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A Symplectic Topology approach to the Poincaré-Birkhoff Theorem and to weak solutions for H-J equations

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Contents

Introduzione

Gli argomenti affrontati in questa Tesi di Dottorato –orbite periodiche di sistemi meccanici Hamiltoniani su $T^*\mathbb{T}^n$ e soluzioni deboli per l'equazione di Hamilton-Jacobi– si inseriscono nelle più ampio panorama delle teorie variazionali di Lusternik-Schnirelman e di Morse e della teoria delle funzioni generatrici in geometria simplettica. Questi due differenti argomenti sono inoltre collegati a tematiche classiche nel Calcolo delle Variazioni, in Meccanica Analitica e Hamiltoniana, in Teoria delle Trasformazioni Canoniche e in Topologia e Geometria Simplettica.

Il primo capitolo è dedicato alla dimostrazione, nell'ambiente delle funzioni generatrici quadratiche all'infinito come introdotto in [52], di una versione della congettura di Arnol'd, precedentemente studiata e risolta da Conley, Zehnder e successivamente da Golé. Tale dimostrazione fornisce una generalizzazione dell' ultimo teorema geometrico di Birkhoff-Poincaré.

Nel secondo capitolo, dopo una discussione introduttiva di due esempi di propagazione delle onde che conducono allo studio di un'equazione di Hamilton-Jacobi, si studiano in dettaglio due tipi di soluzione debole per questa equazione, la soluzione di viscosità e la soluzione minimax.

Infine, nel terzo capitolo, per un'ampia classe di Hamiltoniane p-convesse, si presenta una dimostrazione originale dell'equivalenza della soluzione di viscosità con la soluzione minimax. Nell'ultima sezione di questo capitolo, come applicazione della formula di Lax-Oleinik precedentemente introdotta, si riassumono le linee principali della teoria K.A.M. debole. Quest'ultima è una recente teoria globale e nonperturbativa sviluppata da Fathi per estendere classici risultati della teoria K.A.M.

In maggior dettaglio, la Tesi è organizzata come segue:

CAPITOLO 1. Si introduce la nozione di funzione generatrice quadratica all'infinito (G.F.Q.I.). Questo tipo di funzioni generatrici sono fondamentali nella teoria globale delle sottovarietà Lagrangiane e delle loro parametrizzazioni.

Si riassumono inoltre i principali risultati della teoria di Lusternik-Schnirelman, una teoria topologica che si applica ad una classe più ampia di funzioni f rispetto alla teoria di Morse. La teoria di Lusternik-Schnirelman permette di associare valori critici di f a classi non nulle in coomologia relativa e fornisce una limitazione inferiore al numero dei punti critici di f, in termini della complessità topologica dei suoi insiemi di sottolivello.

Gli ambienti delle funzioni generatrici quadratiche all'infinito e della teoria di Lusternik-Schnirelman risultano infatti quelli giusti per meglio interpretare e risolvere molti problemi in topologia simplettica, come la congettura di Arnol'd. In questo capitolo, assumendo questo punto di vista, si propone una nuova, originale, dimostrazione del seguente

TEOREMA. *Per una Hamiltoniana $H : \mathbb{R} \times T^*\mathbb{T}^n \to \mathbb{R}$, tale che per $|p| \geq C$ il campo vettoriale Hamiltoniano X_H è p-lineare e indipendente da $q \in \mathbb{T}^n$ e da $t \in \mathbb{R}$, il flusso al tempo uno ϕ_H^1 di X_H ammette tanti punti fissi quanti punti critici ha una funzione $f : \mathbb{T}^n \to \mathbb{R}$.*

CAPITOLO 2. In questo capitolo si introducono due nozioni di soluzione debole (la *soluzione minimax* e la *soluzione di viscosità*) per l'equazione di Hamilton-Jacobi:

$$\frac{\partial u}{\partial t}(t,q) + H(t,q,\frac{\partial u}{\partial q}(t,q)) = 0. \tag{1}$$

Questa PDE appare molto spesso in Matematica e in Fisica, ad esempio nei sistemi dinamici, nella teoria del controllo, nei giochi differenziali, nella meccanica dei continui, nell'ottica geometrica e in economia.

Si discutono inizialmente due casi di propagazione delle onde che conducono allo studio di un'equazione di Hamilton-Jacobi.

Successivamente, si introduce la nozione di soluzione *geometrica* per l'equazione di Hamilton-Jacobi. La soluzione geometrica è una sottovarietà Lagrangiana L ottenuta incollando le caratteristiche del campo vettoriale Hamiltoniano $X_{\mathcal{H}}$, dove $\mathcal{H}(t,q,\tau,p) = \tau + H(t,q,p)$. In generale, come spiegato nel capitolo 1, le sottovarietà Lagrangiane sono descritte dalle loro funzioni generatrici: le soluzioni minimax sono soluzioni del problema di Cauchy per (1), costruite partendo dalla funzione generatrice quadratica all'infinito per L. Le soluzioni minimax sono state introdotte da Marc Chaperon nel 1991 e sono state successivamente studiate da Claude Viterbo, Jean-Claude Sikorav e altri autori, cfr. [25], [53] e [45].

La nozione di soluzione di viscosità è stata invece introdotta negli anni '80 da Crandall, Evans e Lions, cfr. [41] [28] e [4]. Successivamente Bardi e Evans [5], utilizzando le formule di Hopf, hanno costruito direttamente soluzioni di viscosità per Hamiltoniane convesse e Liouville-integrabili $H(p)$.

In questo capitolo della Tesi, usando risultati classici della teoria simplettica delle funzioni generatrici, si costruisce un'estensione al caso non integrabile della formula di Hopf, usata nel citato lavoro di Bardi e Evans. Questo risultato viene ottenuto utilizzando (*i*) un importante (ma non largamente noto) teorema di Hamilton (citato anche nel Gantmacher [33], si veda la sezione "Perturbation Theory"), (*ii*) una classica legge di composizione per le funzioni generatrici in geometria simplettica [6] ed infine (*iii*) un teorema di esistenza di Chaperon-Laudenbach-Sikorav-Viterbo per funzioni generatrici globali di sottovarietà Lagrangiane immagini di flussi Hamiltoniani a supporto compatto.

CAPITOLO 3. Sebbene le soluzioni minimax e di viscosità abbiamo le stesse proprietà analitiche, in generale risultano differenti [45]: soltanto nella Tesi di Dottorato della Joukovskaia [38] si indica che, per Hamiltoniane p-convesse, i due tipi di soluzione coincidono.

In questo capitolo, per Hamiltoniane $H : T^*\mathbb{R}^n \to \mathbb{R}$ di tipo meccanico:

$$H(q,p) = \frac{1}{2}|p|^2 + V(q),$$

V a supporto compatto, viene fornita una dimostrazione dettagliata della coincidenza delle sopra citate soluzioni minimax e di viscosità.
Nella presente dimostrazione (originale) di tale equivalenza, risulta cruciale la seguente rappresentazione della soluzione debole $u : (0,T) \times \mathbb{R}^n \to \mathbb{R}$, $(t,q) \mapsto u(t,q)$, dove $(\tilde{q}(\cdot), \tilde{p}(\cdot)) \in H^1((0,T), T^*\mathbb{R}^n)$:

$$u(t,q) := \inf_{\substack{\tilde{q}(\cdot):\\ \tilde{q}:[0,t]\to\mathbb{R}^n\\ \tilde{q}(t)=q}} \quad \sup_{\substack{\tilde{p}(\cdot):\\ \tilde{p}:[0,t]\to\mathbb{R}^n,\\ \tilde{p}(0)=\frac{\partial\sigma}{\partial q}(\tilde{q}(0))}} \left\{ \sigma(\tilde{q}(0)) + \int_0^t (p\dot{q} - H)_{|(\tilde{q},\tilde{p})}\, ds \right\}, \tag{2}$$

Infatti la precedente formula (2) fornisce *sia* la soluzione di viscosità sia la soluzione minimax.

Da un lato, (2) è la versione Hamiltoniana della formula di Lax-Oleinik che produce la soluzione di viscosità à la Crandall-Evans-Lions, vedere per esempio [32] e [31].

Dall'altro lato, una riduzione alla Amann-Conley-Zehnder del funzionale coinvolto in (2) produce una funzione generatrice globale a parametri finiti. Tale funzione, sotto le ipotesi di p-convessità dell'Hamiltoniana, risulta quadratica all'infinito con indice di Morse 0: ammette quindi una minimo globale. Dopo la sup-procedura sulle curve $\tilde{p}$ in (2), che rappresenta la trasformata di Legendre, la inf-procedura sulle curve $\tilde{q}$ in (2) cattura tale minimo, che risulta *esattamente* il valore critico minimax.

La soluzione minimax e la soluzione di viscosità emergono da differenti ambienti della matematica: tale coincidenza, dimostrata per Hamiltoniane di tipo meccanico, sembra indicare un buon modello di soluzione per una larga classe di sistemi dinamici, ad esempio per la propagazione delle onde.

Infine, come applicazione della formula di Lax-Oleinik precedentemente introdotta, si riassumono le linee principali del teorema K.A.M. debole, che può essere applicato, ad esempio, alla teoria di Aubry-Mather (vedere [30] per maggiori dettagli): la dimostrazione di questo teorema è basata sulla convergenza del semigruppo di Lax-Oleinik per Lagrangiane definite sul fibrato tangente di una varietà compatta, strettamente convesse e superlineari nelle fibre. Il teorema K.A.M. debole può essere anche interpretato in termini delle soluzioni di viscosità per l'equazione di Hamilton-Jacobi associata all'Hamiltoniana, trasformata di Legendre della Lagrangiana L. Utilizzando la (precedentemente dimostrata) concidenza tra le due soluzioni deboli, si mette infine in evidenza che la soluzione di viscosità data dal teorema K.A.M. debole è effettivamente una soluzione minimax della topologia simplettica.

I risultati di questa Tesi di Dottorato appaiono in [8], [9] e [10].

Introduction

The threads connecting the arguments of this Thesis, Poincaré-Birkhoff periodic orbits for mechanical Hamiltonian systems on $T^*\mathbb{T}^n$ and weak solutions to Hamilton-Jacobi equation, are the Lusternik-Schnirelman theory, the theory of generating functions and related results in global symplectic geometry and topology. This two different arguments are related to several classical themes in Calculus of Variations, Analytical and Hamiltonian Mechanics, Canonical Transformation Theory, Symplectic Geometry and Topology.

The first chapter is devoted to the proof, inside Viterbo's framework of generating functions quadratic at infinity, of a version of the Arnol'd conjecture, first studied by Conley, Zehnder and Golé and giving a generalization of the Poincaré-Birkhoff last geometrical theorem.

In the second chapter, after a discussion on two routes –weak discontinuity waves and hight frequency asymptotic waves– leading to Hamilton-Jacobi equation, we study in detail two types of weak solutions to this equation, the viscosity and the minimax solutions.

Finally, in the last chapter, for a general class of p-convex Hamiltonians, we present a detailed proof of the equivalence of the viscosity solution with the minimax solution. In the last section of this chapter, as an application of the above introduced Lax-Oleinik formula, we resume the main lines of the interesting background of the weak K.A.M. theory. This is a global and nonperturbative theory recently developed by Fathi and devoted to extend the classical picture of K.A.M. theory into the large.

In some more detail, the structure of the Thesis is organized in the following way:

Chapter 1. We introduce the notion of generating function quadratic at infinity (G.F.Q.I.). This is a special type of generating function which has been crucial in the global theory of Lagrangian submanifolds and their parameterizations.

Moreover, we resume some classical results in Lusternik-Schnirelman theory: it works for a more general class of functions f than the Morse theory and it allows us to associate critical values of f to non-vanishing relative cohomology classes and to give a lower bound to the number of critical points of f in terms of the topological complexity of its sublevel sets.

The framework of generating functions quadratic at infinity and the Lusternik-Schnirelman theory are in fact the right landscapes in which better understand many actual aspects of symplectic topology, like Arnol'd conjecture. In this chapter, by assuming this point of view, we propose a new proof of the following

THEOREM. *For a Hamiltonian $H : \mathbb{R} \times T^*\mathbb{T}^n \to \mathbb{R}$, such that for $|p| \geq C$ the related vector field X_H is p-linear and independent of $q \in \mathbb{T}^n$ and $t \in \mathbb{R}$, the time-one flow ϕ_H^1 of X_H admits at least many fixed points as a function $f : \mathbb{T}^n \to \mathbb{R}$ on $\mathbb{T}^n$ possesses critical points.*

This statement is presented here as a finite variational problem, consisting in the search of critical points of a generating function quadratic at infinity: a suitable application of the Lusternik-Schnirelman theory in the degenerate case, and the Morse theory in the nondegenerate one, produces the expected result.

CHAPTER 2. In this chapter we introduce two notions of weak solution for the Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t}(t,q) + H(t,q,\frac{\partial u}{\partial q}(t,q)) = 0. \tag{3}$$

This PDE appears in many branches of Mathematics and Physics, as for instance dynamical systems, control theory, differential games, continuum mechanics, geometric optics and economy.

We first recollect some fundamental routes to Hamilton-Jacobi equation, even outside the classical arena of analytical mechanics where this equation naturally arises.

Next, we introduce the notion of *geometric* solution for the Hamilton-Jacobi equation, which is a Lagrangian submanifold L obtained by gluing together the characteristics of the Hamiltonian vector field $X_{\mathcal{H}}$, where $\mathcal{H}(t,q,\tau,p) = \tau + H(t,q,p)$. In general, Lagrangian submanifolds are described by their generating functions S: *minimax* solutions are weak solutions of Cauchy problems related to (3), constructed starting from the generating function quadratic at infinity for the geometric solution L. Minimax solutions have been introduced by Marc Chaperon and they have been studied by Claude Viterbo, Jean-Claude Sikorav and other authors.

Afterwards, using classical results concerning the symplectic theory of generating functions, we construct an extension to non-integrable case of Hopf's formula, used by Bardi and Evans to produce *viscosity* solutions of Hamilton-Jacobi equations for p-convex integrable Hamiltonians. This result is caught by utilizing (i) a very fruitful, even though scarcely known, theorem of Hamilton (e.g. quoted by Gantmacher [33] as "Perturbation Theory"), (ii) a classical composition rule of generating functions in symplectic geometry [6], and (*iii*) the existence theorem by Chaperon-Laudenbach-Sikorav-Viterbo of global generating functions for Lagrangian submanifolds related to compact support Hamiltonians.

Chapter 3. In the last chapter, we prove in detail the coincidence of minimax solutions and viscosity solutions for p-convex Hamiltonians of mechanical type:

$$H(q,p) = \frac{1}{2}|p|^2 + V(q),$$

where V is compact support.
In our proof it is crucial the following representation of the weak solution $u : (0,T) \times \mathbb{R}^n \to \mathbb{R}$, $(t,q) \mapsto u(t,q)$, where we take $(\tilde{q}(\cdot), \tilde{p}(\cdot)) \in H^1((0,T), T^*\mathbb{R}^n)$:

$$u(t,q) := \inf_{\substack{\tilde{q}(\cdot):\\ \tilde{q}:[0,t]\to\mathbb{R}^n\\ \tilde{q}(t)=q}} \sup_{\substack{\tilde{p}(\cdot):\\ \tilde{p}:[0,t]\to\mathbb{R}^n,\\ \tilde{p}(0)=\frac{\partial\sigma}{\partial q}(\tilde{q}(0))}} \left\{ \sigma(\tilde{q}(0)) + \int_0^t (p\dot{q} - H)_{|(\tilde{q},\tilde{p})}\, ds \right\}, \tag{4}$$

In fact the above formula (4) provides *both* the viscous *and* the minimax solution.

From the one hand, (4) is the Hamiltonian version of the Lax-Oleinik formula producing the viscosity solution à la Crandall-Evans-Lions, see for example [32], [31] and bibliography quoted therein.

From the other hand, Amann-Conley-Zehnder reduction does work for the Hamilton-Helmholtz functional involved in (4), producing a global generating function with a finite number of parameters. It turns out that such a function, under the p-convexity hypothesis, is quadratic at infinity with Morse index $i = 0$: in other words, it admits global minimum. After the sup-procedure on the curves $\tilde{p}$ in (4) representing the Legendre transformation, the inf-procedure on the curves $\tilde{q}$ in (4) captures the above minimum, which is *exactly* the minimax critical value, proving that (4) is precisely the minimax solution proposed by Chaperon-Sikorav-Viterbo.

Viscous solution and minimax solution emerge from different, separate fields of mathematics: such a coincidence, which does work surely for physically Hamiltonians, seems to mark a sensible step towards the recognition of a robust good model of solution for physical phenomena, as for instance wave propagation.

Finally, as an application of the above introduced Lax-Oleinik formula, we resume the main lines of the "weak K.A.M. theorem" which can also be applied, for instance, to the Aubry-Mather theory (see [30] for more details): the proof is based on the convergence of the Lax-Oleinik semigroup for a

Lagrangian defined on the tangent space of a compact manifold which is strictly convex and superlinear in the fibers. The convergence of the Lax-Oleinik semigroup can also be reinterpreted in terms of viscosity solutions of the underlying Hamilton-Jacobi equation. Using the coincidence proved above, we can now compare the viscous solution given by the weak K.A.M. theorem to the corresponding minimax solution.

The results of the present Thesis has been resumed in the papers [8], [9] and [10].

Chapter 1

On Poincaré-Birkhoff periodic orbits for mechanical Hamiltonian systems on $T^*\mathbb{T}^n$

The connection between fixed points of mappings
and critical points of generating functions
seems to be a deeper fact than the theorem on
mappings of a two-dimensional annulus into itself.
V. I. Arnol'd, Appendix 9 of [2]

Henri Poincaré has been the main pioneer of the modern dynamical systems theory. Among the large multitude of his contributes, he formulated the nowadays said 'Poincaré's last geometrical theorem' in order to schematize a crucial class of problems related to the search of period solutions in Hamiltonian dynamics:
{P} *Any area preserving diffeomorphism of the annulus* $A = \{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$ *into itself, uniformly rotating the two boundary circles of radius* a *and* b *in opposite directions, admits at least two geometrically distinct fixed points.*
The first rigorous proof of this statement was given in the twenties of the past century by Birkhoff by means of a technique which seems not easily extendible to greater dimensional systems. In a following paper [12], he remarked the power of "maximum-minimum considerations" in the existence of periodic orbits. Nowadays, these aspects are well ruled in the Lusternik-Schnirelman setting: in this framework, one can select minimax

critical values (connected to periodic orbits) of suitable generating functions –quadratic at infinity, see below.
In the sixties, in a series of papers Arnol'd proposed his celebrated conjecture:
{A} *Any Hamiltonian diffeomorphism of a compact symplectic manifold* (M, ω) *possesses at least many fixed points as a function* $f : M \to \mathbb{R}$ *on* M *possesses critical points.*
This new and intriguing topological question has been answered by Conley and Zehnder [27] in the case $M = \mathbb{T}^{2n}$; in that same paper they also proved that
{C-Z} *For a Hamiltonian* $H : \mathbb{R} \times T^*\mathbb{T}^n \to \mathbb{R}$*, such that for* $|p| \geq C$ *the related vector field* X_H *is p-linear and independent of* $q \in \mathbb{T}^n$ *and* $t \in \mathbb{R}$*, the time-one flow* ϕ^1_H *of* X_H *admits at least many fixed points as a function* $f : \mathbb{T}^n \to \mathbb{R}$ *on* $\mathbb{T}^n$ *possesses critical points.*
It is interesting to notice that this last statement, directly descending from Poincaré's last geometrical theorem, in a sense, comes back to the original setting of Analytical Mechanics in which it arose. E.g., the above Hamiltonians are at once interpreted as describing a physical landscape in which a number of particles does interact among them only under a suitable energy threshold (low energy scattering):

$$H(q,p) = \frac{1}{2}|p|^2 + f(q,p), \quad q \in \mathbb{T}^n, \quad f \in O(1).$$

Incidentally, we can note that this is quite near to a typical Hamiltonian setting of Nekhoroshev perturbation theory: $H(q,p) = 1/2|p|^2 + \varepsilon f(q,p)$.

Conley and Zehnder introduced a sort of Liapunov-Schmidt reduction technique, now known as *Amann-Conley-Zehnder reduction*, based on a suitable Fourier cut-off on the loop space and giving, at last, a finite dimensional variational problem. Chaperon –see [21]– proposed few time later his new ingenious *broken geodesics reduction*, showing it is not indispensable to start from the infinite dimensional formulation of the problem. In both cases, the estimates on fixed points of ϕ^1_H are proved using the isolated invariant sets and the Morse index, as presented by Conley [26].

More recently, Golé [35], [34], gave an alternative proof of the statement {C-Z}, extending $\mathbb{T}^n$ to any compact manifold and using a variation of Chaperon's argument. The finite variational problem which in such a way he obtained was solved by utilizing techniques based on Conley index and further results on it by Floer. Furthermore, the author pointed out that his function, defining the above finite variational problem, was not a generating function quadratic at infinity, an essential property in order to apply agreeably Lusternik-Schnirelman theory.

Nowadays, a short and nice proof of this theorem can be found in the fine papers [22] and [23] by Chaperon.

After the impressive paper [52], there exists a rather common growing prejudge that the framework of the generating functions quadratic at infinity and Lusternik-Schnirelman theory should be a right environment to better understand many actual aspects of symplectic topology, as Arnol'd conjecture, see p.e. [37], p. 216.

In this chapter, by assuming this point of view, we restart from the original statement {C-Z}, for $\mathbb{T}^n$. In genuine framework of the generating functions quadratic at infinity, and then using now classical results by Chaperon, Chekanov, Laudenbach, Sikorav and Viterbo, we propose a finite variational problem consisting of a generating function quadratic at infinity: a suitable application of Lusternik-Schnirelman theory in the degenerate case, and Morse theory in the nondegenerate one, produces the expected result. By making this goal, I give short recalls on some tools here exploited, often involved in symplectic topology.

