(\hbar,P,x,z), \qquad (5.8)\\ Q(\hbar,P,x,z) &= \frac{z}{2}\cdot\int_0^1 t\int_0^1 \Big[\nabla^2 v(\hbar,P,x+stz) - \nabla^2 v(\hbar,P,x-stz)\Big] z\,ds\,dt. \end{aligned}$$

We next write $W_\hbar\psi_{\hbar,P} = \sum_{k=0}^2 W_{\hbar,P}^{(k)}$ and compute each of these terms as integrals on the fixed periodicity domain $Q_n := [-\pi,\pi]^n$ in the following way

$$\begin{aligned} W_{\hbar,P}^{(0)} &:= (2\pi)^{-n} a^2(\hbar,P,x)\int_{Q_n} e^{2\frac{i}{\hbar}(\xi-P-\nabla_x\hat{v}(\hbar,P,x))\cdot z} dz,\\ W_{\hbar,P}^{(1)} &:= (2\pi)^{-n} a^2(\hbar,P,x)\int_{Q_n} e^{2\frac{i}{\hbar}(\xi-P-\nabla_x\hat{v}(\hbar,P,x))\cdot z}\Big(e^{2\frac{i}{\hbar}Q(\hbar,P,x,z)}-1\Big)dz,\\ W_{\hbar,P}^{(2)} &:= (2\pi)^{-n}\int_{Q_n}\rho(\hbar,P,x,z)\; e^{2\frac{i}{\hbar}(\xi-P-\nabla_x\hat{v}(\hbar,P,x))\cdot z}\; e^{2\frac{i}{\hbar}Q(\hbar,P,x,z)}dz. \end{aligned}$$

The zeroth-order contribution is exactly the probability measure $d\mu_{\hbar,P}$ introduced in (5.6), whereas the sum $W_{\hbar,P}^{(1)} + W_{\hbar,P}^{(2)}$ is a remainder measure $dr_{\hbar,P}(x,\xi)$ asymptotically vanishing as $\hbar = 1/M \to 0^+$ for $M\in\mathbb{N}$. In fact,

$$\begin{aligned} I_0 &:= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\phi(x,\xi)W_{\hbar,P}^{(0)}(x,\xi)dx\\ &= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\phi(x,\xi)(2\pi)^{-n}\; a(\hbar,P,x)^2\int_{Q_n} e^{2\frac{i}{\hbar}(\xi-P-\nabla_x\hat{v}(\hbar,P,x))\cdot z}\;dz\,dx. \end{aligned}$$

By exchanging the sum with the integrals we get

$$\begin{aligned} I_0 \;=\; &\int_{Q_n} a(\hbar,P,x)^2(2\pi)^{-n}\\ &\int_{Q_n}\Big[\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\phi(x,\xi)e^{i\xi\cdot\frac{2z}{\hbar}}\Big]e^{-2\frac{i}{\hbar}(P+\nabla_x v(\hbar,P,x))\cdot z}dz\,dx. \end{aligned}$$

Setting $y := 2z/\hbar$ and $D_{\hbar,n} := 2\hbar^{-1}Q_n$ gives

$$\begin{aligned} I_0 = \int_{Q_n}\Big\{& a(\hbar,P,x)^2(2\pi)^{-n}\\ &\int_{D_{\hbar,n}}\Big[\Big(\frac{\hbar}{2}\Big)^n\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\phi(x,\xi)e^{i\xi\cdot y}\Big]\; e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y}dy\Big\}dx. \end{aligned}$$

Applying property (b) of Lemma 5.4 with $\varepsilon = \hbar/2$ we finally have

$$I_0 = \int_{\mathbb{T}^n} a(\hbar,P,x)^2\,\phi(x,P+\nabla_x v(\hbar,P,x))dx =: \int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\xi)\; d\mu_{\hbar,P}(x,\xi).$$

The other two terms represent remainder measures and we need some estimates in order to prove their vanishing behaviour as $\hbar = 1/M \to 0^+$. We start by considering

$$\begin{aligned} I_1 &:= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\phi(x,\xi)W^{(1)}_{\hbar,P}(x,\xi)dx \\ &= \sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\Big[\phi(x,\xi)(2\pi)^{-n}a\,(\hbar,P,x)^2 \\ &\qquad\times\int_{Q_n} e^{2\frac{i}{\hbar}(\xi-P-\nabla_x v(\hbar,P,x))\cdot z}\left(e^{2\frac{i}{\hbar}Q(\hbar,P,x,z)}-1\right)dz\Big]dx. \end{aligned}$$

Using the change of variable $y = 2z/\hbar$ gives

$$\begin{aligned} I_1 = &\int_{Q_n}\Big[(2\pi)^{-n}a\,(\hbar,P,x)^2\left(\frac{\hbar}{2}\right)^n \\ &\times\int_{D_{\hbar,n}}\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\phi(x,\xi)e^{i\xi\cdot y}\left(e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)}-1\right)e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y}dy\Big]dx. \end{aligned}$$

Now we apply property (a) to have $I_1 = I_1' + I_1''$

$$\begin{aligned} I_1' &= \int_{Q_n}\Big[(2\pi)^{-n}a\,(\hbar,P,x)^2 \\ &\quad\times\int_{D_{\hbar,n}}\mathcal{F}_y^{-1}\phi(x,y)\left(e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)}-1\right)e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y}dy\Big]dx, \\ I_1'' &= \int_{Q_n}\Big[(2\pi)^{-n}a\,(\hbar,P,x)^2 \\ &\quad\times\int_{D_{\hbar,n}}F(\hbar,y)\left(e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)}-1\right)e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y}dy\Big]dx. \end{aligned}$$

By using the fact that $\mathrm{supp}[\mathcal{F}_y^{-1}\phi(x,\cdot)]\subset B_R(0)$ we have

$$\sup_{y\in B_R(0)}\left|e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)}-1\right| \le \sup_{y\in B_R(0)}\frac{4}{\hbar}|Q(\hbar,P,x,\hbar y/2)| \le$$

(by Remark 4.13)

$$\le \sup_{y\in B_R(0)}|y|^2(d(P)+\sqrt{c(P)})\hbar^{1-\alpha} = C(P)\hbar^{1-\alpha}, \quad \text{where} \quad 0<\alpha<1.$$

As a consequence, an upper bound for I_1' is

$$|I_1'| \le (2\pi)^{-n}\mathrm{vol}(B_R(0))\|\mathcal{F}_y^{-1}\phi\|_\infty\; C(P)\hbar^{1-\alpha}.$$

By Lemma 5.4, an upper bound for the second integral is

$$|I_1''| \le (2\pi)^{-n}\mathrm{vol}(\mathrm{supp}F(\hbar,\cdot))\|F(\hbar,\cdot)\|_\infty = (2\pi)^{-n}\mathrm{vol}(\mathrm{supp}F(\hbar,\cdot))C_{\phi,N}\hbar^N,$$

whence the asymptotic behaviour,

$$\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\phi(x,\xi)W^{(1)}_{\hbar,P}(x,\xi)dx \to 0, \quad \text{as } \hbar = 1/M \to 0^+$$

Finally, let us consider the last term

$$I_2 := \sum_{\xi \in \frac{\hbar}{2}\mathbb{Z}^n} \int_{Q_n} \phi(x,\xi) W^{(2)}_{\hbar,P}(x,\xi)dx = I_2' + I_2'',$$

where

$$\begin{aligned} I_2' &= (2\pi)^{-n} \int_{Q_n} \Big[\int_{D_{\hbar,n}} \mathcal{F}_y^{-1}\phi(x,y)\rho(\hbar,P,x,\hbar y/2) \\ &\qquad \times e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)} e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y} dy \Big] dx, \\ I_2'' &= (2\pi)^{-n} \int_{Q_n} \Big[\int_{D_{\hbar,n}} F(\hbar,y)\rho(\hbar,P,x,\hbar y/2) \\ &\qquad \times e^{2\frac{i}{\hbar}Q(\hbar,P,x,\hbar y/2)} e^{-i(P+\nabla_x v(\hbar,P,x))\cdot y} dy \Big] dx. \end{aligned}$$