1.1 Preliminaries

1.1.1 Generating functions

Let N be a compact manifold and $L \subset T^*N$ a Lagrangian submanifold. If L is exact, i.e. $L = \operatorname{im}(df) = L_f$, where $f : N \to \mathbb{R}$ is a C^2 function, then the set $\operatorname{crit}(f)$ of the critical points of f coincides with the intersection of L_f with the zero section $0_N \subset T^*N$:

$$\operatorname{crit}(f) = L_f \cap 0_N.$$

In the more general case, Lagrangian submanifolds are not exact, and a classical argument by Maslov and Hörmander shows that, at least locally, every Lagrangian submanifold is described by some generating function like

$$S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$$
$$(x, \xi) \longmapsto S(x, \xi)$$

in the following way:

$$L_S := \{(x, \frac{\partial S}{\partial x}(x, \xi)) \ : \ \frac{\partial S}{\partial \xi}(x, \xi) = 0\},$$

where 0 is a regular value of the map

$$(x, \xi) \longmapsto \frac{\partial S}{\partial \xi}(x, \xi).$$

Some authors (e.g. Benenti, Tulczyjew, Weinstein) say that in this case the generating function S is a *Morse family*. The direct use of generating functions in the Calculus of Variations pushes to consider and look for conditions guaranteeing the existence of global generating functions. In particular, the following class of generating functions has been decisive in many issues:

Definition 1.1.1 *A generating function $S : N \times \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity (G.F.Q.I.) if for $|\xi| > C$*

$$S(x,\xi) = \xi^T Q \xi, \tag{1.1}$$

where $\xi^T Q \xi$ is a nondegenerate quadratic form.

There were known in literature (see e.g. [54], [40]) two main operations on the generating functions which leave invariant the corresponding Lagrangian submanifolds. The Lemma (1.1.2) and (1.1.3) below recollect these facts. The globalization was realized by Viterbo (see [51]).

Lemma 1.1.2 *Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a G.F.Q.I. and $N \times \mathbb{R}^k \ni (x,\xi) \mapsto (x, \phi(x,\xi)) \in N \times \mathbb{R}^k$ a map such that, $\forall x \in N$,*

$$\mathbb{R}^k \ni \xi \longmapsto \phi(x,\xi) \in \mathbb{R}^k$$

is a diffeomorphism. Then

$$S_1(x,\xi) := S(x, \phi(x,\xi))$$

generates the same Lagrangian submanifold: $L_{S_1} = L_S$.

Proof. Since ϕ is a diffeomorphism,

$$\frac{\partial S_1}{\partial \xi} = \frac{\partial S}{\partial \xi}\frac{\partial \phi}{\partial \xi} = 0 \text{ if and only if } \frac{\partial S}{\partial \xi} = 0.$$

Moreover

$$(x, \frac{\partial S_1}{\partial x}(x,\xi)) = (x, \frac{\partial S}{\partial x}(x,\phi(x,\xi)) + \frac{\partial S}{\partial \xi}(x,\phi(x,\xi))\frac{\partial \phi}{\partial x}(x,\xi)),$$

$$= (x, \frac{\partial S}{\partial x}(x,\phi(x,\xi))).$$

Then $L_{S_1} = L_S$, and it is immediately verified that 0 is also a regular value for $\frac{\partial S_1}{\partial \xi}(x,\xi)$.
□

Lemma 1.1.3 *Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a G.F.Q.I.. Then*

$$S_1(x,\xi,\eta) := S(x,\xi) + \eta^T B \eta,$$

where $\eta \in \mathbb{R}^l$ and $\eta^T B \eta$ is a nondegenerate quadratic form, generates the same Lagrangian submanifold: $L_{S_1} = L_S$.

Proof

$$\frac{\partial S_1}{\partial \xi}(x,\xi,\eta) = 0 \text{ if and only if } \frac{\partial S}{\partial \xi}(x,\xi) = 0,$$

$$\frac{\partial S_1}{\partial \eta}(x,\xi,\eta) = 0 \text{ if and only if } B\eta = 0 \text{ that is } \eta = 0.$$

Thus $(x, \frac{\partial S_1}{\partial x}(x,\xi,\eta))|_{\frac{\partial S_1}{\partial \xi}=0} = (x, \frac{\partial S}{\partial x}(x,\xi))|_{\frac{\partial S}{\partial \xi}=0}$.
□

Finally, as a third –although trivial– invariant operation, we observe that by adding to a generating function S any arbitrary constant $c \in \mathbb{R}$ the described Lagrangian submanifold is invariant: $L_{S+c} = L_S$.

The problems 1 and 2 below have been crucial in the global theory of Lagrangian submanifolds and their parameterizations.

1. When a Lagrangian submanifold $L \subset T^*N$ does admit a FGQI?
2. If L admits a G.F.Q.I., when can we state the uniqueness of it (up to the operations described above)?

The following theorem –see [47]– answers partially to the first question.

Theorem 1.1.4 *(Chaperon-Laudenbach-Sikorav)*
Let 0_N be the zero section of T^*N and $(\phi_t)_{t\in[0,1]}$ a Hamiltonian flow. Then the Lagrangian submanifold $\phi_1(0_N)$ admits a G.F.Q.I.

The answer to the second problem is due to Viterbo:

Theorem 1.1.5 *(Viterbo)*
Let 0_N be the zero section of T^*N and $(\phi_t)_{t\in[0,1]}$ a Hamiltonian flow. Then the Lagrangian submanifold $\phi_1(0_N)$ admits a unique G.F.Q.I.

The theorems above –see also [50]– still hold in $T^*\mathbb{R}^n$, provided that $(\phi_t)_{t\in[0,1]}$ is a flow of a compact support Hamiltonian vector field.

A generalization of Definition 1.1.1 –introduced by Viterbo and studied in detail by Theret [51], [50]– is the following:

Definition 1.1.6 *A generating function $S : N \times \mathbb{R}^k \to \mathbb{R}$, $(q,\xi) \mapsto S(q,\xi)$, is quadratic at infinity if for every fixed $q \in N$*

$$||S\,(q,\cdot) - \mathcal{P}^{(2)}(q,\cdot)||_{C^1} < +\infty, \tag{1.2}$$

where $\mathcal{P}^{(2)}(q,\xi) = Q\,(q,\xi) + A(q)\xi + B(q)$ and $Q\,(q,\xi) = \xi^T Q(q)\xi$ is a nondegenerate quadratic form.

In particular, it can be proved that conditions (1.1) and (1.2) result equivalent.

1.1.2 Lusternik-Schnirelman theory

Let $f : N \to \mathbb{R}$ be a C^2 function. We shall assume that either N is compact or f satisfies the Palais-Smale (PS) condition:
(PS) Any sequence $\{x_n\}$ such that $\nabla f(x_n) \to 0$ and $f(x_n)$ is bounded, admits a converging subsequence.
We recall now some results of the Lusternik-Schnirelman theory, which allows us to associate critical values of f to non-vanishing relative cohomology classes and to give a lower bound to the number of critical points of f in terms of the topological complexity of N.
Let us define the sublevel sets

$$N^\nu := \{x \in N : \ f(x) \leq \nu\}. \tag{1.3}$$

(PS) condition guarantees the well-defined gradient vector field ∇f, which flow realizes an diffeomorphism between N^μ and N^ν whenever no critical values exist in $[\mu, \nu]$:

Proposition 1.1.7 *Let $\mu < \nu$. If f has no critical points in $N^\nu \setminus N^\mu$, then $H^*(N^\nu, N^\mu) = 0$.*

Thus if $H^*(N^\nu, N^\mu) \neq 0$, then in $N^\nu \setminus N^\mu$ there exists at least one critical point of f, with critical value in $[\mu, \nu]$. For $\lambda \in [\mu, \nu]$, let

$$i_\lambda : N^\lambda \hookrightarrow N^\nu$$

be the inclusion.

Definition 1.1.8 *For every $u \in H^*(N^\nu, N^\mu)$, $u \neq 0$, we define:*

$$c(u, f) =: \inf \{\lambda \in [\mu, \nu] : \ i^*_\lambda u \neq 0\},$$

where

$$i^*_\lambda : \ H^*(N^\nu, N^\mu) \longrightarrow H^*(N^\lambda, N^\mu)$$

denotes the pull-back of the inclusion.

This Definition provides a tool to detect critical values, indeed:

Theorem 1.1.9 *$c(u, f)$ is a critical value of f.*

The main result of this construction consists in the following

Theorem 1.1.10 *(Cohomological Lusternik-Schnirelman theory)*
Let $0 \neq u \in H^*(N^\nu, N^\mu)$ and $v \in H^*(N^\nu) \setminus H^0(N^\nu)$.

1.

$$c(u \wedge v, f) \geq c(u, f). \tag{1.4}$$

2. *If (1.4) is an equality* ($c(u \wedge v, f) = c(u, f) =: c$), *set*

$$K_c = \{x \ : \ df(x) = 0, \ f(x) = c\},$$

then, for every neighbourhood U of K_c, v is not vanishing in $H^*(U)$, and the common critical level contains infinite critical points.

Corollary 1.1.11 *Let N be a compact manifold. The function $f : N \to \mathbb{R}$ has at least a number of critical points equal to the cup-lenght of N:*

$$\mathrm{cl}(N) := \max\left\{k : \ \exists v_1, \ldots, v_{k-1} \in H^*(N) \setminus H^0(N) \ s.\,t. \ v_1 \wedge \ldots \wedge v_{k-1} \neq 0\right\}. \tag{1.5}$$

Proof Apply the Theorem 1.1.10 with $\mu < \inf f$, $\sup f < \nu$ and $u = 1 \in H^*(N, \emptyset) = H^*(N)$. $\square$
By the Corollary 1.1.12 below, we verify that the preceding estimate on the number of critical points of f still holds in the non-compact case whenever G.F.Q.I. f are taken into account.

Corollary 1.1.12 *Let N be a compact manifold and $f : N \times \mathbb{R}^n \to \mathbb{R}$ be a G.F.Q.I., $f(x, \xi) = Q(\xi)$ out of a compact set in the parameters ξ. Then, for $c > 0$ large enough, there exist $0 \neq u \in H^*(f^c, f^{-c})$ and $v_1, \ldots, v_{k-1}$ as in (1.5) such that*

$$u \wedge p^* v_1 \wedge \ldots \wedge p^* v_{k-1} \neq 0,$$

where $p : N \times \mathbb{R}^n \to N$. Consequently, the G.F.Q.I. $f : N \times \mathbb{R}^n \to \mathbb{R}$ has at least $\mathrm{cl}(N)$ critical points.

Proof Let us first observe that for $c > 0$ large enough, the sublevel sets of f are invariant from a homotopical point of view:

$$f^{\pm c} = N \times Q^{\pm c},$$

and $f^{\pm \bar{c}}$ retracts on $f^{\pm c}$ for any $\bar{c} > c$. Let $A := Q^{-(c+\epsilon)}$, $\epsilon > 0$ small. Then the isomorphisms below (the first one by excision and the second one by retraction) hold:

$$H^*(Q^c, Q^{-c}) \cong H^*(Q^c \setminus \mathring{A}, Q^{-c} \setminus \mathring{A}) \cong H^*(D^i, \partial D^i),$$

where i is the index of the quadratic form Q and D^i denotes the disk (of radius $\sqrt{c}$) in $\mathbb{R}^i$. Consequently

$$H^h(Q^c, Q^{-c}) \cong H^h(D^i, \partial D^i) = \begin{cases} 0 & \text{if } h \neq i \\ \alpha \cdot \mathbb{R} & \text{if } h = i \end{cases}$$

To conclude in the non-compact case $N \times \mathbb{R}^n$, by Thom's isomorphism

$$H^*(N) \cong H_c^{*+i}(N \times \mathbb{R}^i)$$

and the homotopy argument

$$H_c^*(N \times \mathbb{R}^i) \cong H^*(N \times D^i, N \times \partial D^i),$$

the following isomorphism

$$H^*(N) \ni v \longmapsto q^*\alpha \wedge p^* v \in H^{*+i}(N \times D^i, N \times \partial D^i)$$

holds, where $p : N \times \mathbb{R}^n \to N$, $q = (q_1, q_2) : (N \times D^i, N \times \partial D^i) \to (D^i, \partial D^i)$ are the standard projections. Now we apply the Theorem 1.1.10 with $u = q^*\alpha$; since

$$q^*\alpha \wedge p^* v_1 \wedge \ldots \wedge p^* v_{k-1} = q^*\alpha \wedge p^*(v_1 \wedge \ldots \wedge v_{k-1}) \neq 0$$

whenever $v_1 \wedge \ldots v_{k-1} \neq 0$, then the number of critical points of the G.F.Q.I. $f : N \times \mathbb{R}^n \to \mathbb{R}$ is at least $\mathrm{cl}(N)$. $\square$

1.2 The Hamiltonian setting

Let $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n} = \{(q,p) : \ q \in \mathbb{R}^n, \ p \in \mathbb{R}^n\}$ be endowed with the standard symplectic form $\omega = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq^i$.
On $(\mathbb{R}^{2n}, \omega)$ we consider the time-dependent globally Hamiltonian vector field given by $H(t,q,p) \in C^2(\mathbb{R} \times \mathbb{R}^{2n}; \mathbb{R})$ with the properties:

$$H(t, q + 2\pi k, p) = H(t,q,p), \quad \forall (t,q,p) \in \mathbb{R} \times \mathbb{R}^{2n}, \ \forall k \in \mathbb{Z}^n, \tag{1.6}$$

and

$$H(t,q,p) = \frac{1}{2}|p|^2 \quad \text{if} \quad |p| \geq C > 0. \tag{1.7}$$

Our aim is to draw a new proof of a popular version, due to Arnol'd, of Poincaré's last geometrical theorem (see [27], [34]) inside Viterbo's framework of symplectic topology; we will give a reasonable estimate of the number of fixed points of the symplectic time-one map ϕ_H^1 –on the cotangent of the torus, see below– by noticing that the present construction differs from Viterbo's format by considering non compact support Hamiltonian functions.

1.2.1 Properties of flows on the cotangent of the torus

In connection with the above Hamiltonian H, see (1.6),(1.7), let[1] ϕ_H^t be the flow of the Hamiltonian vector field X_H, $\omega(X_H, \eta) = -dH(\eta)$, so that $X_H = J\nabla H$, where J is the symplectic $2n$-matrix. The n-torus is denoted

[1]Here, as in other analogous circumstances, we mean $\phi_H^t := \phi_H^t$.

by $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Therefore a Hamiltonian $\bar{H}$ and the related flow $\phi^t_{\bar{H}}$ are well defined on $T^*\mathbb{T}^n$, see (1.2.3) below:

$$\begin{tikzcd}
\mathbb{R} \times T^*\mathbb{R}^n \arrow[r, "H"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathbb{R} \arrow[d, "id_{\mathbb{R}}"] & T^*\mathbb{R}^n \arrow[r, "\phi^t_H"] \arrow[d, "\pi"'] & T^*\mathbb{R}^n \arrow[d, "\pi"] \\
\mathbb{R} \times T^*\mathbb{T}^n \arrow[r, "\bar{H}"'] & \mathbb{R} & T^*\mathbb{T}^n \arrow[r, "\phi^t_{\bar{H}}"'] & T^*\mathbb{T}^n
\end{tikzcd}$$

Proposition 1.2.1 *The flow ϕ^t_H associated to H satisfies:*

$$\begin{array}{rcl}
(\phi^t_H)_q(q + 2\pi k, p) & = & (\phi^t_H)_q(q, p) + 2\pi k, \\
(\phi^t_H)_p(q + 2\pi k, p) & = & (\phi^t_H)_p(q, p),
\end{array}$$

$\forall k \in \mathbb{Z}^n$ *and* $\forall (q, p) \in \mathbb{R}^{2n}$.

We first prove the following

Lemma 1.2.2 *Under the above hypothesis on H, the map $\tilde{\phi}^t_H$:*

$$((\tilde{\phi}^t_H)_q(q, p), (\tilde{\phi}^t_H)_p(q, p)) := ((\phi^t_H)_q(q + 2\pi k_1, p) + 2\pi k_2, (\phi^t_H)_p(q + 2\pi k_1, p))$$

$\forall k_1, k_2 \in \mathbb{Z}^n$, *is again a solution of the dynamic system related with the Hamiltonian H.*

Proof

$$\begin{array}{rl}
\frac{d}{dt}(\tilde{\phi}^t_H)_q(q, p) & = \frac{\partial H}{\partial p}((\phi^t_H)_q(q + 2\pi k_1, p), (\phi^t_H)_p(q + 2\pi k_1, p)), \\
& = \frac{\partial H}{\partial p}((\phi^t_H)_q(q + 2\pi k_1, p) + 2\pi k_2, (\phi^t_H)_p(q + 2\pi k_1, p)), \\
& = \frac{\partial H}{\partial p}((\tilde{\phi}^t_H)_q(q, p), (\tilde{\phi}^t_H)_p(q, p)).
\end{array}$$

$$\begin{array}{rl}
\frac{d}{dt}(\tilde{\phi}^t_H)_p(q, p) & = -\frac{\partial H}{\partial q}((\phi^t_H)_q(q + 2\pi k_1, p), (\phi^t_H)_p(q + 2\pi k_1, p)), \\
& = -\frac{\partial H}{\partial q}((\phi^t_H)_q(q + 2\pi k_1, p) + 2\pi k_2, (\phi^t_H)_p(q + 2\pi k_1, p)), \\
& = -\frac{\partial H}{\partial q}((\tilde{\phi}^t_H)_q(q, p), (\tilde{\phi}^t_H)_p(q, p)).
\end{array}$$

□

Proof of the proposition 1.2.1

We have only to determine k_1 and $k_2 \in \mathbb{Z}^n$ such that

$$(\tilde{\phi}^0_H)_q(q, p) = q.$$

We obtain $k_1 = -k_2$, therefore, $\forall k \in \mathbb{Z}^n$,

$$((\tilde{\phi}^t_H)_q(q, p), (\tilde{\phi}^t_H)_p(q, p))|_{k_1=-k_2=k} = ((\phi^t_H)_q(q+2\pi k, p)-2\pi k, (\phi^t_H)_p(q+2\pi k, p))$$

is a flow of the dynamical system related to H. By uniqueness of the flow, we obtain:

$$((\phi^t_H)_q(q + 2\pi k, p) - 2\pi k, (\phi^t_H)_p(q + 2\pi k, p)) = ((\phi^t_H)_q(q, p), (\phi^t_H)_p(q, p)),$$

that this:

$$\begin{array}{rcl} (\phi^t_H)_q(q+2\pi k,p) & = & (\phi^t_H)_q(q,p)+2\pi k, \\ (\phi^t_H)_p(q+2\pi k,p) & = & (\phi^t_H)_p(q,p). \end{array}$$

$\forall k \in \mathbb{Z}^n$ and $\forall (q,p) \in \mathbb{R}^{2n}$.
$\square$
We denote by $[q] \in \mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ the class of $q \in \mathbb{R}^n$. From the above deductions it follows the

Corollary 1.2.3 *The flow of* $X_{\bar{H}}$ *is*

$$\phi^t_{X_{\bar{H}}}([q],p) = \left(\left[\phi^t_{X_H,q}(q,p)\right], \phi^t_{X_H,p}(q,p)\right). \tag{1.8}$$

1.2.2 The splitting $H = H_0 + f$

We remind that the Hamiltonian H coincides with $\frac{1}{2}|p|^2$ if $|p| \geq C > 0$. Consequently, outside of this compact set (in the p variables) the flow associated to the Hamiltonian H reduces to

$$\mathbb{R}^n \times \{p: \ |p| \geq C\} \longrightarrow \mathbb{R}^n \times \{p: \ |p| \geq C\}$$

$$(q,p) \longmapsto \phi^t_H(q,p) = (q+tp,p).$$

We split H as the sum of the Hamiltonian H_0,

$$H_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(t,q,p) \longmapsto H_0(p) := \frac{1}{2}|p|^2$$

and a Hamiltonian f, hence necessarily compact support in the p variables,

$$H = H_0 + f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(t,q,p) \longmapsto H(t,q,p) = H_0(p) + f(t,q,p).$$

By denoting

$$\phi^t_0 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$$

$$(q,p) \longmapsto (q+tp,p)$$

the flow related to H_0, we define the time-depending Hamiltonian K as the pull-back of f with respect to the flow ϕ^t_0:

$$K : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$K := (\phi^t_0)^* f, \text{ i.e. } \quad K(t,q,p) = H(t,q+tp,p) - \frac{|p|^2}{2}.$$

This Hamiltonian K, which is compact support in the p variables like f, it will be essential in the next sections. We indicate now ϕ^t_K the flow of K and write down the following proposition, which is, essentially, a result of Hamilton [36] (see also [33]).

Proposition 1.2.4 *Let ϕ_H^t, ϕ_0^t and ϕ_K^t be the flows of $H = H_0 + (H - H_0)$, H_0 and $K = (\phi_0^t)^*(H - H_0)$ respectively. We have:*

$$\phi_H^t(q,p) = \phi_0^t \circ \phi_K^t(q,p),$$

$\forall (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\forall t \in \mathbb{R}$.

We recall some technical premises to the proof of this fact.

Definition 1.2.5 *(Push-forward) Let N be a manifold and ρ a diffeomorphism of N into itself,*

$$N \xrightarrow{\rho} N$$

$$x \longmapsto y = \rho(x)$$

The push-forward ρ_* of a vector field X is defined as follows:

$$\rho_* X(y) := d\rho(\rho^{-1}(y))X(\rho^{-1}(y)).$$

(Below, we will use this definition with $N = T^*\mathbb{R}^n$ and $\rho = \phi_0^t$.) The following Lemma is a central result of the canonical transformations theory.