We have $|I_2| \leq |I_2'| + |I_2''|$ where

$$\begin{aligned} |I_2'| &\leq (2\pi)^{-n} \int_{Q_n} \int_{D_{\hbar,n}} |\mathcal{F}_y^{-1}\phi(x,y)|\,|\rho(\hbar,P,x,\hbar y/2)|dydx \\ |I_2''| &\leq (2\pi)^{-n} \int_{Q_n} \int_{D_{\hbar,n}} |F(\hbar,y)|\,|\rho(\hbar,P,x,\hbar y/2)|dydx. \end{aligned}$$

To bound the first integral, we consider

$$\begin{aligned} &\int_{B_R(0)} |\rho(\hbar,P,x,\hbar y/2)|dy = 2\int_{B_{2R}(0)} |\rho(\hbar,P,x,\hbar z)|dz \\ &\leq 2\hbar \int_{B_{2R}(0)} |a(\hbar,P,x)||z| \\ &\qquad \times \int_0^1 |\nabla a(\hbar,P,x+t\hbar z)| + |\nabla a(\hbar,P,x-t\hbar z)|dtdz \\ &+ 2\int_{B_{2R}(0)} \int_0^1 \hbar|z||\nabla a(\hbar,P,x+t\hbar z)|dt \int_0^1 \hbar|z||\nabla a(\hbar,P,x-t'\hbar z)|dt'dz \\ &\leq 4R\hbar \int_0^1 \int_{B_{2R}(0)} |a(\hbar,P,x)| \\ &\qquad \times \Big[|\nabla a(\hbar,P,x+t\hbar z)| + |\nabla a(\hbar,P,x-t\hbar z)| \Big] dzdt \\ &+ 8R^2\hbar^2 \int_0^1 \int_0^1 \int_{B_{2R}(0)} |\nabla a(\hbar,P,x+t\hbar z)||\nabla a(\hbar,P,x-t'\hbar z)|dzdtdt'. \end{aligned}$$

Now recall that we have to integrate on Q_n with respect to x-variables. So, we have therefore

$$\begin{aligned} &\int_{Q_n} |a(\hbar,P,x)||\hbar\nabla a(\hbar,P,x\pm tz)|dx \\ &\leq \left(\int_{Q_n} |a(\hbar,P,x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{Q_n} |\hbar\nabla a(\hbar,P,x\pm tz)|^2 dx \right)^{\frac{1}{2}} \leq \quad (5.9) \end{aligned}$$

(by the L^2-normalization of $a(\hbar, P, \cdot)$, its periodicity, and (4.29))

$$\leq (e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha})^{\frac{1}{2}}. \tag{5.10}$$

Similar arguments give

$$\begin{aligned}
&\int_{Q_n} |\hbar\nabla a(\hbar, P, x+tz)||\hbar\nabla a(\hbar, P, x-t'z)|dx \\
\leq\ & \left(\int_{Q_n} |\hbar\nabla a(\hbar, P, x+tz)|^2 dx\right)^{\frac{1}{2}} \left(\int_{Q_n} |\hbar\nabla a(\hbar, P, x-t'z)|^2 dx\right)^{\frac{1}{2}} \\
\leq\ & \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right).
\end{aligned} \tag{5.11}$$

Finally,

$$\begin{aligned}
|I_2'| \leq\ & \|\mathcal{F}_y^{-1}\phi\|_\infty \mathrm{vol}\{B_{2R}(0)\} 4R\hbar \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right)^{\frac{1}{2}} \\
+\ & \|\mathcal{F}_y^{-1}\phi\|_\infty \mathrm{vol}\{B_{2R}(0)\} 8R^2\hbar^2 \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right) \\
\leq\ & C_\phi(P)\hbar^{(1+\alpha/4)}.
\end{aligned}$$

Next, in order to get an estimate for I_2'', we recall that $\|F(\hbar,\cdot)\|_\infty \leq C_{\phi,N}\hbar^N$ and have