Lemma 1.2.6 *Let M be a manifold and ρ a symplectic diffeomorphism of T^*M into itself, then, for every Hamiltonian function $L : T^*M \to \mathbb{R}$,*

$$\rho_* X_L = X_{\rho_* L} = X_{L \circ \rho^{-1}}.$$

Proof of the Proposition 1.2.4

$$\begin{aligned}
\frac{d}{dt}(\phi_0^t \circ \phi_K^t)(q,p) &= X_{H_0}(\phi_0^t \circ \phi_K^t(q,p)) + d\phi_0^t(\phi_K^t(q,p))X_K(\phi_K^t(q,p)), \\
&= X_{H_0}(\phi_0^t \circ \phi_K^t(q,p)) + \\
&\quad d\phi_0^t(\phi_0^{-t} \circ \phi_0^t \circ \phi_K^{t,0}(q,p))X_K(\phi_0^{-t} \circ \phi_0^t \circ \phi_K^t(q,p)), \\
&= X_{H_0}(\phi_0^t \circ \phi_K^t(q,p)) + (\phi_0^t)_* X_K(\phi_0^t \circ \phi_K^t(q,p)), \\
&= [X_{H_0} + X_{(\phi_0^t)_* K}](\phi_0^t \circ \phi_K^t(q,p)) = X_{H_0+f}(\phi_0^t \circ \phi_K^t(q,p)), \\
&= X_H(\phi_0^t \circ \phi_K^t(q,p)).
\end{aligned}$$

□

1.2.3 The 'graph' and the 'cotangent' structures of $\mathbb{R}^{4n}$

We introduce now the following map h, from the 'graph'-structure to the 'cotangent'-structure:

Definition 1.2.7

$$h : (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n}) \longrightarrow (T^*(T^*\mathbb{R}^n), \omega_{\mathbb{R}^{2n}})$$

$$(q, p, Q, P) \longmapsto (q, P, p - P, Q + P - q).$$

Proposition 1.2.8 *The linear map h is a symplectic isomorphism.*

Proof

$$\begin{aligned} h^*\omega_{\mathbb{R}^{2n}} &= d(p-P) \wedge dq + d(Q+P-q) \wedge dP = \\ &= dp \wedge dq - dP \wedge dq + dQ \wedge dP - dq \wedge dP, \\ &= dp \wedge dq - dP \wedge dq + dP \wedge dq - dP \wedge dQ, \\ &= dp \wedge dq - dP \wedge dQ = \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n}. \end{aligned}$$

□

The following Lagrangian submanifold F of $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n})$,

$$F := \{(q, p, q - p, p) : \ (q, p) \in T^*\mathbb{R}^n\}, \tag{1.9}$$

is mapped by h to the zero section $0_{\mathbb{R}^{2n}}$:

$$h(F) = 0_{\mathbb{R}^{2n}} \subset \mathbb{R}^{4n}. \tag{1.10}$$

Since we are looking for fixed points of ϕ^1_H, we denote by Γ_H and Γ_K the graphs of ϕ^1_H and ϕ^1_K in $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n \ , \ \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n})$ respectively, and by Δ the diagonal of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n = \mathbb{R}^{4n}$.
It comes out that:

$$(\bar{q}, \bar{p}) \in T^*\mathbb{R}^n \text{ is a fixed point of } \phi^1_H,$$

that is

$$(\bar{q}, \bar{p}, (\phi^1_H)_q(\bar{q}, \bar{p}), (\phi^1_H)_p(\bar{q}, \bar{p})) \in \Gamma_H \cap \Delta,$$

that is, by Prop. 1.2.4,

$$(\bar{q}, \bar{p}, (\phi^1_0 \circ \phi^1_K)_q(\bar{q}, \bar{p}), (\phi^1_0 \circ \phi^1_K)_p(\bar{q}, \bar{p})) \in \Gamma_H \cap \Delta,$$

if and only if, setting:

$$\hat{\phi}_0^{-1}(q, p, Q, P) := id_{\mathbb{R}^{2n}} \times \phi_0^{-1}(q, p, Q, P) = (q, p, Q - P, P),$$

and using F in (1.9),

$$(\bar{q}, \bar{p}, (\phi^1_K)_q(\bar{q}, \bar{p}), (\phi^1_K)_p(\bar{q}, \bar{p})) \in \hat{\phi}_0^{-1}(\Gamma_H) \cap \hat{\phi}_0^{-1}(\Delta) = \Gamma_K \cap F$$

if and only if, using h of Def. 1.2.7 and (1.10),

$$h(\bar{q},\bar{p},(\phi^1_K)_q(\bar{q},\bar{p}),(\phi^1_K)_p(\bar{q},\bar{p})) \in h(\Gamma_K)\cap h(F) = h(\Gamma_K)\cap 0_{\mathbb{R}^{2n}}$$

that is

$$(\bar{q},(\phi^1_K)_p(\bar{q},\bar{p}),\bar{p}-(\phi^1_K)_p(\bar{q},\bar{p}),(\phi^1_K)_q(\bar{q},\bar{p})+(\phi^1_K)_p(\bar{q},\bar{p})-\bar{q})$$
$$\in h(\Gamma_K)\cap 0_{\mathbb{R}^{2n}}.$$

Thus, we claim that the periodic time-one solutions, corresponding to fixed points of ϕ^1_H, are caught by the critical points of a (possible) generating function for $h(\Gamma_K)$. Furthermore, they are contained in the region $\mathbb{T}^n\times\{p: |p|<C\}$. In fact, on $\mathbb{T}^n\times\{p: \ |p|\geq C\}$ the Hamiltonian system is trivially integrable and in such a case the tori $\mathbb{T}^n\times\{p\}$ are invariant under the flow $\phi^t_H:(q,p)\mapsto(q+tp,p)$. Consequently, the non trivial periodic solutions of $\phi^1_{\bar{H}}$, corresponding precisely to the fixed points of ϕ^1_H, must lie in $\mathbb{T}^n\times\{p: \ |p|<C\}$ and are contractible loops on $\mathbb{T}^n$.

1.3 Existence for generating functions

Our original problem has been translated into the investigation of $h(\Gamma_K)\cap 0_{\mathbb{R}^{2n}}$. The Lagrangian submanifold $h(\Gamma_K)$,

$$h(\Gamma_K)=\{(q,(\phi^1_K)_p(q,p),p-(\phi^1_K)_p(q,p),(\phi^1_K)_p(q,p)+(\phi^1_K)_q(q,p)-q),$$
$$\forall (q,p)\in T^*\mathbb{R}^n\}\subset (T^*\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}}),$$

in a neighbourhood of infinity (in the p variables) results:

$$h(\Gamma_K)=\{(q,p,0,p), \ \forall q\in\mathbb{R}^n, \ \forall p\in\mathbb{R}^n \ : \ |p|\geq C\}.$$

In this section we study its structure, proving that it is the image (through a suitable symplectic isomorphism ψ of $(T^*\mathbb{R}^{2n},\omega_{\mathbb{R}^{2n}})$) of another Lagrangian submanifold, denoted by $\bar{h}(\Gamma_K)$, which is isotopic to the zero section of $T^*\mathbb{R}^{2n}$, so that it admits a G.F.Q.I. (Theorem 1.1.4). This is crucial in order to gain the existence of a generating function for $h(\Gamma_K)$. In fact, by means of a natural composition of the above generating functions for $\bar{h}(\Gamma_K)$ and for ψ, we will be able to construct a G.F.Q.I. for $h(\Gamma_K)$.

1.3.1 The factorization of the map h

We introduce the following linear two maps $\bar{h}$ (introduced by Viterbo in [52]) and ψ:

$$\bar{h}:(T^*\mathbb{R}^n\times T^*\mathbb{R}^n,\omega_{\mathbb{R}^n}\ominus\omega_{\mathbb{R}^n})\longrightarrow(T^*(T^*\mathbb{R}^n),\omega_{\mathbb{R}^{2n}}) \qquad (1.11)$$

$$(q,p,Q,P)\longmapsto(\frac{q+Q}{2},\frac{p+P}{2},p-P,Q-q),$$

$$\psi : (T^*(T^*\mathbb{R}^n) = T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \longrightarrow (T^*(T^*\mathbb{R}^n) = T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \tag{1.12}$$
$$(q,p,Q,P) := (x_0,y_0) \longmapsto (x_1,y_1) =: (\frac{2q-P}{2}, \frac{2p-Q}{2}, Q, \frac{2P+2p-Q}{2}).$$

It results well-defined the following map on the quotient tori structures

$$\widetilde{\psi} : T^*(T^*\mathbb{T}^n) \longrightarrow T^*(T^*\mathbb{T}^n)$$

$$([q],p,Q,P) \longmapsto ([\frac{2q-P}{2}], \frac{2p-Q}{2}, Q, \frac{2P+2p-Q}{2})$$

and the following diagram is commutative

$$\begin{tikzcd} T^*(T^*\mathbb{R}^n) \arrow[r, "\psi"] \arrow[d, "\pi"'] & T^*(T^*\mathbb{R}^n) \arrow[d, "\pi"] \\ T^*(T^*\mathbb{T}^n) \arrow[r, "\widetilde{\psi}"'] & T^*(T^*\mathbb{T}^n) \end{tikzcd}$$

Lemma 1.3.1 *The map ψ is a symplectic isomorphism.*

Proof

$$\psi^*\omega_{\mathbb{R}^{2n}} = dQ \wedge d(\frac{2q-P}{2}) + d(\frac{2P+2p-Q}{2}) \wedge d(\frac{2p-Q}{2}) =$$

$$= dQ \wedge dq - \frac{dQ \wedge dP}{2} + dP \wedge dp - \frac{dP \wedge dQ}{2} - \frac{dp \wedge dQ}{2} - \frac{dQ \wedge dp}{2} =$$

$$= dQ \wedge dq + dP \wedge dp = \omega_{\mathbb{R}^{2n}}.$$

□

Also the map $\bar{h}$ is a symplectic diffeomorphism (see [52]) and it is easy to check that the factorization $h = \psi \circ \bar{h}$ holds:

$$\begin{tikzcd} T^*\mathbb{R}^n \times T^*\mathbb{R}^n \arrow[r, "h"] \arrow[dr, "\bar{h}"'] & T^*(T^*\mathbb{R}^n) \\ & T^*(T^*\mathbb{R}^n) \arrow[u, "\psi"'] \end{tikzcd}$$

1.3.2 The Lagrangian submanifold $\bar{h}(\Gamma_K)$

This section is devoted to the proof of the following

Proposition 1.3.2 *The Lagrangian submanifold $\bar{h}(\Gamma_K) \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ admits the G.F.Q.I., $S_1(q,p;\xi)$, 2π-periodic in the q variables.*

Proof. We observe that like H also the Hamiltonian K is periodic of 2π-period in the q variables. Moreover the flow $\phi^t_K = \phi_0^{-t} \circ \phi^t_H$ inherits from the flow ϕ^t_H (see Prop. 1.2.4) the following properties

$$\begin{aligned}(\phi^t_K)_q(q+2\pi k, p) &= (\phi^t_K)_q(q,p) + 2\pi k,\\ (\phi^t_K)_p(q+2\pi k, p) &= (\phi^t_K)_p(q,p)\end{aligned}$$

$\forall (q,p) \in \mathbb{R}^{2n},\ \forall k \in \mathbb{Z}^n$.
Consequently, for all fixed $t \in \mathbb{R}$ a flow $\widetilde{\phi}^t_K$ in $T^*\mathbb{T}^n$ results well-defined, in particular, the following definition is independent of the choice of q in the class $[q]$:

$$\widetilde{\phi}^t_K([q],p) = ((\widetilde{\phi}^t_K)_q([q],p), (\widetilde{\phi}^t_K)_p([q],p)) := ([(\phi^t_K)_q(q,p)], (\phi^t_K)_p(q,p)),$$

$$\begin{tikzcd} T^*\mathbb{R}^n \arrow[r, "\phi^t_K"] \arrow[d, "\pi"'] & T^*\mathbb{R}^n \arrow[d, "\pi"] \\ T^*\mathbb{T}^n \arrow[r, "\widetilde{\phi}^t_K"'] & T^*\mathbb{T}^n \end{tikzcd}$$

Here we mean $\pi : (q,p) \to ([q],p)$.
Similarly to Γ_K, we indicate by $\widetilde{\Gamma}_K$ the graph of $\widetilde{\phi}^1_K$:

$$\widetilde{\Gamma}_K \subset (T^*\mathbb{T}^n \times T^*\mathbb{T}^n, \omega_{\mathbb{T}^n} \ominus \omega_{\mathbb{T}^n}).$$

The Lagrangian submanifold $\bar{h}(\Gamma_K)$:

$$\bar{h}(\Gamma_K) = \Big\{\Big(\frac{q + (\phi^1_K)_q(q,p)}{2}, \frac{p + (\phi^1_K)_p(q,p)}{2}, p - (\phi^1_K)_p(q,p), (\phi^1_K)_q(q,p) - q\Big),$$

$$\forall (q,p) \in T^*\mathbb{R}^n\Big\} \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}),$$

in a neighbourhood of infinity (in the p variables) results:

$$\bar{h}(\Gamma_K) = \{(q,p,0,0),\ \forall q \in \mathbb{R}^n,\ \forall p \in \mathbb{R}^n\ :\ |p| \geq C\}.$$

It is easy to verify that if $(q,p,Q,P) \in \bar{h}(\Gamma_K)$, then $\forall k \in \mathbb{Z}^n$ $(q+2\pi k, p, Q, P) \in \bar{h}(\Gamma_K)$. Therefore the Lagrangian submanifold $\bar{h}(\Gamma_K) \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ has a natural inclusion into $(T^*(\mathbb{T}^n \times \mathbb{R}^n), \omega_{\mathbb{T}^n \times \mathbb{R}^n})$. Now, we prove that $\bar{h}(\Gamma_K)$ coincides, up to the symplectic morphism $\widetilde{h}$ below from $\widetilde{\Gamma}_K$ to $T^*(\mathbb{T}^n \times \mathbb{R}^n)$, with the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}^1_K$. In order to see this, we introduce the following well-defined (independent of the choice of q in $[q]$)[2] map

$$\widetilde{h} : \widetilde{\Gamma}_K \longrightarrow T^*(\mathbb{T}^n \times \mathbb{R}^n)$$

$$([q], p, [(\phi_K^1)_q(q,p)], (\phi_K^1)_p(q,p)) \longmapsto$$

$$([\frac{q + (\phi_K^1)_q(q,p)}{2}], \frac{p + (\phi_K^1)_p(q,p)}{2}, p - (\phi_K^1)_p(q,p), (\phi_K^1)_q(q,p) - q).$$

Therefore the following diagram results commutative

$$\begin{tikzcd}
\Gamma_K \arrow[r, "\bar{h}"] \arrow[d, "\pi_1"'] & T^*\mathbb{R}^{2n} \arrow[d, "\pi_2"] \\
\widetilde{\Gamma}_K \arrow[r, "\widetilde{h}"'] & T^*(\mathbb{T}^n \times \mathbb{R}^n)
\end{tikzcd}$$

here we mean $\pi_1 : (q,p,Q,P) \to ([q], p, [Q], P)$, $\pi_2 : (q,p,Q,P) \to ([q], p, Q, P)$. Thus we have proved that $\bar{h}(\Gamma_K)$ results, up to the symplectic diffeomorphism $\widetilde{h}$, the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}_K^1$. On the other hand, the manifold $\widetilde{h}(\widetilde{\Gamma}_K)$ is essentially the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}_K^1$. In such hypothesis (see Theorem 1.1.4) the manifold $\widetilde{h}(\widetilde{\Gamma}_K)$ admits a G.F.Q.I., say $s([q], p, \xi)$. Then a G.F.Q.I. for $\bar{h}(\Gamma_K)$, say $S_1(q,p,\xi)$, can be obtained extending periodically (in the q variables) $s([q], p, \xi)$. □

1.3.3 A generating function for $h(\Gamma_K)$

In this section we build (see Lemma 1.3.3 below) a generating function for the linear symplectomorphism ψ. Combining it with the one above (see Proposition 1.3.2), we will state the existence of a generating function for $h(\Gamma_K)$ (see Proposition 1.3.4).
The following composition rule is popular in symplectic geometry and mechanics, see e.g. [6], [7], and it has been handled by Laudenbach and Sikorav in meaningful problems in symplectic topology (see [47]).

Lemma 1.3.3 *The linear symplectomorphism ψ –see (1.12)– admits the generating function $S_2(x_0, x_1)$:*

$$S_2(x_0,x_1) = \frac{1}{2}\left\langle x_0, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} x_0 \right\rangle - \left\langle x_0, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} x_1 \right\rangle +$$

$$+\frac{1}{2}\left\langle x_1, \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix} x_1 \right\rangle.$$

[2]We note that, unlike the map $\bar{h}$, it does not exist a natural definition of $\widetilde{h}$ from $T^*\mathbb{T}^n \times T^*\mathbb{T}^n$ in $T^*(\mathbb{T}^n \times \mathbb{R}^n)$, since it is essential the property: $(\phi_K^1)_q(q + 2\pi k, p) = (\phi_K^1)_q(q,p) + 2\pi k$.

(See also [44], p. 280).
Proof
Recalling the map ψ in details,

$$\begin{aligned} x_0 &= (q,p) \\ x_1 &= (\frac{2q-P}{2}, \frac{2p-Q}{2}) \\ y_0 &= (Q,P) \\ y_1 &= (Q, \frac{2P+2p-Q}{2}), \end{aligned}$$

we proceed to verify by direct calculation:

$$-\frac{\partial S_2}{\partial x_0}(x_0,x_1)\big|_{x_0=(q,p),\, x_1=(\frac{2q-P}{2},\frac{2p-Q}{2})} = -\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \frac{2q-P}{2} \\ \frac{2p-Q}{2} \end{pmatrix} =$$

$$= (2p,2q) + (Q-2p, P-2q) = (Q,P) = y_0,$$

$$\frac{\partial S_2}{\partial x_1}(x_0,x_1)\big|_{x_0=(q,p),\, x_1=(\frac{2q-P}{2},\frac{2p-Q}{2})} = -(q,p)\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{2q-P}{2} \\ \frac{2p-Q}{2} \end{pmatrix} =$$

$$= (2p,2q) + (Q-2p, P-2q+\frac{2p-Q}{2}) = (Q, \frac{2P+2p-Q}{2}) = y_1.$$

□
We can now prove the following

Proposition 1.3.4 *The Lagrangian submanifold $h(\Gamma_K)$ admits the generating function $S(x_1;x_0,\xi)$:*

$$S(x_1;x_0,\xi) = S_1(x_0,\xi) + S_2(x_0,x_1).$$

Remark. Note that the variables x_0 now are interpreted as auxiliary parameters, at the same level of ξ.
Proof
The symplectomorphism ψ is generated by $S_2(x_0,x_1)$, that is

$$\psi(x_0,y_0) = (x_1,y_1) \text{ iff } \begin{cases} y_0 = -\frac{\partial S_2}{\partial x_0}(x_0,x_1) \\ y_1 = \frac{\partial S_2}{\partial x_1}(x_0,x_1) \end{cases}$$

$$\frac{\partial S}{\partial x_0}(x_1;x_0,\xi) = 0 \text{ means } \frac{\partial S_1}{\partial x_0}(x_0,\xi) + \frac{\partial S_2}{\partial x_0}(x_0,x_1) = 0, \text{ that is, } y_0 = \frac{\partial S_1}{\partial x_0}(x_0,\xi).$$

Furthermore,

$$\frac{\partial S}{\partial \xi}(x_1;x_0,\xi) = 0 \quad \text{iff} \quad \frac{\partial S_1}{\partial \xi}(x_0,\xi) = 0.$$

Therefore the Lagrangian submanifold generated by $S(x_1;x_0,\xi)$ results

$$\left\{(x_1,y_1)=(x_1,\frac{\partial S}{\partial x_1}(x_1;x_0,\xi)):\ \frac{\partial S}{\partial x_0}(x_1;x_0,\xi)=0\ ,\ \frac{\partial S}{\partial \xi}(x_1;x_0,\xi)=0\right\}=$$

$$=\left\{(x_1,y_1)=(x_1,\frac{\partial S}{\partial x_1}(x_1;x_0,\xi)):\ y_0=\frac{\partial S_1}{\partial x_0}(x_0,\xi)\ ,\ \frac{\partial S_1}{\partial \xi}(x_0,\xi)=0\right\}=$$

$$=\left\{(x_1,y_1)=(x_1,\frac{\partial S_2}{\partial x_1}(x_0,x_1)):\ y_0=\frac{\partial S_1}{\partial x_0}(x_0,\xi)\ ,\ \frac{\partial S_1}{\partial \xi}(x_0,\xi)=0\right\}=$$

$$=\left\{(x_1,y_1):\ (x_1,y_1)=\psi(x_0,y_0) \text{ with } (x_0,y_0)\in\bar{h}(\Gamma_K)\right\}=\psi(\bar{h}(\Gamma_K))=h(\Gamma_K).$$

□

1.3.4 The Quadratic at Infinity property

We are ready to look for fixed points of ϕ_H^1, that is, to estimate

$$\#\ (h(\Gamma_K)\cap 0_{\mathbb{R}^{2n}}).$$

These intersection points are exactly the global critical points of the generating function S for $h(\Gamma_K)$. More precisely, by the Proposition 1.3.5 below, we show that they are essentially (that is to say, up to periodicity) the critical points for a G.F.Q.I. f defined on a domain contracting to the torus $\mathbb{T}^n$: this is crucial in order to gain, in the Lusternik-Schnirelman format, a lower bound estimate of the number of fixed points of ϕ_H^1.
Although in the previous Section we managed with a formal expression of S_2, by a straightforward computation we easily find out the simplified structure[3] of it:

$$S_2(x_0,x_1)=S_2(q_0,p_0,q_1,p_1)=2(p_0-p_1)\cdot(q_1-q_0)+\frac{p_1^2}{2}.$$

Proposition 1.3.5 *The fixed points of ϕ_H^1 correspond to the critical points of the G.F.Q.I.*

$$f:\mathbb{T}^n\times\mathbb{R}^{3n+k}\longrightarrow\mathbb{R}$$

$$([q_1],p_1,v,p_0,\xi)\overset{f}{\longmapsto}S_1([q_1-v],p_0+p_1,\xi)+2p_0\cdot v+\frac{p_1^2}{2}\qquad(1.13)$$

Proof. Using the notation $x_1=(q_1,p_1)$ and $x_0=(q_0,p_0)$ we can rewrite S as

$$S:\mathbb{R}^{4n+k}\longrightarrow\mathbb{R}$$

$$(q_1,p_1,q_0,p_0,\xi)\longmapsto S_1(q_0,p_0,\xi)+2(p_0-p_1)\cdot(q_1-q_0)+\frac{p_1^2}{2}.$$

[3]here, for opportunity, we write $S_2(q_0,p_0,....)$ instead of $S_2(q,p,....)$

There is an evident invariance property:

$$S(q_1+2\pi k, p_1, q_0+2\pi k, p_0, \xi) = S(q_1, p_1; q_0, p_0, \xi)$$

$\forall (q_1, p_1, q_0, p_0, \xi) \in \mathbb{R}^{4n+k}$ and $\forall k \in \mathbb{Z}^n$. This fact is the same as saying that S is constant over the fibers of the surjective map Π below, thus it results well-defined the following real-valued function $\widetilde{S}$:

$$\begin{tikzcd} \mathbb{R}^{4n+k} \arrow[r, "S"] \arrow[d, "\Pi"'] & \mathbb{R} \\ \mathbb{T}^n \times \mathbb{R}^{3n+k} \arrow[ur, "\widetilde{S}"'] & \end{tikzcd} \tag{1.14}$$

$$\begin{tikzcd} (q_1, p_1, q_0, p_0, \xi) \arrow[r, "S"] \arrow[d, "\Pi"'] & \mathbb{R} \\ ([q_1], p_1, q_1 - q_0, p_0, \xi) \arrow[ur, "\widetilde{S}"'] & \end{tikzcd} \tag{1.15}$$

$$\Pi^{-1}([q_1], p_1, v, p_0, \xi) = \{(q_1+2\pi k, p_1, q_1 - v + 2\pi k, p_1, \xi) : \ k \in \mathbb{Z}^n\} \tag{1.16}$$

$$\widetilde{S} : \mathbb{T}^n \times \mathbb{R}^{3n+k} \longrightarrow \mathbb{R}$$

$$([q_1], p_1, v, p_0, \xi) \longmapsto S_1([q_1 - v], p_0, \xi) + 2(p_0 - p_1)\cdot v + \frac{p_1^2}{2} \tag{1.17}$$

satisfying the property:

$$\widetilde{S} \circ \Pi = S \tag{1.18}$$

Furthermore, since $d\widetilde{S}(y)_{|_{y=\Pi(x)}} \circ d\Pi(x) = dS(x)$, we have that ($rk\ d\Pi =$ max): $\Pi^{-1}(\text{Crit }(\widetilde{S})) = \text{Crit }(S)$. Now $S_1([q_1 - v], p_0 + p_1, \xi)$ coincides for $|\xi| > C$ with a nondegenerate quadratic form $(A\xi, \xi)$, then for $|p_1|$, $|v|$, $|p_0|$, $|\xi| > C$ and for any fixed $[q_1] \in \mathbb{T}^n$, $f([q_1], p_1, v, p_0, \xi) = Q(p_1, v, p_0, \xi)$ where $Q(p_1, v, p_0, \xi)$ is the nondegenerate quadratic form

$$Q(p_1, v, p_0, \xi) := \begin{pmatrix} \frac{1}{2} & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & A \end{pmatrix} \begin{pmatrix} p_1 \\ v \\ p_0 \\ \xi \end{pmatrix} \begin{pmatrix} p_1 \\ v \\ p_0 \\ \xi \end{pmatrix}.$$

Therefore f is a G.F.Q.I. $\square$

1.3.5 Fixed points: Degenerate case

We conclude this Section with the estimate in the possible degenerate case, first proved by Conley and Zehnder [27], of which we propose a proof based on the Quadratic at Infinity property of the generating function f.

Theorem 1.3.6 *Let ϕ_H^1 be the time-one map of a Hamiltonian $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ with the properties:*

$$H(t, q + 2\pi k, p) = H(t, q, p), \quad \forall (t, q, p) \in \mathbb{R} \times \mathbb{R}^{2n},\ \forall k \in \mathbb{Z}^n,$$

and

$$H(t, q, p) = \frac{1}{2}|p|^2 \quad \textit{if} \quad |p| \geq C > 0.$$

Then ϕ_H^1 has at least $n+1$ fixed points and they correspond to homotopically trivial closed orbits of the Hamiltonian flow.

Proof Fixed points of ϕ_H^1 correspond to critical points of f (see Proposition 1.3.5). Moreover, via the Lusternik-Schnirelman theory (see Theorem 1.1.10), critical values of f can be detected involving non-vanishing relative cohomology classes in $H^*(f^c, f^{-c})$. As a consequence, and since $f : \mathbb{T}^n \times \mathbb{R}^{3n+k} \to \mathbb{R}$ is a G.F.Q.I., Corollary 1.1.12 does work, so that we obtain the well-known estimate:

$$\#\ \mathrm{fix}(\phi_H^1) = \#\ \mathrm{crit}(f) \geq \mathrm{cl}(\mathbb{T}^n) = n + 1$$

□

1.4 Fixed points: Nondegenerate case

Whenever all the fixed points of ϕ_H^1 are *a priori* nondegenerate, so that the corresponding critical points of f are, it happens that the G.F.Q.I. f becomes also a so-called *Morse function*, and in this case we caught a rather better estimate.

Definition 1.4.1 *Let N be a smooth manifold. A fixed point $x \in N$ of a diffeomorphism $\Phi : N \to N$ is said nondegenerate if the graph of Φ intersects the diagonal of $N \times N$ transversally at (x, x), that is,*

$$\det\big(d\Phi(x) - \mathbb{I}\big) \neq 0.$$

The notion of nondegeneracy for fixed points of diffeomorphisms corresponds to the notion of nondegeneracy for critical points of functions, originally due to Morse.

Definition 1.4.2 *Let N be a smooth manifold and $f : N \to \mathbb{R}$ be a C^2 function. A critical point x for f, $\nabla f(x) = 0$, is said nondegenerate if the Hessian $\frac{\partial^2 f}{\partial x^i \partial x^j}(x)$ of f at x is nondegenerate.*

(Recall that the Hessian of a scalar function f at its critical points is a well-defined tensorial object.) Starting from the study of the sublevel sets N^ν (see (1.3)), where ν is not a critical value of f, Morse proved the following famous lower bound on the number of critical points of f.

Theorem 1.4.3 *(Morse inequality) Let N be a compact manifold and $f : N \to \mathbb{R}$ be a Morse function. Then*

$$\mathrm{crit}(f) \geq \sum_{k=0}^{dimN} H^k(N) =: \sum_{k=0}^{dimN} b_k(N),$$

where the values $b_k(N)$ are called the Betti numbers of N.

As in the degenerate case, the preceding estimate still holds when $f : N \times \mathbb{R}^n \to \mathbb{R}$ is a G.F.Q.I. (see for example [24]):

Theorem 1.4.4 *Let N be a compact manifold and $f : N \times \mathbb{R}^n \to \mathbb{R}$ be a G.F.Q.I. If all the critical points of f are nondegenerate, then*

$$\mathrm{crit}(f) \geq \sum_{k=0}^{dimN} b_k(N).$$

The expected estimate on the number of nondegenerate fixed points for the Hamiltonian flow ϕ^1_H is a straight consequence of the above Theorem 1.4.4.

Theorem 1.4.5 *Same hypothesis of the Theorem 1.3.6. Then ϕ^1_H has at least 2^n nondegenerate fixed points and they correspond to homotopically trivial closed orbits of the Hamiltonian flow.*

Proof Nondegenerate fixed points of ϕ^1_H correspond (via the diffeomorphisms h and ψ) to transversal intersections between $h(\Gamma_K)$ and $0_{\mathbb{R}^{2n}}$. We observe now that the Lagrangian submanifold $h(\Gamma_K)$ intersects transversally $0_{\mathbb{R}^{2n}}$ in the point $(\bar{q}, \bar{p}, \bar{u}) := (\bar{x}, \bar{u}) \in h(\Gamma_K)$ if

$$\det\big(\frac{\partial^2 S}{\partial x^i \partial x^j}\big)(\bar{x}, \bar{u}) \neq 0. \tag{1.19}$$

Moreover, since the point $(\bar{x}, \bar{u}) \in h(\Gamma_K)$, the transversality condition guarantees that

$$\mathrm{rk}\big(\frac{\partial^2 S}{\partial x^i \partial u^j}, \frac{\partial^2 S}{\partial u^i \partial u^j}\big)(\bar{x}, \bar{u}) = \max. \tag{1.20}$$

Then, from the conditions (1.19) and (1.20), we conclude that the nondegenerate fixed points of ϕ^1_H correspond exactly to the nondegenerate critical points of S, which are essentially (that is up to periodicity) the nondegenerate critical points of f. Now $f : \mathbb{T}^n \times \mathbb{R}^{3n+k} \longrightarrow \mathbb{R}$ is a G.F.Q.I., then, as a consequence of Theorem 1.4.4, we obtain

$$\# \text{ nondeg-fix}(\phi^1_H) = \# \text{ nondeg-crit}(f) \geq 2^n.$$

□

Chapter 2

Viscosity and minimax solutions to Hamilton-Jacobi equations

We review some aspects of the Cauchy Problem (CP) for Hamilton-Jacobi equations of evolutive type:

$$(CP)\begin{cases} \frac{\partial S}{\partial t}(t,q) + H\left(t,q,\frac{\partial S}{\partial q}(t,q)\right) = 0, \\ \\ S(0,q) = \sigma(q), \end{cases}$$

$t \in [0,T]$, $q \in N$, where N is a smooth connected manifold without boundary.
For T small enough, the unique classical solution to (CP) is determined using the characteristics method. However, even though H and σ are smooth, in general there exists a critical time in which the classical solution breaks down: it becomes multivalued, i.e. the q-components of some characteristics cross each other. Hence, it turns out the question how to define, and then to determine, weak (e.g., continuous and almost everywhere differentiable) global solutions of (CP).

In the eighties, Crandall, Evans and Lions introduced the notion of viscosity solution for Hamilton-Jacobi equations, see [41] and [4] for a detailed review on the subject. Bardi and Evans [5], using Hopf's formulas, directly constructed viscosity solutions for convex Liouville-integrable Hamiltonians like $H = H(p)$.

Afterwards, in 1991 Chaperon and Sikorav proposed in a geometric framework a new type of weak solutions for (CP), called minimax solutions (see [25], [53], [45]). Their definition is based on generating functions

quadratic at infinity (G.F.Q.I.) of the Lagrangian submanifold L obtained by gluing together the characteristics of the Hamiltonian vector field $X_{\mathcal{H}}$ where $\mathcal{H}(t,q,\tau,p) = \tau + H(t,q,p)$. This global object L resumes geometrically the multi-valued features of the Hamilton-Jacobi problem, like a sort of Riemann surface (see e.g. [54]) occurring in complex analysis. A discussion on the construction of global generating functions of L related to viscosity solutions has been made in [16] in the very case of existence of a complete solution ("complete integral") of Hamilton-Jacobi equation.
In this new topological framework, a lot of examples can be found and produced, even outside the classical mechanics: e.g. in control theory [11], or in multi-time theory of Hamilton-Jacobi equation [19].

Viscosity and minimax solutions have the same analytic properties, namely, theorems of existence and uniqueness hold, but in general they are different, see [45]. In [38] Joukovskaia indicated that viscosity and minimax solutions of (CP) coincide, provided that the Hamiltonian H is convex in the p variables. The task of chapter 4 is to furnish a detailed proof of this fact.

In this chapter, after a discussion on two fundamental routes —weak discontinuity waves and high frequency asymptotic waves— both leading to Hamilton-Jacobi equation, we review the two above mentioned notions of weak solution for it, the minimax solution and the viscosity solution. Furthermore, we construct an extension of the generating function involving in the Hopf's formula:

$$u_{visc}(t,q) = \inf_{\chi\in\mathbb{R}^n} \sup_{v\in\mathbb{R}^n} \{-H(v)\,t + (q-\chi)\cdot v + \sigma(\chi)\}\,. \tag{2.1}$$

for more general non-integrable Hamiltonians; this is performed on the torus $N = \mathbb{T}^n$. This result is caught by utilizing (i) a very fruitful, even though scarcely known, theorem of Hamilton (e.g. quoted by Gantmacher [33] as "Perturbation Theory"), (ii) a classical composition rule of generating functions in symplectic geometry [6], and (iii) the existence theorem by Chaperon-Laudenbach-Sikorav-Viterbo of global generating functions for Lagrangian submanifolds related to compact support Hamiltonians.

The sequel is organized as follows.
In Section 2.1 we consider discontinuity and asymptotic wave propagation, leading us to Hamilton-Jacobi equation. In Section 2.2 we recall the notion of viscosity solution for the Hamilton-Jacobi equation. Sections 2.3-2.4 are devoted to the construction of the minimax solution of (CP). In Section 2.5, for $N = \mathbb{T}^n$, we explicitly write down a generating function with finite parameters for the Lagrangian submanifold geometric solution of (CP) for the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$, $q \in \mathbb{T}^n$, f compact support.

We finally note that in literature the term minimax solution is often used to indicate a third approach to generalized solutions of (CP); this alternative approach –which lies outside the tasks of the present Thesis– has been clarified to be equivalent to the concept of viscosity solution, see for example [48], [49] and [46].

2.1 Some routes to Hamilton-Jacobi equation

The work reviewed in this section is aimed at recollecting some fundamental routes to Hamilton-Jacobi equations, even outside the classical arena of analytical mechanics where this equation naturally arises.

We reconsider discontinuity and asymptotic wave propagation –see 2.1.1 and 2.1.2 below– leading us to Hamilton-Jacobi equation.

2.1.1 Discontinuities

Let consider a general semi-linear evolution system of partial differential equations:

$$\frac{\partial u^i}{\partial t} + \sum_{L=1,\, j=1}^{d\, n} A_j^{i\,L}(t,q) \frac{\partial u^j}{\partial q^L} = b^i(t,q) \tag{2.2}$$

As is well known, it modelizes e.g. Maxwell equations or non-homogeneous linear elasticity. Weak discontinuities have support on propagating wave in the space-time $\mathbb{R}^{d+1}$ described by

$$\Phi(t, q^L) = 0 \qquad (\Phi : \mathbb{R}^{d+1} \to \mathbb{R}) \tag{2.3}$$

by denoting, as usual, $\left[\frac{\partial u^i}{\partial t}\right]$ and $\left[\frac{\partial u^i}{\partial q^L}\right]$ possible discontinuities of the derivatives of u^i through $\Phi = 0$, we obtain

$$\left[\frac{\partial u^i}{\partial t}\right] + \sum_{L=1,\, j=1}^{d\, n} A_j^{i\,L}(t,q) \left[\frac{\partial u^j}{\partial q^L}\right] = 0 \tag{2.4}$$

By recalling the Hugoniot-Hadamard compatibility conditions,

$$\left[\frac{\partial u^i}{\partial t}\right] = -\lambda^i v\,, \qquad \left[\frac{\partial u^i}{\partial q^L}\right] = -\lambda^i n_L\,, \tag{2.5}$$

where λ^i is the size of the jump, v the normal velocity of the wave, and n_L is the normal unit vector of $\Phi = 0$, in some more detail,

$$v = -\frac{\frac{\partial \Phi}{\partial t}}{\sqrt{\sum_{L=1}^{d} \left(\frac{\partial \Phi}{\partial q^L}\right)^2}}\,, \qquad n_L = \frac{\frac{\partial \Phi}{\partial q^L}}{\sqrt{\sum_{L=1}^{d} \left(\frac{\partial \Phi}{\partial q^L}\right)^2}}\,, \tag{2.6}$$

we write

$$\sum_{j=1}^{n} \left[\delta_j^i \frac{\partial \Phi}{\partial t} - \sum_{L=1}^{d} A_j^{i\,L}(t,q) \frac{\partial \Phi}{\partial q^L}\right] \lambda^j = 0\,, \tag{2.7}$$

hence non trivial solutions occur if

$$\det\left(\delta^i_j\frac{\partial\Phi}{\partial t}-\sum_{L=1}^{d}A^{i\,L}_j(t,q)\frac{\partial\Phi}{\partial q^L}\right)=0 \tag{2.8}$$

A standard irreducible factorization of (2.8), like $...\cdot\mathcal{F}_{\alpha-1}\cdot\mathcal{F}_\alpha\cdot\mathcal{F}_{\alpha+1}\cdot...=0$, produces Hamilton-Jacobi equations $\mathcal{F}_\alpha=0$ for unknown functions Φ describing waves:

$$\mathcal{F}_\alpha\left(t,q,\frac{\partial\Phi}{\partial t},\frac{\partial\Phi}{\partial q}\right)=0. \tag{2.9}$$

For example, we ascertain in this framework *longitudinal* and *transversal* propagation waves of linear elasticity. And the universal suggestion for the reader is to make for the beautiful booklets by Levi Civita [39] and Boillat [13].

2.1.2 Asymptotics

We can discover another route to H-J equation belonging to the high frequency asymptotic approximation of semi-linear partial differential equations, like Schrödinger equation in quantum mechanics; for a particle of mass $m=1$ in a field generated by the potential energy $V(t,q)$ (here $\varepsilon=h/2\pi$, the Planck constant, is the 'small' parameter):

$$i\varepsilon\frac{\partial\psi}{\partial t}(t,q)=-\frac{\varepsilon^2}{2}\Delta\psi(t,q)+V(t,q)\psi(t,q), \tag{2.10}$$

$t\in\mathbb{R},\ q\in\mathbb{R}^n$. Trying to solve (2.10) by a (highly) oscillating integral like

$$I(t,q;\varepsilon)=\int_{u\in U}b(t,q,u;\varepsilon)e^{\frac{i}{\varepsilon}\Phi(t,q,u)}du,$$

($U\subset\mathbb{R}^k$), which is a sort of superposition of oscillating functions, we produce, for some *amplitude* b and (real) *phase* Φ, independent of ε,

$$\int_{u\in U}\left(\frac{\partial\Phi}{\partial t}+\frac{1}{2}|\nabla_q\Phi|^2+V\right)b\,e^{\frac{i}{\varepsilon}\Phi(t,q,u)}du\ +O(\varepsilon)=0$$

Non trivial amplitudes are admissible if the phase satisfies the H-J equation

$$\frac{\partial\Phi}{\partial t}+\frac{1}{2}|\nabla_q\Phi|^2+V=0 \tag{2.11}$$

which is exactly a H-J evolution equation, related to the Hamiltonian of the connected classical model of the physical system: $H(t,q,p)=\frac{1}{2}|p|^2+V$.

2.2 Viscosity solutions of H-J equations

In this section we review some aspects of the basic theory of continuous viscosity solutions of the Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t} + H(t, q, \frac{\partial u}{\partial q}) = 0, \tag{2.12}$$

$t \in (0, T)$, $q \in N$. Special attention will be devoted later to the case where $H = H(p)$ and $p \mapsto H(p)$ is convex.

Definition 2.2.1 *A function $u \in C((0, T) \times N)$ is a viscosity subsolution [supersolution] of (2.12) if, for any $\phi \in C^1((0, T) \times N)$,*

$$\frac{\partial \phi}{\partial t}(\bar{t}, \bar{q}) + H(\bar{t}, \bar{q}, \frac{\partial \phi}{\partial q}(\bar{t}, \bar{q})) \leq 0 \qquad [\geq 0] \tag{2.13}$$

at any local maximum [minimum] point $(\bar{t}, \bar{q}) \in (0, T) \times N$ of $u - \phi$. Finally, u is a viscosity solution of (2.12) if it is simultaneously a viscosity sub- and supersolution.

The origin of the term "viscosity solution" is going back to the vanishing viscosity method:

$$-\varepsilon \Delta u_\varepsilon(q) + H(q, \frac{\partial u_\varepsilon}{\partial q}(q)) = 0, \qquad q \in N. \tag{2.14}$$

In this case, the Hamiltonian of the problem is given by

$$H_\varepsilon(q, p, M) = -\varepsilon \mathrm{tr}(M) + H(q, p),$$

converging in $C(N \times \mathbb{R}^n \times Sym_{n \times n})$ to $H(q, p)$. Giving a solution of (2.14), a natural question arises: if $\varepsilon \to 0$ does u_ε tends to a function u, solution (in some sense) of the limit equation $H(q, \frac{\partial u}{\partial q}(q)) = 0$?
The question is not so easy because the regularizing effect of the term $\varepsilon \Delta u_\varepsilon$ vanishes as $\varepsilon \to 0$ and we end up with an equation that has easily non regular solutions. The answer is that if $u_\varepsilon \to u$ uniformly on every compact sets, then u is a viscosity solution. This is actually the motivation for the terminology "viscosity solution", used in the original paper of Crandall and Lions [28].

Analogously for minimax solutions, existence and uniqueness theorems hold for viscous ones. Moreover, Bardi and Evans [5] directly constructed viscosity solutions for Liouville-integrable and convex Hamiltonians $H(p)$. Their representation of solutions is based on a Hopf's formula and on an inf-sup procedure on auxiliary parameters:

$$u_{visc}(t, q) = \inf_{\chi \in \mathbb{R}^n} \sup_{v \in \mathbb{R}^n} \left\{ -H(v) t + (q - \chi) \cdot v + \sigma(\chi) \right\}. \tag{2.15}$$

The generating function involved in this representation formula,

$$S(t,q,\underbrace{(\chi,v)}_{\xi}) = -H(v)t + (q-\chi)\cdot v + \sigma(\chi)\,,$$

results quadratic at infinity under auxiliary hypothesis: for example, σ compact support and $H(p)=\frac{1}{2}|p|^2$.
The plan of construct viscosity solutions starting from generating functions has been rather fruitless; nevertheless, we can find similar representation formulas for state-dependent Hamiltonians, see [15] and [43], although they hold only under suitable restrictive assumptions.

2.3 Extension of exact Lagrangian isotopies

Conventions. The considered applications are of class C^∞ and the spaces of applications are assumed to be endowed with the C^∞ topology. Given a space of applications E, a *path* into E is a map $t\mapsto f_t$, denoted by (f_t), from $I:=[0,1]$ into E, such that the application $(t,x)\mapsto f_t(x)$ is of class C^∞. For a generic manifold P, we denote by $E(P)$ the space of functions $f:P\to\mathbb{R}$.