$$\begin{aligned}
&\int_{D_{\hbar,n}} |\rho(\hbar, P, x, \hbar y/2)|dy \\
=\ & 2\int_{2D_{\hbar,n}} |\rho(\hbar, P, x, \hbar z)|dz \\
\leq\ & 2\hbar \int_{2D_{\hbar,n}} |a(\hbar, P, x)||z| \int_0^1 |\nabla a(\hbar, P, x+t\hbar z)| + |\nabla a(\hbar, P, x-t\hbar z)|dtdz \\
+\ & 2\int_{2D_{\hbar,n}} \int_0^1 \hbar|z||\nabla a\left(\hbar, P, x+t\hbar z\right)|dt \int_0^1 \hbar|z||\nabla a\left(\hbar, P, x-t'\hbar z\right)|dt'dz \\
\leq\ & 8\pi \int_0^1 \int_{2D_{\hbar,n}} |a(\hbar, P, x)| \Big\{|\nabla a(\hbar, P, x+t\hbar z)| + |\nabla a(\hbar, P, x-t\hbar z)|\Big\} dzdt \\
+\ & 32\pi^2 \int_0^1 \int_0^1 \int_{2D_{\hbar,n}} |\nabla a(\hbar, P, x+t\hbar z)||\nabla a(\hbar, P, x-t'\hbar z)|dzdtdt'.
\end{aligned}$$

Now use (5.10), (5.11), $\mathrm{vol}(2D_{\hbar,n}) = 16\pi^2\hbar^{-n}$ and get

$$\begin{aligned}
|I_2''| \leq\ & (2\pi)^{-n}\mathrm{vol}(2D_{\hbar,n})\|F(\hbar,\cdot)\|_\infty 8\pi 2 \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right)^{\frac{1}{2}} + \\
+\ & (2\pi)^{-n}\mathrm{vol}(2D_{\hbar,n})\|F(\hbar,\cdot)\|_\infty 32\pi^2\hbar^{-2} \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right) \\
\leq\ & (2\pi)^{-n} 16\pi^2\hbar^{-n} C_{f,N}\hbar^N \Big[16\pi \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right)^{\frac{1}{2}} \\
+\ & 32\pi^2 \frac{1}{\hbar^2} \left(e(P)\hbar^{\alpha/2} + 2\hbar^{\alpha}\right) \Big] \\
\leq\ & \hat{C}_{\phi,N}\hbar^{N-n-2+\alpha/4}.
\end{aligned}$$

We therefore conclude that

$$\sum_{\xi\in\frac{\hbar}{2}\mathbb{Z}^n}\int_{Q_n}\phi(x,\xi)W^{(2)}_{\hbar,P}(x,\xi)dx\to 0^+\quad\text{as }\hbar=1/M\to 0^+$$

□

Now we are ready to prove the second main result of the paper.

Proof of Theorem 1.2 We have to show the weak* convergence for some subsequence of the family of measures

$$\int_{\mathbb{T}^n\times\mathbb{R}^n}\phi(x,\xi)d\mu_{\varepsilon,P}(x,\xi):=\int_{\mathbb{T}^n}\phi(x,P+\nabla\hat{v}(\varepsilon,P,x))\varphi_0(\varepsilon,P,\hat{v})(x)^2dx.$$

for every fixed $P\in\mathbb{Z}^n$, and $\varepsilon=1/N$ with $N\in\mathbb{N}$. In the first place, we recall property (ii) of Definition 4.12, namely the $C^{0,1}$-convergence of functions in $\hat{v}\in\mathbf{\Gamma}_{\varepsilon,P}$ to critical subsolutions $\tilde{v}(P,\cdot)$ of the stationary Hamilton-Jacobi equation. Next, we observe that the image measures on $\mathbb{T}^n$ through the canonical projection $\pi:\mathbb{T}^n\times\mathbb{R}^n\to\mathbb{T}^n$,

$$\pi_\star d\mu_{\varepsilon,P}=\varphi_0(\varepsilon,P,v(\varepsilon,P,x))(x)^2dx,$$

have the same support, namely $\mathbb{T}^n$, and so they are in fact a family of tight probability measures (see [Bi]). As a consequence, there exists a weak* convergent subsequence $\{\pi_\star d\mu_{\varepsilon(P,m),P}\}_{m\in\mathbb{N}}$ such that, as $m\to+\infty$,

$$\varphi_0(\varepsilon(P,m),P,v(\varepsilon(P,m),P,x))(x)^2dx\rightharpoonup d\sigma_P(x),\tag{5.12}$$

where $d\sigma_P(x)$ is a Radon probability measure on $\mathbb{T}^n$. We can deduce immediately that for $m\to+\infty$ one has

$$d\mu_{\varepsilon(P,m),P}\rightharpoonup d\tilde{\mu}_P,$$

where the limit is a probability measure on $\mathbb{T}^n\times\mathbb{R}^n$ defined by

$$d\tilde{\mu}_P(x,\xi):=\delta(\xi-P-\nabla_x\tilde{v}(P,x))d\sigma_P(x).\tag{5.13}$$

To see this, write

$$\begin{aligned}
&\int_{\mathbb{T}^n}\phi(x,P+\nabla\hat{v}(\varepsilon,P,x))\varphi_0(\varepsilon,P,\hat{v})(x)^2dx\\
&=\int_{\mathbb{T}^n}\Big[\phi(x,P+\nabla\hat{v}(\varepsilon,P,x))-\phi(x,P+\nabla\tilde{v}(P,x))\Big]\varphi_0(\varepsilon,P,\hat{v})(x)^2dx\\
&+\int_{\mathbb{T}^n}\phi(x,P+\nabla\tilde{v}(P,x))\varphi_0(\varepsilon,P,\hat{v})(x)^2dx-\int_{\mathbb{T}^n}\phi(x,P+\nabla\tilde{v}(P,x))d\sigma_P(x)\\
&+\int_{\mathbb{T}^n}\phi(x,P+\nabla\tilde{v}(P,x))d\sigma_P(x).
\end{aligned}\tag{5.14}$$

Now, by defining for $0<\alpha<1$, $P\in\mathbb{Z}^n$ and $0\le\varepsilon\le 1$ the constants

$$R_\alpha(\varepsilon,P):=\|\hat{v}(\varepsilon,P,\cdot)-\tilde{v}(P,\cdot)\|_{C^{0,1}}\le c_{2,\alpha}(P)\varepsilon^{\alpha/2},$$

we get the upper uniform bound

$$|\phi(x,P+\nabla\hat{v}(\varepsilon,P,x))-\phi(x,P+\nabla\tilde{v}(P,x))|\le\|\nabla_\xi\phi\|_\infty\ R_\alpha(\varepsilon,P).$$

As a consequence, the first integral in (5.14) tends to zero as $\varepsilon \to 0^+$. The difference between the second and the third integral is also vanishing as $\varepsilon \to 0^+$ thanks to (5.12) and by using the density of $C^\infty(\mathbb{T}^n;\mathbb{R})$ in $C^{0,1}(\mathbb{T}^n;\mathbb{R})$. The last term is exactly the integral of the test function ϕ with respect to the measure $d\tilde{\mu}_P$.
Now, recalling Proposition 4.15, hence for a subsequence $\varepsilon = \varepsilon(P,m)$ as above, we have

$$\bar{H}_\varepsilon(P) \le \int_{\mathbb{T}^n} H(P+\nabla_x v(\varepsilon,P,\cdot),x)\varphi_0(\varepsilon,P,v)(x)^2 dx \le \bar{H}(P) + e(P)\varepsilon^{\alpha/2}.$$

Now we apply again the weak* convergence of $\varphi_0(\varepsilon,P,v)(x)^2dx$ by a subsequence of $\varepsilon = \varepsilon(P,m)$ in such a way the limit $d\sigma_P$ has support contained in the graph of $\nabla_x\tilde{v}(P,\cdot)$. This gives

$$\int_{\mathbb{T}^n} H(P+\nabla_x\tilde{v}(P,x))d\sigma_P(x) = \bar{H}(P). \tag{5.15}$$

Now, thanks to Lemma 4.3 and property (iii) of Definition 4.12, we get

$$\int_{\mathbb{T}^n} \nabla_x g(x)\cdot\Big(P+\nabla_x\tilde{v}(P,x)\Big)d\sigma_P(x) = 0, \quad \forall g \in C^\infty(\mathbb{T}^n;\mathbb{R}), \tag{5.16}$$

and

$$\int_{\mathbb{T}^n} |P+\nabla_x\tilde{v}(P,x)|d\sigma_P(x) \le P + \|\tilde{v}(P,\cdot)\|_{C^{0,1}} < +\infty,$$

that is, the measure $d\tilde{\mu}_P$ is closed. Finally, by (5.15) and (5.16), we can apply Proposition 3.1 and deduce that $d\tilde{\mu}_P$ is a Mather measure. □

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