Let now (P,ω) be a smooth, connected, symplectic manifold of dimension $2n$. An embedding $j:\Lambda\to P$ is called *Lagrangian* if Λ is of dimension n and $j^*\omega=0$. In this case, $j(\Lambda)$ is called Lagrangian submanifold of (P,ω). The symplectic manifold (P,ω) is *exact* when ω admits a global primitive λ; given a primitive λ, an embedding $j:\Lambda\to P$ such that $j^*\lambda$ is exact is called *exact Lagrangian embedding*.

The following results in symplectic geometry will be used in Section 2.4.

Theorem 2.3.1 *(Weinstein) For every Lagrangian embedding $j:\Lambda\to P$, there exists an open neighbourhood U of the zero section of $T^*\Lambda$ and an embedding $J:U\to P$ such that $J^*\omega = d\lambda_\Lambda|_U$, where λ_Λ denotes the Liouville 1-form on $T^*\Lambda$. Moreover, if $0_\Lambda:\Lambda\to T^*\Lambda$ is the zero 1-form on Λ, we have $j=J\circ 0_\Lambda$.*

We call J a *tubular neighbourhood of j for ω* when the open set $U\cap T^*_x\Lambda$ is star-shaped with respect to the origin for every $x\in\Lambda$.
We denote $Emb(\Lambda,\omega)$ the space of Lagrangian embeddings of Λ into P. A *Lagrangian isotopy* of a manifold Λ in (P,ω) is a path into $Emb(\Lambda,\omega)$.

Definition 2.3.2 *(Exact Lagrangian isotopy) Let (j_t), $j_t:\Lambda\to P$, be a Lagrangian isotopy of a manifold Λ in (P,ω). Then (j_t) is called exact when, for every $t\in I$ and every local primitive λ of ω in a neighbourhood of $j_t(\Lambda)$, the 1-form $\frac{d}{dt}j_t^*\lambda$ is exact on Λ.*

We note that a Lagrangian isotopy (j_t) such that j_t is exact for every $t \in I$, results an exact Lagrangian isotopy, in fact, in such a case:

$$\frac{d}{dt} j_t^* \lambda = d_x \left[\frac{\partial f}{\partial t} (t, x) \right].$$

We refer to [20] for a detailed proof of the following

Theorem 2.3.3 *(Extension of exact Lagrangian isotopies) For every isotopy (j_t) of a compact manifold Λ in (P, ω), the following two properties are equivalent:*
a) (j_t) is an exact Lagrangian isotopy of Λ in (P, ω).
b) j_0 is Lagrangian and there exists a Hamiltonian isotopy (ϕ_t), with compact support, such that $j_t = \phi_t \circ j_0$ for every $t \in I$.

Proof. $b) \Rightarrow a)$. Let (h_t) be the Hamiltonian associated to (ϕ_t). For every local primitive λ of ω in a neighbourhood of $j_t(\Lambda)$, we have

$$\frac{d}{dt} j_t^* \lambda = \frac{d}{dt} (\phi_t \circ j_0)^* \lambda = j_0^* \frac{d}{dt} \phi_t^* \lambda = j_0^* \phi_t^* L_{\dot{\phi}_t} \lambda =$$

$$= j_t^* \left(i_{\dot{\phi}_t} \omega + d i_{\dot{\phi}_t} \lambda \right) = d \left(j_t^* \left(h_t + i_{\dot{\phi}_t} \lambda \right) \right),$$

hence a) holds.
$a) \Rightarrow b)$. Following an idea of Moser, we define for every $t \in I$ a vector field ξ_t on P through

$$\xi_t \circ j_t = \frac{d}{dt} j_t.$$

If λ is a primitive of ω in a neighborhood of $j_t(\Lambda)$, zero on $j_t(\Lambda)$ (see Theorem 2.3.1), we have

$$j_t^* \left(i_{\xi_t} \omega (\cdot) \right) = j_t^* \omega (\xi_t, \cdot) = j_t^* \omega \left(j_t^* \xi_t, j_t^* (\cdot) \right) =$$

$$= j_t^* \omega \left(\frac{d}{dt} j_t, j_t^* (\cdot) \right) = i_{\frac{d}{dt} j_t} j_t^* \omega \left(j_t^* (\cdot) \right) =$$

$$= L_{\frac{d}{dt} j_t} \lambda \left(j_t^* (\cdot) \right) - d i_{\frac{d}{dt} j_t} \lambda \left(j_t^* (\cdot) \right).$$

Now the term $d i_{\frac{d}{dt} j_t} \lambda \left(j_t^* (\cdot) \right) = 0$, because we have assume that λ is 0 on $j_t(\Lambda)$. Hence

$$j_t^* \left(i_{\xi_t} \omega \right) = L_{\xi_t \circ j_t} \lambda \left(j_t^* \right) = L_{j_t^* (\xi_t)} \lambda \left(j_t^* \right) = j_t^* L_{\xi_t} \lambda = \frac{d}{dt} \left(j_t^* \lambda \right),$$

that is

$$j_t^* \left(i_{\xi_t} \omega \right) = \frac{d}{dt} \left(j_t^* \lambda \right).$$

Now we define $\tilde{j}(t,x) := (t, j_t(x))$. Since j_t is exact (i.e. the 1-form $\frac{d}{dt}j_t^*\lambda$ is exact on Λ), we deduce the existence of a function $h^0 : \tilde{j}(I \times \Lambda) \to \mathbb{R}$ with the following property: for every $t \in I$, $j_t^*(i_{\xi_t}\omega) = \frac{d}{dt}(j_t^*\lambda) = j_t^* dh_t^0$, where $h_t^0 : j_t(\Lambda) \to \mathbb{R}$ is defined by $h_t^0(x) := h^0(t,x)$. Then we have constructed a path (h_t) with compact support in $E(P)$, such that $h_t^0 = h_t|_{j_t(\Lambda)}$ and $i_{\xi_t}\omega = dh_t|_{j_t(\Lambda)}$, for every $t \in I$. The Hamiltonian flow related with (h_t) satisfy b). $\square$

2.4 Geometric and minimax solutions

Before discussing the second announced type of weak solution –the minimax solution– we recall the concept of *geometric solution* of Hamilton-Jacobi equation: it is a *Lagrangian* submanifold L obtained by gluing together the characteristics of the Hamiltonian vector field $X_{\mathcal{H}}$, where $\mathcal{H}(t,q,\tau,p) = \tau + H(t,q,p)$. The geometric solution L was intended to be a global object, showing, among other things, the multivalue features of the H-J problem. Multivalues, if any, produce in turn singularities, which were studied in past decades by theoretical physicists and, mainly, by mathematicians like Thom, Arnol'd and Mather, producing *singularity theory*[1], see [3].
From the one side, by means of several operations, recalled in Section 1.3 of the previous chapter, it was possible to make a theory of *local* classification of Lagrangian singularities.
From the other side, in the more recent *global* theory of generating functions –namely, *symplectic topology*– their use appears more extensive: up to these operations, uniqueness is reached for such mathematical tools –the generating functions, describing our involved Lagrangian submanifolds.

Let N be a smooth, connected and closed (i.e. compact and without boundary) manifold. Let us consider the Cauchy problem (CP). We suppose that the Hamiltonian $H : \mathbb{R} \times T^*N \to \mathbb{R}$ is of class C^2 and the initial condition $\sigma : N \to \mathbb{R}$ is of class C^1.
Let $\mathbb{R} \times N$ be the "space-time", $T^*(\mathbb{R} \times N) = \{(t,q,\tau,p)\}$ its cotangent bundle (endowed with the standard symplectic form $dp \wedge dq + d\tau \wedge dt$) and $\mathcal{H}(t,q,\tau,p) = \tau + H(t,q,p)$.
In order to overcome the difficulties arising from the obstruction to existence of global solutions, we search for Lagrangian submanifolds $L \subset T^*(\mathbb{R} \times N)$ satisfying the following geometric version of Hamilton-Jacobi equation:

$$L \subset \mathcal{H}^{-1}(0).$$

But how to obtain such an L? We explain now the procedure.
Let $\Phi^t : \mathbb{R} \times T^*(\mathbb{R} \times N) \to T^*(\mathbb{R} \times N)$ be the flow generated by the Hamil-

[1] sometimes called catastrophe theory

tonian $\mathcal{H} : T^* (\mathbb{R} \times N) \to \mathbb{R}$, $\mathcal{H} (t, q, \tau, p) = \tau + H (t, q, p)$:

$$\begin{cases} \dot{t} = & 1 \\ \dot{q} = & \frac{\partial H}{\partial p} \\ \dot{\tau} = & -\frac{dH}{dt} \\ \dot{p} = & -\frac{\partial H}{\partial q} \end{cases}$$

and Γ_σ be the initial data submanifold:

$$\Gamma_\sigma := \{(0, q, -H (0, q, d\sigma (q)), d\sigma (q)) : \ q \in N\} \subset \mathcal{H}^{-1} (0) \subset T^* (\mathbb{R} \times N).$$

We note that Γ_σ is the intersection of the Lagrangian submanifold $\Lambda_\sigma = \{(0, q, t, d\sigma (q)) : \ (t, q) \in \mathbb{R} \times N\}$ with the hypersurface $\mathcal{H}^{-1} (0)$:

$$\Gamma_\sigma = \Lambda_\sigma \cap \mathcal{H}^{-1} (0).$$

Definition 2.4.1 *The geometric solution to (CP) is the submanifold*

$$L := \bigcup_{0 \le t \le T} \Phi^t (\Gamma_\sigma) \subset T^* (\mathbb{R} \times N).$$

Proposition 2.4.2 *The geometric solution L is an exact Lagrangian submanifold, contained into the hypersurface $\mathcal{H}^{-1} (0)$ and Hamiltonian isotopic to the zero section $\mathcal{O}_{T^*([0,T] \times N)} = \{(t, q, 0, 0) : \ 0 \le t \le T, \ q \in N\}$ of $T^* ([0, T] \times N)$.*

Proof. A direct computation shows that every geometric solution is an exact Lagrangian submanifold. In order to prove that L is Hamiltonian isotopic to the zero section $\mathcal{O}_{T^*([0,T] \times N)} = \{(t, q, 0, 0) : \ 0 \le t \le T, \ q \in N\}$ of $T^* ([0, T] \times N)$, we determine a continuous path of exact Lagrangian submanifolds in $T^* (\mathbb{R} \times N)$ connecting the zero section to L. Hence we conclude using Theorem 2.3.3.
Let us consider the following 1-parameter family of Cauchy problems related to Hamilton-Jacobi equations:

$$(CP)_\lambda \begin{cases} \frac{\partial S}{\partial t} (t, q) + \lambda H \left(t, q, \frac{\partial S}{\partial q} (t, q)\right) = 0 \\ \\ S (0, q) = \lambda \sigma (q) \end{cases}$$

The initial data submanifold related to $(CP)_\lambda$ is:

$$\Gamma_{\lambda\sigma} = \{(0, q, -\lambda H (0, q, \lambda d\sigma (q)), \lambda d\sigma (q))\}$$

and the geometric solution to $(CP)_\lambda$ is

$$L_\lambda = \bigcup_{0\le t\le T} \Phi^t_\lambda\left(\Gamma_{\lambda\sigma}\right) = \left\{\left(t, \tilde{q}_\lambda\left(t\right), \tilde{\tau}_\lambda\left(t\right), \tilde{p}_\lambda\left(t\right)\right)\right\}$$

with
1) Φ^t_λ the flow of $\mathcal{H}_\lambda = \tau + \lambda H$,
2) $\left(\tilde{q}_\lambda\left(t\right), \tilde{p}_\lambda\left(t\right)\right)$ the characteristics of $X_{\lambda H}$ such that $\tilde{q}_\lambda\left(0\right) = q_0$ and $\tilde{p}_\lambda\left(0\right) = \lambda d\sigma\left(q_0\right)$,
3) $\tilde{\tau}_\lambda\left(t\right) = -\lambda H\left(t, \tilde{q}_\lambda\left(t\right), \tilde{p}_\lambda\left(t\right)\right)$.
We point out that every L_λ, geometric solution to $\left(CP\right)_\lambda$, results an exact Lagrangian submanifold of $T^*\left(\mathbb{R}\times N\right)$ and that $L_1 = L$. On the other hand $L_0 = \mathcal{O}_{T^*([0,T]\times N)}$. Hence we have defined a continuous path $\lambda \mapsto L_\lambda$ connecting the zero section $\mathcal{O}_{T^*([0,T]\times N)}$ to the Lagrangian submanifold L. As a consequence of Theorem 2.3.3, this fact results equivalent to the existence of a Hamiltonian isotopy connecting the zero section $\mathcal{O}_{T^*([0,T]\times N)}$ to L. $\square$

As a consequence of previous Proposition 2.4.2 and of the compactness of N, Theorem 1.1.5 of Viterbo guarantees that the Lagrangian submanifold L admits essentially (that is, up to the three operations described in Section 1.3) a unique G.F.Q.I. $S : [0,T]\times N\times\mathbb{R}^k \to \mathbb{R}$, $(t, q; \xi) \mapsto S\left(t, q; \xi\right)$.
Up to a suitable constant, we can assume that the graph of $S\left(t, q; \xi\right)$ at $t = 0$ coincides with Γ_σ:

$$\Gamma_\sigma = \left\{\left(0, q, \frac{\partial S}{\partial t}\left(0, q; \xi\right), \frac{\partial S}{\partial q}\left(0, q; \xi\right)\right) : \ \frac{\partial S}{\partial \xi}\left(0, q; \xi\right) = 0\right\}.$$

The quadraticity at infinity property of $S\left(t, q; \xi\right)$ is crucial: minimax solutions arise from the application of the Lusternik-Schnirelman method to the G.F.Q.I. $S\left(t, q; \xi\right)$. In some more detail, let us consider the sublevel sets

$$S^c_{(t,q)} := \left\{\xi\in\mathbb{R}^k : \ S\left(t, q; \xi\right) \le c\right\}, \ \left(t, q\right)\in[0,T]\times N \text{ fixed},$$

$$Q^c := \left\{\xi\in\mathbb{R}^k : \ Q\left(\xi\right) \le c\right\}.$$

We observe that for $c > 0$ large enough, $S^c_{(t,q)}$ and Q^c are invariant from a homotopical point of view:

$$S^{\pm c}_{(t,q)} = Q^{\pm c},$$

and $S^{\pm\bar{c}}_{(t,q)}$ retracts on $S^{\pm c}_{(t,q)}$ for every $\bar{c} > c$. Let $A := Q^{(c-\epsilon)}$, $\epsilon > 0$ small. Then the isomorphisms below (the first one by excision and the second one by retraction) hold:

$$H^*\left(Q^c, Q^{-c}\right) \cong H^*\left(Q^c \setminus \mathring{A}, Q^{-c}\setminus \mathring{A}\right) \cong H^*\left(D^i, \partial D^i\right),$$

where i is the index of the quadratic form Q (that is, the number of negative eigenvalues of Q) and D^i denotes the disk (of radius $\sqrt{c}$) in $\mathbb{R}^i$. Hence $H^*\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ is 1-dimensional:

$$H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right) \cong H^h\left(D^i, \partial D^i\right) = \left\{ \begin{array}{cl} 0 & \text{if } h \neq i \\ \alpha \cdot \mathbb{R} & \text{if } h = i \end{array} \right. \tag{2.16}$$

We remark that (2.16) holds also for generalized G.F.Q.I. of Definition 1.1.6, because the relative coomology $H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ is invariant.

Definition 2.4.3 *(Minimax solution) Let $S(t,q;\xi)$ be the unique G.F.Q.I. for L, $S(t,q;\xi) = Q(\xi)$ out of a compact set in the parameters $\xi \in \mathbb{R}^k$. For $c > 0$ large enough and for every $(t,q) \in [0,T] \times N$, let $0 \neq \alpha \in H^i\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ be the unique generator (up to a constant factor) as in (2.16) and*

$$i_\lambda : S^\lambda_{(t,q)} \hookrightarrow S^c_{(t,q)}.$$

The function

$$(t,q) \mapsto u(t,q) := \inf\left\{\lambda \in [-c,+c] : \ i^*_\lambda \alpha \neq 0\right\} \tag{2.17}$$

is the minimax solution of (CP).

The following fundamental Theorem has been proved by Chaperon, see [25].

Theorem 2.4.4 *The minimax solution $u(t,q)$ is a weak solution to (CP), Lipschitz on finite times, which does not depend on the choice of the G.F.Q.I.*

We observe that the definition of minimax solutions arises naturally in the compact case, when the Uniqueness Theorem of Viterbo is satisfied. Moreover, for a fixed point on the manifold $[0,T] \times N$, the minimax critical value is unique and is determined by the Morse index of the quadratic form Q. We conclude with the following Proposition, which will be useful in the sequel.

Proposition 2.4.5 *Let $S(t,q;\xi)$ and $u(t,q)$ as in Definition 2.4.3. Let us suppose that the Morse index of the quadratic form Q is 0. Then*

$$u(t,q) = \min_{\xi \in \mathbb{R}^k} S(t,q;\xi).$$

Proof. Let us fix a point $(t,q) \in [0,T] \times N$. Since Q is positive definite, $S^{-c}_{(t,q)} = \emptyset$, and for $c > 0$ large enough, it results (see (2.16))

$$H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right) = H^h\left(S^c_{(t,q)}\right) = \left\{ \begin{array}{cl} 0 & \text{if } h \neq 0 \\ 1 \cdot \mathbb{R} & \text{if } h = 0 \end{array} \right.$$

where 1 is the generator of $H^0\left(S^c_{(t,q)}\right)$. Consequently, the minimax solution (2.17)

$$u(t,q) = \inf\{\lambda \in [-c,+c] : \; i^*_\lambda 1 \neq 0\}$$

coincides with the minimum of the function $\xi \mapsto S(t,q;\xi)$, that is

$$u(t,q) = \min_{\xi \in \mathbb{R}^k} S(t,q;\xi).$$

□

2.5 A global generating function for the geometric solution for $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$ on $T^*\mathbb{T}^n$

Let us consider the Hamiltonian $H(q,p) \in C^2(T^*\mathbb{T}^n;\mathbb{R})$:

$$H(q,p) = \frac{1}{2}|p|^2 + f(q,p), \tag{2.18}$$

f compact support in the p variables, and the Cauchy Problem $(CP)_H$:

$$(CP)_H \begin{cases} \frac{\partial S}{\partial t}(t,q) + H\left(q, \frac{\partial S}{\partial q}(t,q)\right) = 0, \\ \\ S(0,q) = \sigma(q), \end{cases}$$

where $t \in [0,T]$, $q \in \mathbb{T}^n$ and $\sigma \in C^1(\mathbb{T}^n;\mathbb{R})$.
In this section we investigate around the structure of the generating function for the geometric solution of $(CP)_H$, showing that its structure is naturally interpreted as an improvement of the Hopf's formula utilized by Bardi and Evans in order to build the viscosity solution for Liouville-integrable Hamiltonians.
It turns out useful to introduce the compact support Hamiltonian $K(t,q,p)$:

$$K(t,q,p) = \left(\left(\phi^t_0\right)^* f\right)(q,p) = H(q+tp,p) - \frac{1}{2}|p|^2, \tag{2.19}$$

where ϕ^t_0 is the flow of $H_0(p) := \frac{1}{2}|p|^2$.
We recall now the following Proposition, which has been proved in details in Subsection 1.2.2 and is, essentially, a result of Hamilton (see [36] and also [33], [8]).

Proposition 2.5.1 *Let ϕ^t_H, ϕ^t_K and ϕ^t_0 be the flows of H, K and H_0 respectively. We have:*

$$\phi^t_H(q,p) = \phi^t_0 \circ \phi^t_K(q,p), \tag{2.20}$$

$\forall (q,p) \in T^*\mathbb{T}^n$ and $\forall t \in \mathbb{R}$.

Now let us consider the Cauchy Problem $(CP)_K$ related to K:

$$(CP)_K \begin{cases} \frac{\partial S}{\partial t}(t,q) + K\left(t, q, \frac{\partial S}{\partial q}(t,q)\right) = 0 \\ \\ S(0,q) = \sigma(q) \end{cases}$$

and define $\mathcal{K}(t,q,\tau,p) := \tau + K(t,q,p)$, Φ_K^t its flow, and $(\Gamma_K)_\sigma$ the initial data submanifold

$$(\Gamma_K)_\sigma := \{(0, q, -K(0, q, d\sigma(q)), d\sigma(q)) : \ q \in \mathbb{T}^n\} \subset \mathcal{K}^{-1}(0).$$

Since the manifold $\mathbb{T}^n$ is compact, a consequence of Proposition 2.4.2 and Theorem 1.1.5 is the existence of a unique G.F.Q.I. $S_K(t,q;u)$ for the geometric solution L_K of $(CP)_K$:

$$L_K := \bigcup_{0\le t\le T} \Phi_K^t\left((\Gamma_K)_\sigma\right).$$

Proposition 2.5.2 *Let $\mathcal{H}(q,\tau,p) := \tau + H(q,p)$, Φ_H^t its flow and $(\Gamma_H)_\sigma$ the initial data submanifold*

$$(\Gamma_H)_\sigma := \{(0, q, -H(q, d\sigma(q)), d\sigma(q)) : \ q \in \mathbb{T}^n\} \subset \mathcal{H}^{-1}(0).$$

Then the Lagrangian submanifold L_H

$$L_H := \bigcup_{0\le t\le T} \Phi_H^t\left((\Gamma_H)_\sigma\right),$$

geometric solution of $(CP)_H$, is generated by the function

$$\widetilde{S}(t,q;\xi,u,v) := -\frac{1}{2}v^2 t + (q-\xi)\cdot v + S_K(t,\xi;u). \tag{2.21}$$

Proof. The generating function $S_K(t,\xi;u)$ generates the Lagrangian submanifold L_K, which can be written, more explicitly

$$L_K := \bigcup_{0\le t\le T} \Phi_K^t\left((\Gamma_K)_\sigma\right) = \{(t, \widetilde{q}(t), \widetilde{\tau}(t), \widetilde{p}(t)) : \ 0 \le t \le T\},$$

where $\widetilde{q}$ and $\widetilde{p}$ are the characteristics of X_K such that $\widetilde{q}(0) = q_0$ and $\widetilde{p}(0) = d\sigma(q_0)$, and $\widetilde{\tau}(t) = -K(t, \widetilde{q}(t), \widetilde{p}(t))$.
Now, by a direct computation, we prove that the Lagrangian submanifold generated by $\widetilde{S}(t,q;\xi,u,v)$ coincides with L_H.

$$L_{\widetilde{S}} = \left\{\left(t, q, \frac{\partial \widetilde{S}}{\partial t}, \frac{\partial \widetilde{S}}{\partial q}\right) : \ \frac{\partial \widetilde{S}}{\partial \xi} = 0, \ \frac{\partial \widetilde{S}}{\partial u} = 0, \ \frac{\partial \widetilde{S}}{\partial v} = 0\right\}.$$

More precisely,

$$\frac{\partial \widetilde{S}}{\partial t}(t,q;\xi,u,v) = -\frac{1}{2}v^2 + \frac{\partial S_K}{\partial t}(t,\xi;u),$$

$$\frac{\partial \widetilde{S}}{\partial q}(t,q;\xi,u,v) = v,$$

$$\frac{\partial \widetilde{S}}{\partial \xi}(t,q;\xi,u,v) = 0 \text{ is and only if } -v + \frac{\partial S_K}{\partial \xi}(t,\xi;u) = 0$$

$$\text{if and only if } v = \frac{\partial S_K}{\partial \xi}(t,\xi;u),$$

$$\frac{\partial \widetilde{S}}{\partial u}(t,q;\xi,u,v) = 0 \text{ if and only if } \frac{\partial S_K}{\partial u}(t,\xi;u) = 0,$$

$$\frac{\partial \widetilde{S}}{\partial v}(t,q;\xi,u,v) = 0 \text{ if and only if } -vt + q - \xi = 0 \text{ if and only if } q = \xi + vt.$$

Hence $L_{\widetilde{S}}$ is equivalent to

$$L_{\widetilde{S}} = \left\{ \left(t, q, -\frac{1}{2}v^2 + \frac{\partial S_K}{\partial t}(t,\xi;u), v \right) : \; v = \frac{\partial S_K}{\partial \xi}(t,\xi;u), \; \frac{\partial S_K}{\partial u}(t,\xi;u) = 0, \; q = \xi + vt \right\}.$$

Now we remind that $S_K(t,\xi;u)$ generates the Lagrangian submanifold L_K, hence

$$L_{\widetilde{S}} = \left\{ \left(t, q, -\frac{1}{2}v^2 - K(t,\xi,v), v \right) : \; (\xi, v) \in \phi_K^t(\mathrm{Im}(d\sigma)), \; q = \xi + vt \right\}.$$

But $K(t,\xi,v) = H(\xi + tv, v) - \frac{1}{2}v^2$, then $-\frac{1}{2}v^2 - K(t,\xi,v) = -H(\xi + tv, v)$. Hence

$$L_{\widetilde{S}} = \left\{ (t, \xi + tv, -H(\xi + tv, v), v) : \; (\xi, v) \in \phi_K^t(\mathrm{Im}(d\sigma)) \right\}.$$

Now we also note that $(\xi + vt, v) = \phi_0^t(\xi, v)$, therefore, since $\phi_H^t = \phi_0^t \circ \phi_K^t$,

$$L_{\widetilde{S}} = \{ (t, \bar{q}(t), -H(\bar{q}(t), \bar{p}(t)), \bar{p}(t)) : \; 0 \le t \le T \},$$

where $\bar{q}$ and $\bar{p}$ are the characteristics of X_H such that $\bar{q}(0) = q_0$ and $\bar{p}(0) = d\sigma(q_0)$. Equivalently

$$L_{\widetilde{S}} = L_H.$$

□

We remark the following interesting fact: the structure of the generating function (2.21) recalls the Hopf's formula used in 1984 by Bardi and Evans in order to construct viscosity solutions for Liouville-integrable and convex

Hamiltonians $H(p)$.
In fact, their formula is

$$u_{visc}(t,q) = \inf_{\xi} \sup_{v} \{-H(v)\,t + (q-\xi)\cdot v + \sigma(\xi)\}\,. \tag{2.22}$$

The above formula (2.22), in the case $H(p) = \frac{1}{2}|p|^2$, becomes

$$u_{visc}(t,q) = \inf_{\xi} \sup_{v} \left\{-\frac{1}{2}v^2 t + (q-\xi)\cdot v + \sigma(\xi)\right\}. \tag{2.23}$$

Therefore, the generating function (2.21) can be considered the improvement of $-\frac{1}{2}v^2 t + (q-\xi)\cdot v + \sigma(\xi)$ in (2.23) when we take into account the perturbed non-integrable Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$, f compact support. This correction is just provided by the term $S_K(t,\xi;u)$.
We finally note that the plan of construct viscosity solutions starting from generating functions has been rather fruitless; nevertheless, we can find similar representation formulas for state-dependent Hamiltonians, see [43] and [15], although they hold only under suitable restrictive assumptions.

Chapter 3

A relationship between minimax and viscosity solutions in the p-convex case

Here we prove in detail the coincidence of minimax and viscosity solutions for p-convex Hamiltonians of mechanical type. The equivalence is essentially established through an Amann-Conley-Zehnder reduction of an infinite parameters generating function arising from Hamilton-Helmholtz variational principle and through the Hamiltonian version of the Lax-Oleinik formula.

As an application of this formula, we finally resume the main lines of the interesting background of the weak K.A.M. theory. The goal of this recent theory is the employing of dynamical systems, variational and PDE methods to find "integrable structures" within general Hamiltonian dynamics. The result of the weak K.A.M. theorem can also be reinterpreted in terms of viscosity solutions of the underlying Hamilton-Jacobi equation. Using the coincidence proved above, we can now compare the viscous solution given by the weak K.A.M. theorem to the corresponding minimax solution.

3.1 A global generating function for the geometric solution for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ on $T^*\mathbb{R}^n$

We consider the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + V(q) \in C^2(T^*\mathbb{R}^n; \mathbb{R})$, V compact support, and its related Cauchy problem $(CP)_H$:

$$(CP)_H \begin{cases} \frac{\partial S}{\partial t}(t,q) + \frac{1}{2}|\frac{\partial S}{\partial q}(t,q)|^2 + V(q) = 0, \\ \\ S(0,q) = \sigma(q), \end{cases}$$

where $t \in [0,T]$, $q \in \mathbb{R}^n$, σ compact support. The starting point consists to take into account the below global generating function W for the geometric solution for H –see Theorem 3.1.2– arising from Hamilton-Helmholtz functional.

Let us consider the set of curves:

$$\Gamma := \left\{\gamma(\cdot) = (q(\cdot), p(\cdot)) \in H^1\left([0,T], \mathbb{R}^{2n}\right) : \; p(0) = d\sigma(q(0))\right\}.$$

By Sobolev imbedding theorem,

$$H^1\left((0,T), \mathbb{R}^{2n}\right) \hookrightarrow C^0\left([0,T], \mathbb{R}^{2n}\right)$$

compactly, so in the above definition the elements of Γ are the natural continuous extensions of the curves of $H^1\left((0,T), \mathbb{R}^{2n}\right)$ (i.e. the continuous curves $t \mapsto \gamma(t)$, starting from the graph of $d\sigma$, such that $\dot{\gamma} = \frac{d\gamma}{dt} \in L^2 := L^2\left((0,T), \mathbb{R}^{2n}\right)$). Moreover, the set Γ has a natural structure of linear space, and then $T_\gamma\Gamma = \Gamma$, for all $\gamma \in \Gamma$.
An equivalent way to describe the curves of Γ is to assign the q-projection at time t, $q = q(t) \in \mathbb{R}^n$, and the velocity $\dot{\gamma}$ of the curve γ by means of a function $\Phi \in L^2$. This is summarized by the following bijection g.

Lemma 3.1.1 *(The bijection g)*
For all $\Phi \in L^2$ set $\Phi = (\Phi_q, \Phi_p)$. The map g,

$$g : [0,T] \times \mathbb{R}^n \times L^2\left((0,T), \mathbb{R}^{2n}\right) \longrightarrow [0,T] \times \Gamma$$

$$(t, q, \Phi) \longmapsto g(t, q, \Phi) = (t, \gamma(\cdot)),$$

$$\gamma(s) = (pr_\Gamma \circ g)(t, q, \Phi)(s) = (q(s), p(s)) =$$

$$= \left(q - \int_s^t \Phi_q(r)\, dr, d\sigma(q(0)) + \int_0^s \Phi_p(r)\, dr\right) =$$

$$= \left(q - \int_s^t \Phi_q(r)\, dr, d\sigma\left(q - \int_0^t \Phi_q(r)\, dr\right) + \int_0^s \Phi_p(r)\, dr\right). \tag{3.1}$$

is a bijection.

Proof. Let $\gamma(\cdot) = (q(\cdot), p(\cdot)) \in \Gamma$; since $(\dot{q}(\cdot), \dot{p}(\cdot)) \in L^2$, then $(t, \gamma(s)) = g(t, q(t), (\dot{q}(s), \dot{p}(s)))$. This proves that g is surjective.
Now, let $q, \bar{q} \in \mathbb{R}^n$, $\Phi, \bar{\Phi} \in L^2$, such that $g(t, q, \Phi) = g\left(t, \bar{q}, \bar{\Phi}\right)$. In other words,

$$\left(q - \int_s^t \Phi_q(r)\, dr, d\sigma\left(q - \int_0^t \Phi_q(r)\, dr\right) + \int_0^s \Phi_p(r)\, dr\right) =$$

$$= \left(\bar{q} - \int_s^t \bar{\Phi}_q(r)\, dr, d\sigma\left(\bar{q} - \int_0^t \bar{\Phi}_q(r)\, dr\right) + \int_0^s \bar{\Phi}_p(r)\, dr\right).$$

Thus, for all $s \in [0,t]$ one has

$$q - \bar{q} - \int_s^t \left(\Phi_q(r) - \bar{\Phi}_q(r)\right) dr = 0,$$

$$d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) - d\sigma\left(\bar{q} - \int_0^t \bar{\Phi}_q(r)\,dr\right) + \int_0^s \left(\Phi_p(r) - \bar{\Phi}_p(r)\right) dr = 0.$$

Hence

$$q = \bar{q},\ \ \Phi_q = \bar{\Phi}_q,\ \ \Phi_p = \bar{\Phi}_p.$$

This shows that g is injective. □

To be more clear, we remark that the second value of the map $g(t,q,\Phi)$ is the curve $\gamma(\cdot) = (q(\cdot), p(\cdot))$ which is
1) starting from $(q(0), d\sigma(q(0)))$, such that
2) $\dot{\gamma}(\cdot) = \Phi(\cdot)$, and
3) $q(t) = q$.

By componing the Hamilton-Helmholtz functional:

$$A : [0,T] \times \Gamma \longrightarrow \mathbb{R}$$

$$(t, \gamma(\cdot)) \mapsto A[t, \gamma(\cdot)] := \sigma(q(0)) + \int_0^t [p(r) \cdot \dot{q}(r) - H(r, q(r), p(r))]\,dr.$$

with the bijection g, we obtain the following global generating function $W = A \circ g$:

Theorem 3.1.2 *The infinite-parameters function:*

$$W := A \circ g : [0,T] \times \mathbb{R}^n \times L^2 \longrightarrow \mathbb{R}, \tag{3.2}$$

$$(t,q,\Phi) \longmapsto W(t,q,\Phi) := A \circ g(t,q,\Phi),$$

generates $L_H = \bigcup_{0 \le t \le T} \Phi_H^t((\Gamma_H)_\sigma)$*, the geometric solution for the Hamiltonian* $H(q,p) = \frac{1}{2}|p|^2 + V(q)$*.*

Proof. We first explicitly write down W:

$$\begin{aligned}
W(t,q,\Phi) &= \sigma(q(0)) + \int_0^t \left[\left(d\sigma(q(0)) + \int_0^s \Phi_p(r)\,dr\right) \cdot \Phi_q(s) - \right. \\
&\left. - H\left(s, q - \int_s^t \Phi_q(r)\,dr, d\sigma(q(0)) + \int_0^s \Phi_p(r)\,dr\right)\right] ds, \\
&= \sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) + \int_0^t \left[\left(d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) + \right.\right. \\
&\left.\left. + \int_0^s \Phi_p(r)\,dr\right) \cdot \Phi_q(s)\right] ds - \\
&- \int_0^t \left[H\left(s, q - \int_s^t \Phi_q(r)\,dr, d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) + \int_0^s \Phi_p(r)\,dr\right)\right] ds.
\end{aligned}$$

Then, for $\frac{DW}{D\Phi}=0$, we compute $\frac{\partial W}{\partial q}$ and $\frac{\partial W}{\partial t}$.

$$\begin{aligned}
\frac{\partial W}{\partial q} &= d\sigma\left(q\left(0\right)\right)+\int_0^t d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(s\right)ds-\int_0^t\frac{\partial H}{\partial q}ds-\int_0^t\frac{\partial H}{\partial p}\cdot d^2\sigma\left(q\left(0\right)\right)ds,\\
&= d\sigma\left(q\left(0\right)\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\Phi_q\left(s\right)ds+\int_0^t\dot{p}\left(s\right)ds-d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds,\\
&= d\sigma\left(q\left(0\right)\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds+\int_0^t\dot{p}\left(s\right)ds-d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds,\\
&= d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(s\right)ds=p\left(t\right).
\end{aligned}$$

Finally, we compute $\frac{\partial W}{\partial t}$.

$$\begin{aligned}
\frac{\partial W}{\partial t} &= d\sigma\left(q\left(0\right)\right)\cdot\frac{\partial q\left(0\right)}{\partial t}+\left(d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(r\right)dr\right)\cdot\Phi_q\left(t\right)-\\
&-H\left(t,q,d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(r\right)dr\right)+\int_0^t d^2\sigma\cdot\left(-\Phi_q\left(t\right)\right)\cdot\Phi_q\left(s\right)ds+\\
&+\int_0^t\frac{\partial H}{\partial q}\left(s,q-\int_0^T\Phi_q\left(r\right)dr,d\sigma\left(q\left(0\right)\right)+\int_0^s\Phi_p\left(r\right)dr\right)\cdot\Phi_q\left(t\right)ds-\\
&-\int_0^t\frac{\partial H}{\partial p}\left(s,q-\int_0^t\Phi_q\left(r\right)dr,d\sigma\left(q\left(0\right)\right)+\int_0^s\Phi_p\left(r\right)dr\right)\cdot\frac{\partial^2\sigma}{\partial q^2}\left(q\left(0\right)\right)\cdot\left(-\Phi_q\left(t\right)\right)ds,\\
&= d\sigma\left(q\left(0\right)\right)\cdot\left(-\Phi_q\left(t\right)\right)+p\left(t\right)\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),p\left(t\right)\right)-d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(t\right)\cdot\int_0^t\Phi_q\left(s\right)ds+\\
&+\int_0^t\frac{\partial H}{\partial p}\left(s,q\left(s\right),p\left(s\right)\right)ds\cdot\Phi_q\left(t\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(t\right)\cdot\int_0^t\frac{\partial H}{\partial p}\left(s,q\left(s\right),p\left(s\right)\right)ds,\\
&= -p\left(0\right)\cdot\dot{q}\left(t\right)+p\left(t\right)\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),p\left(t\right)\right)-d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\Phi_q\left(s\right)ds+\\
&+\int_0^t\frac{\partial H}{\partial q}\left(s,q\left(s\right),p\left(s\right)\right)ds\cdot\dot{q}\left(t\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds,\\
&= -p\left(0\right)\cdot\dot{q}\left(t\right)+\frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds-\\
&-\frac{\partial}{\partial q}\left(\int_0^t\left[p\dot{q}-H\right]d\tau\right)\cdot\dot{q}\left(t\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds,\\
&= \frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-\frac{\partial}{\partial q}\left[\sigma\left(q\left(0\right)\right)+\int_0^t\left(p\dot{q}-H\right)d\tau\right]\cdot\dot{q}\left(t\right),\\
&= \frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-\frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right),\\
&= -H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right).
\end{aligned}$$

$\square$

3.1.1 Fourier expansion and fixed point

Hamilton equations related to X_H are

$$\begin{cases} \dot{q} = p \\ \dot{p} = -V'(q) \end{cases} \tag{3.3}$$

Using the p-components of the bijection g, (3.3) can be rewritten, almost everywhere, as

$$\begin{cases} \Phi_q(s) = d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) + \int_0^s \Phi_p(r)\,dr \\ \Phi_p(s) = -V'\left(q - \int_s^t \Phi_q(r)\,dr\right) \end{cases} \tag{3.4}$$

Hence

$$\Phi_q(s) = d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) - \int_0^s V'\left(q - \int_r^t \Phi_q(\tau)\,d\tau\right)dr \tag{3.5}$$

Note that the reduction of (3.4) into (3.5) is equivalent to the displacement from the Hamiltonian formalism to the Lagrangian formalism through the Legendre transformation.
For every $\Phi_q \in L^2((0,T), \mathbb{R}^n)$, let us consider the Fourier expansion

$$\Phi_q(s) = \sum_{k\in\mathbb{Z}} (\Phi_q)_k\, e^{i(2\pi k/T)s}.$$

For each fixed $N \in \mathbb{N}$, let us consider the projection maps on the basis $\left\{e^{i(2\pi k/T)s}\right\}_{k\in\mathbb{Z}}$ of $L^2((0,T), \mathbb{R}^n)$,

$$\mathbb{P}_N\Phi_q(s) := \sum_{|k|\leq N} (\Phi_q)_k\, e^{i(2\pi k/T)s},\ \ \mathbb{Q}_N\Phi_q(s) := \sum_{|k|>N} (\Phi_q)_k\, e^{i(2\pi k/T)s}.$$

Clearly,

$$\mathbb{P}_N L^2((0,T), \mathbb{R}^n) \oplus \mathbb{Q}_N L^2((0,T), \mathbb{R}^n) = L^2((0,T), \mathbb{R}^n),$$

and for $\Phi_q \in L^2((0,T), \mathbb{R}^n)$ we will write $u := \mathbb{P}_N\Phi_q$ and $v := \mathbb{Q}_N\Phi_q$.

We will try to solve (3.5) by a fixed point procedure.

Proposition 3.1.3 *(Lipschitz) Let* $\sup_{q\in\mathbb{R}^n} |V''(q)| = C\ \ (< +\infty)$.
For fixed $(t,q) \in [0,T] \times \mathbb{R}^n$ *and* $u \in \mathbb{P}_N L^2((0,T), \mathbb{R}^n)$*, the map*

$$F : \mathbb{Q}_N L^2((0,T), \mathbb{R}^n) \longrightarrow \mathbb{Q}_N L^2((0,T), \mathbb{R}^n)$$

$$v \longmapsto \mathbb{Q}_N \left\{ -\int_0^s V' \left(q - \int_r^t (u+v)(\tau)\, d\tau \right) dr \right\}$$

is Lipschitz with constant

$$Lip\,(F) \leq \frac{T^2 C}{2\pi N} \left(1 + \sqrt{2N}\right).$$

Before the proof of the Proposition 3.1.3, we premise some technical results.

Lemma 3.1.4 *Let $f \in L^1((0,T),\mathbb{R}^n) \cap L^2((0,T),\mathbb{R}^n)$. Then the function $\int_0^t f(s)\,ds \in L^2((0,T),\mathbb{R}^n)$ and*

$$|| \int_0^t f(s)\,ds ||_{L^2((0,T),\mathbb{R}^n)} \leq T \cdot ||f||_{L^2((0,T),\mathbb{R}^n)} \tag{3.6}$$

Proof. We recall that $\langle f, g \rangle_{L^2((0,T),\mathbb{R}^n)} := \frac{1}{T}\int_0^T fg dt$, $||f||_{L^1((0,T),\mathbb{R}^n)} := \frac{1}{T}\int_0^T |f(t)|dt$ and $||f||_{L^1((0,T),\mathbb{R}^n)} \leq ||f||_{L^2((0,T),\mathbb{R}^n)}$.

$$|| \int_0^t f(s)\,ds ||^2_{L^2((0,T),\mathbb{R}^n)} = \frac{1}{T}\int_0^T \left(\int_0^t f(s)\,ds \right)^2 dt \leq$$
$$\leq \frac{1}{T}\int_0^T \left(\int_0^t |f(s)|ds \right)^2 dt \leq \frac{1}{T}\int_0^T \left(\int_0^T |f(s)|ds \right)^2 dt =$$
$$= T^2 \cdot ||f||^2_{L^1((0,T),\mathbb{R}^n)} \leq T^2 \cdot ||f||^2_{L^2((0,T),\mathbb{R}^n)}$$

hence (3.6) follows. □

Proof of Proposition 3.1.3 For each $v_1, v_2 \in \mathbb{Q}_N L^2((0,T),\mathbb{R}^n)$, let us consider the Fourier expansion

$$v := v_2 - v_1 = \sum_{|k|>N} v_k e^{i(2\pi k/T)\tau}.$$

We compute $F(v_2) - F(v_1)$:

$$F(v_2) - F(v_1) =$$
$$\mathbb{Q}_N \left\{ -\int_0^s \left[V'\left(q - \int_r^t (u+v_2)(\tau)\, d\tau \right) dr \right] + \int_0^s \left[V'\left(q - \int_r^t (u+v_1)(\tau)\, d\tau \right) dr \right] \right\},$$
$$= \mathbb{Q}_N \left\{ -\int_0^s \left[V'\left(q - \int_r^t (u+v_2)(\tau)\, d\tau \right) - V'\left(q - \int_r^t (u+v_1)(\tau)\, d\tau \right) dr \right] \right\}.$$

Therefore

$$
\begin{aligned}
&||F\left(v_2\right)-F\left(v_1\right)||_{L^2((0,T),\mathbb{R}^n)} \leq \\
&\leq T\cdot||\mathbb{Q}_N\left\{V'\left(q-\int_r^t\left(u+v_2\right)(\tau)\,d\tau\right)-V'\left(q-\int_r^t\left(u+v_1\right)(\tau)\,d\tau\right)\right\}||_{L^2((0,T),\mathbb{R}^n)} \leq \\
&\leq TC\cdot||-\int_r^t\sum_{|k|>N}v_k e^{i(2\pi k/T)\tau}d\tau||_{L^2((0,T),\mathbb{R}^n)} \leq \\
&\leq TC\cdot\left(||\sum_{|k|>N}v_k e^{i(2\pi k/T)r}\cdot\frac{T}{i2\pi k}||_{L^2((0,T),\mathbb{R}^n)}+||\sum_{|k|>N}v_k e^{i(2\pi k/T)t}\cdot\frac{T}{i2\pi k}||_{L^2((0,T),\mathbb{R}^n)}\right) \leq \\
&\leq \frac{T^2C}{2\pi N}\cdot||v||_{L^2((0,T),\mathbb{R}^n)}+T^2C\cdot||\sum_{|k|>N}\frac{|v_k|}{2\pi k}||_{L^2((0,T),\mathbb{R}^n)}.
\end{aligned}
$$

We now use Cauchy-Schwartz inequality in $l^2 := L^2(\mathbb{Z},\mathbb{C})$ as follows

$$
\sum_{|k|>N}\frac{|v_k|}{k}=\left\langle(|v_k|)_{k\in\mathbb{Z}},\left(\frac{1}{k}\right)_{|k|>N}\right\rangle_{l^2}\leq||(|v_k|)_{k\in\mathbb{Z}}||_{l^2}\cdot||\left(\frac{1}{k}\right)_{|k|>N}||_{l^2}.
$$

Hence

$$
\sum_{|k|>N}\frac{|v_k|}{2\pi k}\leq\frac{1}{2\pi}||v||_{L^2((0,T),\mathbb{R}^n)}\sqrt{2\sum_{|k|>N}\frac{1}{k^2}}\leq\frac{1}{2\pi}||v||_{L^2((0,T),\mathbb{R}^n)}\sqrt{\frac{2}{N}},
$$

obtaining

$$
\begin{aligned}
&\frac{T^2C}{2\pi N}\cdot||v||_{L^2((0,T),\mathbb{R}^n)}+T^2C\cdot||\sum_{|k|>N}\frac{|v_k|}{2\pi k}||_{L^2((0,T),\mathbb{R}^n)} \leq \\
&\leq \frac{T^2C}{2\pi N}\cdot||v||_{L^2((0,T),\mathbb{R}^n)}+T^2C\cdot\frac{1}{2\pi}\sqrt{\frac{2}{N}}||v||_{L^2((0,T),\mathbb{R}^n)} = \\
&= \frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right)\cdot||v||_{L^2((0,T),\mathbb{R}^n)},
\end{aligned}
$$

that is

$$
\mathrm{Lip}\,(F)\leq\frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right).
$$

□

Corollary 3.1.5 *(Contraction map) Let* $\sup_{q\in\mathbb{R}^n}|V''(q)| = C \ (<+\infty)$. *For fixed* $(t,q)\in[0,T]\times\mathbb{R}^n$, $u\in\mathbb{P}_N L^2((0,T),\mathbb{R}^n)$ *and* N *large enough:*

$$
\frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right)<1,
$$

the map $s \mapsto F(t,q,u)(s)$

$$F : \mathbb{Q}_N L^2((0,T),\mathbb{R}^n) \longrightarrow \mathbb{Q}_N L^2((0,T),\mathbb{R}^n)$$

$$v \longmapsto \mathbb{Q}_N \left\{ -\int_0^s V'\left(q - \int_r^t (u+v)(\tau)\, d\tau\right) dr \right\}$$

is a contraction.

By Banach-Cacciopoli Theorem, for fixed $(t,q) \in [0,T] \times \mathbb{R}^n$ and $u \in \mathbb{P}_N L^2((0,T),\mathbb{R}^n)$, there exists one and only one fixed point $\mathcal{F}(t,q,u)(s)$, shortly $\mathcal{F}(u)$, for the above contraction:

$$\mathcal{F}(u) = \mathbb{Q}_N \left\{ -\int_0^s V'\left(q - \int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right) dr \right\}. \tag{3.7}$$

Beside (3.7), let us consider the finite-dimensional equation of unknown $u \in \mathbb{P}_N L^2((0,T),\mathbb{R}^n)$:

$$u = \mathbb{P}_N \left\{ d\sigma\left(q - \int_0^t (u+\mathcal{F}(u))(r)\, dr\right) - \int_0^s V'\left(q - \int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right) dr \right\}. \tag{3.8}$$

By adding (3.7) and (3.8), in correspondence to any solution u of (3.8), we gain

$$u+\mathcal{F}(u) = d\sigma\left(q - \int_0^t (u+\mathcal{F}(u))(r)\, dr\right) - \int_0^s V'\left(q - \int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right) dr \tag{3.9}$$

in other words, the curve (see (3.1))

$$\gamma(s) := \mathrm{pr}_\Gamma \circ g\left(t, q, \left([u+\mathcal{F}(u)](s), -V'\left(q - \int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right)\right)\right)$$

solves the Hamilton canonical equations starting from the graph of $d\sigma$ (so that $\gamma \in \Gamma$).
Furthermore, we point out that $\dim(\mathbb{P}_N L^2((0,T),\mathbb{R}^n)) = n(2N+1)$.
As a consequence, substantially following the line of thought in [1], [17] and [18], we get that the geometric solution of Hamilton-Jacobi problem for H admits a finite-parameters generating function, denoted by $\overline{W}(t,q,u)$:

Theorem 3.1.6 *The finite-parameters function:*

$$\overline{W} := A \circ g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n(2N+1)} \longrightarrow \mathbb{R},$$

$$(t,q,u) \longmapsto \overline{W}(t,q,u) =$$

$$= \left\{ \sigma \left(q \left(0 \right) \right) + \int_0^t \left[p \left(s \right) \cdot \dot{q} \left(s \right) - H \left(s, q \left(s \right), p \left(s \right) \right) \right] ds \right\} |_{\left(q(s), p(s) \right)},$$

$\left(q \left(s \right), p \left(s \right) \right) = pr_{\Gamma} \circ g \left(t, q, \left(\left[u + \mathcal{F} \left(u \right) \right] \left(s \right), -V' \left(q - \int_r^t \left(u + \mathcal{F} \left(u \right) \right) \left(\tau \right) d\tau \right) \right) \right),$
generates $L_H = \bigcup_{0 \le t \le T} \Phi_H^t \left(\left(\Gamma_H \right)_\sigma \right)$, *the geometric solution for the Hamiltonian* $H \left(q, p \right) = \frac{1}{2} |p|^2 + V \left(q \right)$.

3.1.2 The quadraticity at infinity property

We check the quadraticity at infinity property of $\overline{W} \left(t, q, u \right)$ with respect to u: this is a crucial step in order to catch the minimax critical point in the Lusternik-Schnirelman format. We premise the following technical

Lemma 3.1.7 *For fixed* $\left(t, q \right) \in [0, T] \times \mathbb{R}^n$, *the function* $u \mapsto \mathcal{F} \left(u \right)$ *and its derivatives* $u \mapsto \frac{\partial \mathcal{F}}{\partial u} \left(u \right)$ *are uniformly bounded.*

Proof. We immediately get from (3.7) that $|\mathcal{F} \left(u \right)| \le TC$, where $C = \sup_{q \in \mathbb{R}^n} |V'' \left(q \right)| < +\infty$. Moreover, by a direct computation, it can be proved that the derivatives $\frac{\partial \mathcal{F}}{\partial u}$ are uniformly bounded. In fact, the fixed point function $\mathcal{F}$ solves the equation of unknown v:

$$\mathcal{G} \left(t, q, u, v \right) := \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\} - v = 0.$$

The implicit function theorem does work, since

$$\frac{\partial \mathcal{G}}{\partial v} \left(t, q, u, v \right) = \frac{\partial}{\partial v} \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\} - \mathbb{I},$$

and, by a classical argument, it can be proved that

$$\left[\frac{\partial \mathcal{G}}{\partial v} \left(t, q, u, v \right) \right]^{-1} = - \sum_{k=0}^{+\infty} \left[\frac{\partial}{\partial v} \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\} \right]^k.$$

Since a bound for the derivatives $\frac{\partial}{\partial v} \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\}$ is given by the Lipschitz constant $\alpha := \frac{T^2 C}{2\pi N} (1 + \sqrt{2N})$,

$$\left| \frac{\partial}{\partial v} \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\} \right| \le \alpha < 1,$$

we obtain that

$$\left| \left[\frac{\partial \mathcal{G} \left(t, q, u, v \right)}{\partial v} \right]^{-1} \right| \le \sum_{k=0}^{+\infty} \left| \frac{\partial}{\partial v} \mathbb{Q}_N \left\{ - \int_0^s V' \left(q - \int_r^t \left(u + v \right) \left(\tau \right) d\tau \right) dr \right\} \right|^k =$$

$$= \frac{1}{1-\alpha} < +\infty.$$

$\mathcal{G}(t,q,u,\mathcal{F}(u)) = 0$ implies $\frac{\partial \mathcal{G}}{\partial u} + \frac{\partial \mathcal{G}}{\partial v}\frac{\partial \mathcal{F}}{\partial u} = 0$, therefore the derivatives $\frac{\partial \mathcal{F}}{\partial u} = -\left(\frac{\partial \mathcal{G}}{\partial v}\right)^{-1}\frac{\partial \mathcal{G}}{\partial u}$ result uniformly bounded by the constant $\frac{\alpha}{1-\alpha}$:

$$|\frac{\partial \mathcal{F}}{\partial u}| \leq |\left(\frac{\partial \mathcal{G}}{\partial v}\right)^{-1}| \cdot |\frac{\partial \mathcal{G}}{\partial u}| \leq \frac{\alpha}{1-\alpha} < +\infty.$$

□

Theorem 3.1.8 *The finite-parameters function*

$$\overline{W} := A \circ g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n(2N+1)} \longrightarrow \mathbb{R},$$

$$(t,q,u) \longmapsto \bar{W}(t,q,u) =$$

$$= \left\{\sigma(q(0)) + \int_0^t [p(s)\cdot \dot{q}(s) - H(s,q(s),p(s))]\, ds\right\}|_{(q(s),p(s))},$$

$$(q(s),p(s)) = pr_\Gamma \circ g\left(t,q,\left([u+\mathcal{F}(u)](s), -V'\left(q - \textstyle\int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right)\right)\right),$$

is a G.F.Q.I: there exists an u-polynomial $\mathcal{P}^{(2)}(t,q,u)$ *such that for any fixed* $(t,q) \in [0,T] \times \mathbb{R}^n$

$$||\overline{W}(t,q,\cdot) - \mathcal{P}^{(2)}(t,q,\cdot)||_{C^1} < +\infty$$

and its leader term is positive defined (Morse index 0*).*

Proof.

$$\overline{W}(t,q,u) = \left\{\sigma(q(0)) + \int_0^t [p(s)\cdot \dot{q}(s) - H(s,q(s),p(s))]\, ds\right\}|_{(q(s),p(s))},$$

$$(q(s),p(s)) = \mathrm{pr}_\Gamma \circ g\left(t,q,\left([u+\mathcal{F}(u)](s), -V'\left(q - \textstyle\int_r^t (u+\mathcal{F}(u))(\tau)\, d\tau\right)\right)\right),$$

that is (through the Legendre transformation)

$$\overline{W}(t,q,u) = \left\{\sigma(q(0)) + \int_0^t \left[\frac{1}{2}|\dot{q}(s)|^2 - V(q(s))\right] ds\right\}|_{q(s)=q-\int_s^t [u(r)+(\mathcal{F}(u))(r)]dr},$$

$$= \sigma\left(q - \int_0^t [u(r) + (\mathcal{F}(u))(r)]\, dr\right) +$$

$$+ \int_0^t \left\{\frac{1}{2}|u(s) + (\mathcal{F}(u))(s)|^2 - V\left(q - \int_s^t [u(r) + (\mathcal{F}(u))(r)]\, dr\right)\right\} ds.$$

As a consequence of the technical Lemma 3.1.7 above and the compactness of σ and V, for fixed $(t,q) \in [0,T] \times \mathbb{R}^n$ we obtain that

$$||\overline{W}(t,q,\cdot) - \mathcal{P}^{(2)}(t,q,\cdot)||_{C^1} < +\infty,$$

where $\mathcal{P}^{(2)}(t,q,u)$ is a function with positive defined leader term $\frac{1}{2}\int_0^t |u(s)|^2 ds$ (hence with Morse index 0) and linear term with uniformly bounded coefficient, that is (see Definition 1.1.6) $\overline{W}(t,q,u)$ is a G.F.Q.I. □

3.1.3 Minimax and viscosity solutions for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$

We finally prove the main result: the equivalence of minimax and viscosity solutions for a large class of p-convex mechanical Hamiltonians.
Preliminarily, we point out the following technical fact:

Lemma 3.1.9 *Let $H(t,q,p)$ be a C^2-uniformly p-convex Hamiltonian function:*

$$\exists C \geq c > 0 : \quad c\,|\lambda|^2 \leq \frac{\partial^2 H}{\partial p_i \partial p_j}(t,q,p)\lambda_i\lambda_j \leq C\,|\lambda|^2, \tag{3.10}$$

$\forall \lambda \in \mathbb{R}^n$, $\forall t \in [0,T]$, $\forall (q,p) \in \mathbb{R}^{2n}$. *Then, for every fixed $q(\cdot) \in H^1([0,T],\mathbb{R}^n)$,*

$$\sup_{\substack{p(\cdot) \in H^1([0,T],\mathbb{R}^n) : \\ p(0) = \frac{\partial \sigma}{\partial q}(q(0))}} \int_0^T [p(t) \cdot \dot{q}(t) - H(t,q(t),p(t))]\, dt =$$

$$= \int_0^T L(t,q(t),\dot{q}(t))\, dt, \tag{3.11}$$

where

$$L(t,q,v) = \sup_{p \in \mathbb{R}^n} \{p \cdot v - H(t,q,p)\} \tag{3.12}$$

Proof. Convexity (3.10) guarantees us that global Legendre transformation holds and that, for any fixed $q(\cdot)$, the (unique, see below) critical curve of the functional:

$$\hat{A} : \left\{ H^1([0,T],\mathbb{R}^n) : \ p(0) = \frac{\partial \sigma}{\partial q}(q(0)) \right\} \longrightarrow \mathbb{R}$$

$$p(\cdot) \longmapsto \int_0^T [p(t) \cdot \dot{q}(t) - H(t,q(t),p(t))]\, dt$$

realizes the strong maximum (in the uniform convergence topology). In fact, $p(\cdot)$ is a critical curve iff, $\forall \triangle p \in H^1([0,T],\mathbb{R}^n)$ such that $\triangle p(0) = 0$,

$$d\hat{A}(p)\triangle p = d\Big\{ \int_0^T [p(t) \cdot \dot{q}(t) - H(t,q(t),p(t))]\, dt \Big\} \triangle p = 0.$$

$$\frac{d}{d\varepsilon}\Big\{ \int_0^T [(p(t) + \varepsilon \triangle p(t)) \cdot \dot{q}(t) - H(t,q(t),p(t) + \varepsilon \triangle p(t))]\, dt \Big\}\Big|_{\varepsilon=0} = 0,$$

$$\int_0^T \left[\dot{q}(t) - \frac{\partial H}{\partial p}(t,q(t),p(t)) \right] \triangle p(t) dt = 0,$$

that is,

$$\dot{q}(t) = \frac{\partial H}{\partial p}\left(t, q(t), p(t)\right), \tag{3.13}$$

and, by a standard argument laying on Legendre transformation, the unique solution $p(t)$ of (3.13), for *any* time t, is given by

$$p(t) = \frac{\partial L}{\partial v}(t, q(t), \dot{q}(t)).$$

Finally, $d^2\hat{A}(p)\big(\triangle p, \triangle p\big)$ is given by

$$d^2\Big\{\int_0^T \left[p(t)\cdot\dot{q}(t) - H\left(t, q(t), p(t)\right)\right] dt\Big\}\big(\triangle p, \triangle p\big) \leq -cT \sup_{t\in[0,T]} |\triangle p(t)|^2.$$

From the identity

$$\hat{A}(p+\triangle p) =$$

$$\hat{A}(p) + \sum_{i=1}^{n} \frac{\partial \hat{A}}{\partial p_i}(p)\triangle p_i + \int_0^1 s \sum_{i,j=1}^{n} \frac{\partial^2 \hat{A}}{\partial p_i \partial p_j}\big((1-s)(p+\triangle p)+sp\big)\triangle p_i, \triangle p_j ds,$$

we gain, at the critical p,

$$\hat{A}(p+\triangle p) - \hat{A}(p) \leq -cT||\triangle p||^2_{C^0} \leq 0$$

that is, p realizes the maximum of $\hat{A}$ in C^0, and then in $H^1(\hookrightarrow C^0)$. $\square$

Theorem 3.1.10 *Let us consider $H(q,p) = \frac{1}{2}|p|^2 + V(q)$, V compact support and the related Cauchy Problem $(CP)_H$:*

$$(CP)_H \begin{cases} \frac{\partial S}{\partial t}(t,q) + \frac{1}{2}|\frac{\partial S}{\partial q}(t,q)|^2 + V(q) = 0, \\ \\ S(0,q) = \sigma(q), \end{cases}$$

where $t \in [0,T], q \in \mathbb{R}^n$ and σ compact support.
The minimax and the viscosity solution of $(CP)_H$ coincide with the function

$$S(t,q) := \inf_{\substack{\tilde{q}(\cdot):\\ \tilde{q}:[0,t]\to\mathbb{R}^n\\ \tilde{q}(t)=q}} \sup_{\substack{\tilde{p}(\cdot):\\ \tilde{p}:[0,t]\to\mathbb{R}^n,\\ \tilde{p}(0)=\frac{\partial\sigma}{\partial q}(\tilde{q}(0))}} \Big\{\sigma(\tilde{q}(0)) + \int_0^t (p\dot{q} - H)_{|(\tilde{q},\tilde{p})} ds\Big\}. \tag{3.14}$$

Proof. In Subsection 3.1 we have proved that the Hamilton-Helmholtz functional involved in (3.14) can be interpreted as a global generating function W (with infinite parameters) for the geometric solution for the Hamiltonian H (Theorem 3.1.2). By Lemma 3.1.9, the sup-procedure on the curves $\tilde{p}$ in

(3.14) represents exactly the Legendre transformation. Moreover, the fixed point technique described in Subsection 3.1.1 reduces the function W to a finite parameters G.F.Q.I., $\overline{W}$, with Morse index 0 (Theorems 3.1.6 and 3.1.8). As a consequence of Proposition 2.4.5, for such a function $\overline{W}$, the minimax critical value coincides with the minimum (which explains the inf-procedure on the curves $\tilde{q}$ in (3.14)). Hence the function $S(t,q)$ furnishes the minimax solution of $(CP)_H$.

On the other hand (see [31], [32] and bibliography quoted therein), the function $S(t,q)$ is the Hamiltonian version of the Lax-Oleinik formula producing the viscosity solution of $(CP)_H$.

Therefore (3.14) establishes the equivalence of the two solutions. □

3.2 The weak K.A.M. theorem

In this Section we resume the main lines of the "weak K.A.M. theorem" which can also be applied, for instance, to the Aubry-Mather theory (see [30] for more details): the proof is based on the convergence of the Lax-Oleinik semigroup for a Lagrangian defined on the tangent bundle of a compact manifold which is strictly convex and superlinear in the fibers.

3.2.1 Notations

Let us denote by N a compact and connected manifold. We also suppose that N is provided with a fixed Riemannian metric. If $x \in N$, the norm $||\cdot||_x$ on T_xN is the one induced by the Riemannian metric.
We suppose given a C^r Lagrangian $L : TN \to \mathbb{R}$, with $r \geq 2$, such that

1. for each $(x,v) \in TN$, the second partial derivative $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite as a quadratic form (strict convexity in each fiber),

2. L is superlinear in each fiber of the tangent bundle $\pi : TN \to N$, that is $\forall x \in N$
$$\lim_{v\to\infty} \frac{L(x,v)}{||v||_x} = +\infty \tag{3.15}$$
$\forall v \in T_xN$.

3.2.2 A priori compactness

Lemma 3.2.1 *Let $t > 0$ be given. There exists a constant $C_t < +\infty$, such that, for each $x, y \in N$, we can find a C^∞ curve $\gamma : [0,t] \to N$ with $\gamma(0) = x$, $\gamma(t) = y$ and*
$$\mathbb{L}(\gamma) := \int_0^t L(\gamma(s), \dot{\gamma}(s))ds \leq C_t.$$

Proof. By the compactness of N, we can find a geodesic between x and y. The length of this geodesic will be $d(x, y)$. Let us parametrize this geodesic by the interval $[0, t]$ with a speed of constant norm and denote by $\gamma : [0, t] \to N$ this parameterization. As the length of this curve is $d(x, y)$, we find that

$$\forall s \in [0, t], \ ||\dot{\gamma}(s)||_{\gamma(s)} = \frac{d(x, y)}{t}. \tag{3.16}$$

Since the manifold N is compact, the diameter $\mathrm{diam}(N)$ of N for the metric d is finite and hence the set

$$A_t = \{(x, v) \in TN : \ ||v||_x \leq \frac{\mathrm{diam}(N)}{t}\}$$

is compact. From (3.16) we have that $(\gamma(s), (\dot{\gamma}(s))) \in A_t$, for all $s \in [0, t]$. By compactness of A_t and continuity of L, we can find a constant $\widetilde{C}_t < +\infty$ such that

$$\forall (x, v) \in A_t, \ L(x, v) \leq \widetilde{C}_t,$$

where $\widetilde{C}_t := \max_{(x,v)\in A_t} L(x, v)$.
If we set $C_t = t\widetilde{C}_t$, by the mean value theorem, there exists a $s_0 \in [0, t]$ such that

$$\mathbb{L}(\gamma) = \int_0^t L(\gamma(s), \dot{\gamma}(s))ds = tL(\gamma(s_0), \dot{\gamma}(s_0)) \leq t\widetilde{C}_t = C_t.$$

□

In the next Corollary 3.2.3 we make use of a fundamental result of Tonelli's theory. The main lines of this theory are summarized in the following

Theorem 3.2.2 *(Tonelli) Let $L : TN \to \mathbb{R}$ be a C^r Lagrangian, with $r \geq 2$, where N is a compact manifold. We suppose that L is strictly convex and superlinear in each fiber of the tangent bundle $\pi : TN \to N$. Then, we have:*

- *the Eulero-Lagrange flow is complete and C^{r-1},*
- *the extremal curves[1] are of class C^r,*
- *for each $x, y \in N$, each $a, b \in \mathbb{R}$, with $a < b$, there exists an extremal curve $\gamma : [a, b] \to N$ with $\gamma(a) = x$, $\gamma(b) = y$ and such that for all other absolutely continuous curves $\gamma_1 : [a, b] \to N$, with $\gamma_1(a) = x$ and $\gamma_1(b) = y$, we have $\mathbb{L}(\gamma) \leq \mathbb{L}(\gamma_1)$.*
- *if $[a, b] \to N$ is an absolutely continuous curve such that for each other absolutely continuous curve $\gamma_1 : [a, b] \to N$, with $\gamma_1(a) = \gamma(a)$ and $\gamma_1(b) = \gamma(b)$, we have $\mathbb{L}(\gamma) \leq \mathbb{L}(\gamma_1)$, then the curve γ is an extremal curve. In particular, it is of class C^r.*

[1]An extremal curve for the Lagrangian L is a piecewise C^1 curve $\gamma : [a, b] \to N$ such that $\frac{d}{dt}\mathbb{L}(\gamma + t\gamma_1)_{t=0} = 0$, for every C^∞ curve $\gamma_1 : [a, b] \to \mathbb{R}^n$ satisfying $\gamma_1 = 0$ in the neighborhood of a and b.

Corollary 3.2.3 *(A priori compactness) If $t > 0$ is fixed, there exists a compact subset $K_t \subset TN$ such that for every minimizing extremal curve $\gamma : [a, b] \to N$, with $b - a \geq t$, we have*

$$\forall s \in [a, b],\ (\gamma(s), \dot{\gamma}(s)) \in K_t.$$

Proof. We first observe that it is enough to show the Corollary if $[a, b] = [0, t]$. Indeed, if $t_0 \in [a, b]$, we can find an interval of the form $[c, c + t]$, with $t_0 \in [c, c + t] \subset [a, b]$. The curve $\gamma_c : [0, t] \to N$, $s \mapsto \gamma(c + s)$ satisfies the assumptions of the Corollary with $[0, t]$ in the place of $[a, b]$.
Thus let us give the proof of the Corollary with $[a, b] = [0, t]$. As a consequence of previous Lemma 3.2.1 and Theorem 3.2.2, for every minimizing extremal curve $\gamma : [a, b] \to N$, with $b - a \geq t$, we necessarily have

$$\mathbb{L}(\gamma) = \int_0^t L(\gamma(s), \dot{\gamma}(s))ds \leq C_t.$$

Since $s \mapsto L(\gamma(s), \dot{\gamma}(s))$ is continuous on the bounded set $[0, t]$, by the mean value theorem, we can find $s_0 \in [0, t]$ such that

$$L(\gamma(s_0), \dot{\gamma}(s_0)) \leq \frac{C_t}{t}. \tag{3.17}$$

The set $B = \{(x, v) \in TN :\ L(x, v) \leq \frac{C_t}{t}\}$ is a compact subset of TN. By continuity of the flow ϕ_t, the set $K_t = \bigcup_{|s| \leq t} \phi_s(B)$ is also a compact subset of TN. Moreover the inequality in (3.17) means

$$(\gamma(s_0), \dot{\gamma}(s_0)) \in B,$$

consequently

$$\phi_{s-s_0}(\gamma(s_0), \dot{\gamma}(s_0)) = (\gamma(s), \dot{\gamma}(s)) \in \phi_{s-s_0}(B) \subset K_t$$

$\forall s \in [0, t]$. □

3.2.3 The weak K.A.M. theorem

We introduce a semigroup of non-linear operators $(T_t^-)_{t \geq 0}$ from $C^0(N, \mathbb{R})$ into itself. This semigroup is known as the Lax-Oleinik semigroup. To define it let us fix $u \in C^0(N, \mathbb{R})$ and $t > 0$. For $x \in N$, we set

$$T_t^- u(x) = \inf_{\gamma} \{u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s))ds\},$$

where the infimum is taken over all the absolutely continuous curves $\gamma : [0, t] \to N$ such that $\gamma(t) = x$. As a consequence of the compactness of N and the superlinearity of L, we have the following

Lemma 3.2.4 *For every $x \in N$, there exists a constant $l_0 \in \mathbb{R}$ (depending on L) such that*

$$T_t^- u(x) \geq l_0 t - ||u||_\infty, \tag{3.18}$$

where $||u||_\infty = \sup_{x \in N} |u(x)|$.

Proof. The inequality (3.18) is a consequence of the following considerations:

$$u(\gamma(0)) \geq -\sup_{x \in N} |u(x)| = -||u||_\infty \tag{3.19}$$

where $||u||_\infty$ is the L^∞ norm of the continuous function u, which is of course finite by the compactness of N.
The Lagrangian $L : TN \to \mathbb{R}$ is superlinear in each fiber of the cotangent bundle $\pi : TN \to N$: for every $K < +\infty$, there exists $c(K) > -\infty$ such that[2]

$$L(\gamma(s), \dot{\gamma}(s)) \geq K||\dot{\gamma}(s)||_{\gamma(s)} + c(K).$$

Consequently

$$\int_0^t L(\gamma(s), \dot{\gamma}(s))ds \geq \int_0^t K||\dot{\gamma}(s)||_{\gamma(s)} + c(K)ds \geq$$

$$\geq t \sup_{s \in [0,t]} (K||\dot{\gamma}(s)||_{\gamma(s)} + c(K)) =: t\ l_0.$$

Using now also (3.19), we prove that $\forall x \in N$ there exists a constant $l_0 \in \mathbb{R}$ such that

$$T_t^- u(x) \geq l_0 t - ||u||_\infty$$

□
The following Lemma 3.2.5 guarantees that the infimum is a minimum, realized by a extremal curve $\gamma : [0, t] \to N$.

Lemma 3.2.5 *If $t > 0$, $u \in C^0(N, \mathbb{R})$ and $x \in N$ are given, there exists an extremal curve $\gamma : [0, t] \to N$ such that $\gamma(t) = x$ and*

$$T_t^- u(x) = u(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s))ds.$$

Such an extremal curve γ minimizes the action among all absolutely continuous curves $\gamma_1 : [0, t] \to N$ with $\gamma_1(0) = \gamma(0)$ and $\gamma_1(t) = x$. In particular, we have

$$\forall s \in [0, t],\ (\gamma(s), \dot{\gamma}(s)) \in K_t,$$

where $K_t \subset TN$ is the compact set gives by the Corollary 3.2.3.

[2] This property results equivalent to (3.15) for continuous function.

Proof. Let us fix $y \in N$ and denote by $\gamma_y : [0,t] \to N$ an extremal curve minimizing the action among all absolutely continuous curves $\gamma : [0,t] \to N$ such that $\gamma(t) = x$ and $\gamma(0) = y$. We have

$$T_t^- u(x) = \inf_{y \in N} \{u(y) + \int_0^t L(\phi_s(y, \dot\gamma_y(0)))ds\}.$$

By the a priori compactness given by the Corollary 3.2.3, the points $(\gamma_y(0), \dot\gamma_y(0)) = (y, \dot\gamma_y(0))$ are all in the same compact subset $K_t \subset TN$. We can then find a sequence of $y_n \in M$ such that

$$(y_n, \dot\gamma_{y_n}(0))_{n\in\mathbb{N}} \text{ is a subsequence of } (y, \dot\gamma_y(0))_{y\in N},$$

$$(y_n, \dot\gamma_{y_n}(0)) \to (y_\infty, v_\infty) \text{ and}$$

$$u(y_n) + \int_0^t L[\phi_s(y_n, \dot\gamma_{y_n}(0))]ds \to T_t^- u(x).$$

By continuity of u, of L and of the flow ϕ_t, we have

$$T_t^- u(x) = u(y_\infty) + \int_0^t L[\phi_s(y_\infty, v_\infty)]ds.$$

The fact that $\gamma(s) = \pi \circ \phi_s(y_\infty, v_\infty)$ minimizes the action is obvious from the definition of T_t^-. $\square$

Since the extremal curves have the same regularity of the Lagrangian (thus are C^r, with $r \geq 2$) –see Tonelli's Theorem 3.2.2– it is possible, in the definition of the semigroup $T_t^- u$, replace the absolutely continuous curves by (continuous) piecewise C^1 (respectively of class C^1 or even C^r) curves without changing the value of $T_t^- u(x)$.

The following Lemma 3.2.6 proves that, for $t > 0$ fixed, the family of function $T_t^- u : N \to \mathbb{R}$ is equi-Lipschitzian (and thus equi-continuous) –see [32] for a detailed proof.

Lemma 3.2.6 *For each $t > 0$, there exists a constant K_t such that $T_t^- u : N \to \mathbb{R}$ is K_t-Lipschitzian for each $u \in C^0(N, \mathbb{R})$.*

We recall now the well-known definition

Definition 3.2.7 *(Non-expansive map) A map $\phi : X \to Y$, between the metric spaces X and Y, is said to be non-expansive if it is Lipschitzian with Lipschitz constant ≤ 1.*

Corollary 3.2.8 *(Properties of T_t^-)*

1. *Each T_t^- maps $C^0(N, \mathbb{R})$ into itself.*

2. *(Semigroup property) We have*

$$T^-_{t+\bar{t}} = T^-_t \circ T^-_{\bar{t}},$$

for each $t, \bar{t} > 0$.

3. *(Monotony) For each* $u, v \in C^0(N, \mathbb{R})$ *and all* $t > 0$, *we have*

$$u \leq v \Longrightarrow T^-_t u \leq T^-_t v.$$

4. *If* c *is a constant and* $u \in C^0(N, \mathbb{R})$, *we have* $T^-_t(c + u) = c + T^-_t u$.

5. *(Non-expansiveness) The maps* T^-_t *are non-expansive:* $\forall u, v \in C^0(N, \mathbb{R})$, $\forall t \geq 0$

$$||T^-_t u - T^-_t v||_\infty \leq ||u - v||_\infty.$$

Proof. Assertion 1. is a consequence of the previous Lemma 3.2.6. The assertions 2., 3. and 4. result from the definition of T^-_t. To show 5., we notice that $-||u - v||_\infty + v \leq u \leq ||u - v||_\infty + v$ and we apply 3. and 4. □

The proof of the weak K.A.M. theorem is based on Lemma 3.2.6 and on these fixed points results.

Proposition 3.2.9 *(Fixed points results)*

1. *Let* E *be a normed space and* $K \subset E$ *a compact convex subset. We suppose that the map* $\phi : K \to K$ *is non-expansive. Then* ϕ *has a fixed point.*

2. *Let* E *be a Banach space and let* $C \subset E$ *be a compact subset. Then the closed convex envelope of* C *in* E *is itself compact.*

3. *Let* E *be a Banach space. If* $\phi : E \to E$ *is a non-expansive map such that* $\phi(E)$ *has a relatively compact image in* E, *the map* ϕ *admits a fixed point.*

4. *Let* E *be a Banach space and* $\phi_t : E \to E$ *be a family of maps defined for* $t \in [0, \infty[$. *We suppose that the following conditions are satisfied*

   - *For each* $t, \bar{t} \in [0, \infty[$, *we have* $\phi_{t+\bar{t}} = \phi_t \circ \phi_{\bar{t}}$.
   - *For each* $t \in [0, \infty[$, *the map* ϕ_t *is non-expansive.*
   - *For each* $t > 0$, *the image* $\phi_t(E)$ *ie relatively compact in* E.
   - *For each* $x \in E$, *the maps* $t \mapsto \phi_t(x)$ *is continuous on* $[0, \infty[$.

   Then the maps ϕ_t *have a common fixed point.*

Proof. 1. We can always assume that $0 \in K$. Let us consider a parameter $\lambda \in (0,1)$. Then the map

$$K \ni x \mapsto \lambda\phi(x) \in K$$

is a contraction: it admits an unique fixed point $x_\lambda \in K$ ($x_\lambda = \lambda\phi(x_\lambda)$). Let $(\lambda_n)_n \in (0,1)$ be a sequence converging to 1. Then for every $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$x_n = \lambda_n \phi(x_n).$$

Up to a converging subsequence, we obtain $x_n \to \bar{x} \in K$, therefore

$$\bar{x} = \lim_n x_n = \lim_n \lambda_n \phi(x_n) = \phi(\bar{x}).$$

2. We define the following continuous map

$$f : C \times C \times [0,1] \to E$$

$$(x, y, t) \mapsto tx + (1-t)y.$$

The set $C \times C \times [0,1]$ is compact, therefore $f(C \times C \times [0,1])$, coinciding to the closed convex envelope of C, is compact.
3. Let us define K the closed convex envelope of $\overline{\phi(E)}$, which results, for the previous point, a compact set. Therefore the non-expansive map $\phi_{|_K} : K \to K$, from the convex and compact set K into itself, admits a fixed point (see point 1.).
4. We first observe that, if $t > 0, h > 0$, then

$$\phi_{t+h}(E) = \phi_t(\phi_h(E)) \subset \phi_t(E). \tag{3.20}$$

Moreover, if x is a fixed point for the map ϕ_t, therefore it is a fixed point for every ϕ_{nt}, $n \in \mathbb{N}$, in fact

$$\phi_{nt}(x) = \phi_t \circ \phi_t \circ \ldots \circ \phi_t(x) \text{ (n times)} = x. \tag{3.21}$$

For every $n \in \mathbb{N}$, let x_n be a fixed point for $\phi_{\frac{1}{2^n}}$: $\phi_{\frac{1}{2^n}}(x_n) = x_n$. For (3.21), we obtain that

$$\phi_{\frac{1}{2^{kn}}}(x_n) = x_n \qquad \forall k \in \mathbb{N}. \tag{3.22}$$

In particular –see (3.20)– $(x_n)_n \subset \phi_{\frac{1}{2}}(E)$. As a consequence of the compactness of $\overline{\phi_{\frac{1}{2}}(E)}$, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ converging to a value $\bar{x}$. Therefore

$$\phi_{\frac{1}{2^n}}(\bar{x}) = \lim_{i \to +\infty} \phi_{\frac{1}{2^n}}(x_{n_i}) = \lim_{i \to +\infty} x_{n_i} = \bar{x} \qquad \forall n \in \mathbb{N},$$

i.e. $\bar{x}$ is a common fixed point for every $\phi_{\frac{1}{2^n}}$. It is easy to prove now that $\bar{x}$ is a fixed point for $\phi_{\sum_n c_n \frac{1}{2^n}}$, $c_n \in \mathbb{N}^*$.
The expected result is finally obtained using continuity of $t \mapsto \phi_t(\bar{x})$ and the density of the set $\{\sum_n c_n \frac{1}{2^n} : \; c_n \in \mathbb{N}^*\}$ in $[0, +\infty[$. □

We claim now the main result of this section.

Theorem 3.2.10 *(Weak K.A.M.) There exists a Lipschitz function $u_- : N \to \mathbb{R}$ and a constant c such that*

$$T_t^- u_- + ct = u_-,$$

for each $t \in [0, \infty[$.

Proof. Let us denote by $\mathbf{1}$ the constant function equal to 1 everywhere on N and consider the quotient $E = C^0(N, \mathbb{R})/\mathbb{R}\cdot\mathbf{1}$. This quotient space E is a Banach space for the quotient norm

$$||[u]|| = \inf_{a\in\mathbb{R}} ||u + a\cdot\mathbf{1}||_\infty,$$

where $[u]$ is the class in E of $u \in C^0(N, \mathbb{R})$.
Since $T_t^-(u + a\cdot\mathbf{1}) = T_t^- u + a\cdot\mathbf{1}$, when $a \in \mathbb{R}$, the maps T_t^- pass to the quotient to a semigroup $\overline{T}_t^- : E \to E$ consisting of non-expansive maps. Now we apply Ascoli's theorem to the equi-Lipschitzian family of maps T_t^- (here $t > 0$ is fixed) concluding that the image of $\overline{T}_t^-$ is relatively compact in E.
Using part 4. of Proposition 3.2.9 above, we find a common fixed point for all the $\overline{T}_t^-$ (indipendence of the fixed point on t). Then we deduce that there exists $u_- \in C^0(N, \mathbb{R})$ such that $T_t^- u_- = u_- + c_t$, where c_t is a constant. The semigroup property gives $c_{t+\bar{t}} = c_t + c_{\bar{t}}$; since $t \mapsto T_t^- u$ is continuos, we obtain $c_t = -tc$ with $c = -c_1$. We thus have $T_t^- u_- + ct = u_-$. □

3.2.4 Weak K.A.M. theory: Hamilton-Jacobi PDE

In this Subsection, we reinterpreted the previous result of the weak K.A.M. theorem in terms of the theory of Hamilton-Jacobi PDE. Using the coincidence of viscous and minimax solutions for p-convex Hamiltonians proved in Subsection 3.1.3, we can finally compare the viscous solution given by the weak K.A.M. theorem to the corresponding minimax solution.

Definition 3.2.11 *(Dominated function)*
Let $u : N \to \mathbb{R}$ be a continuous function. If $c \in \mathbb{R}$, we say that u is dominated by $L + c$, and we write $u \prec L + c$, if for each continuous Lipschitz curve $\gamma : [a, b] \to N$ we have

$$u(\gamma(b)) - u(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot{\gamma}(s))ds + c(b - a).$$

We note that if $u \prec L + c$, the map u results Lipschitz with Lipschitz constant $\leq A + c$, where $A = \sup\{L(x, v)|\ (x, v) \in TN,\ ||v||_x = 1\}$. Hence, by Rademacher's theorem, a continuous function $u : N \to \mathbb{R}$ such that $u \prec L + c$ is almost everywhere differentiable.
The lemma below follows immediately from the definitions:

Lemma 3.2.12 *Let $u : N \to \mathbb{R}$. We have $u \prec L + c$ if and only if $u \leq ct + T_t^- u$, for each $t \geq 0$.*

In view of the strict convexity and superlinearity of L, we can uniquely and smoothly solve the equation

$$p = \partial_v L(x, v)$$

for $v = \mathbf{v}(x, p)$.

We define the Hamiltonian

$$H(x, p) := p \cdot \mathbf{v}(x, p) - L(x, \mathbf{v}(x, p)).$$

Equivalently (L is superlinear),

$$H(x, p) = \max_{v \in T_x N} (p \cdot v - L(x, v)).$$

In the following theorem we reinterpret the ideas underlying the previous weak K.A.M. theorem in terms of the theory of viscosity solutions related to the Hamiltonian H.

Theorem 3.2.13 *The following statements hold:*

1. *If $u \prec L + c$ and the gradient $Du(x)$ exists at a point $x \in N$, then*

$$H(x, Du(x)) \leq c.$$

2. *Conversely, if u is Lipschitz continuous and $H(x, Du(x)) \leq c$ a.e., then*

$$u \prec L + c.$$

3. *Finally, for the function u_- defined above, we have*

$$H(x, Du_-(x)) = c \qquad a.e.$$

Proof.
1. We select a curve γ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Then

$$\frac{u(\gamma(t)) - u(x)}{t} \leq \frac{1}{t} \int_0^t L(\dot{\gamma}, \gamma) ds + c.$$

Let $t \to 0$, to discover

$$Du(x) \cdot v \leq L(x, v) + c,$$

and therefore

$$H(x, Du(x)) = \max_{v \in T_x N} (Du(x) \cdot v - L(x, v)) \leq c.$$

2. We prove the result when u is smooth. See [32] for the case in which u is only Lipschitz. If u is smooth, we can compute

$$\begin{aligned} u(\gamma(b)) - u(\gamma(a)) &= \int_a^b \frac{d}{dt} u(\gamma(t)) ds \\ &= \int_a^b Du(\gamma) \cdot \dot{\gamma} dt \\ &\leq \int_a^b L(\gamma, \dot{\gamma}) + H(\gamma, Du(\gamma)) dt \\ &\leq \int_a^b L(\gamma, \dot{\gamma}) dt + c(b - a). \end{aligned}$$

3. In view of the weak K.A.M. theorem 3.2.10, there exists a minimizing curve $\gamma : [0, +\infty) \to N$ such that $\gamma(t) = x$ and

$$u_-(x) = u_-(\gamma(0)) + \int_0^t L(\gamma, \dot{\gamma}) d\tau + ct.$$

If u_- is differentiable at $x = \gamma(t)$, then we deduce as before that

$$\frac{d}{ds}[u_-(\gamma(t+s)) - u_-(\gamma(0))]_{|s=0} =$$

$$\frac{d}{ds}[\int_0^{s+t} L(\gamma, \dot{\gamma}) d\tau + c(t+s)]_{|s=0},$$

that is

$$Du_-(x) \cdot \dot{\gamma}(t) = L(x, \dot{\gamma}(t)) + c,$$

and this implies $H(x, Du_-(x)) \geq c$. But we have seen in 1. that always $H(x, Du_-(x)) \leq c$, therefore $H(x, Du_-(x)) = c$ a.e. $\Box$

The function u_- actually solves

$$H(x, Du_-(x)) = c$$

in the viscosity sense. In the case where $N = \mathbb{T}^n$, this theorem has been first proved by Lions, Papanicolaou and Varadhan (see [42]). In general, the assertions about u_- being viscosity solutions follow as in Chapter 10 of [29].

Let us consider the case $N = \mathbb{T}^n$. We define the time-dependent function

$$S_-(t, x) := u_-(x) - ct. \tag{3.23}$$

As a consequence of the weak K.A.M. theorem 3.2.10, $S_-(t,x) = T_t^- u_-$. Since the infimum is reached by an extremal curve with the same regularity of the Lagrangian, it is possible, in the definition of the semigroup $T_t^- u$, replace the absolutely continuous curves by curves of class C^1, without changing the value of $T_t^- u(x)$. This fact implies the following

Lemma 3.2.14 *Let $N = \mathbb{T}^n$.*

$$T_t^- u_- = \inf_{\substack{\gamma \in C^1([0,t],N),\\ \gamma(t)=x}} \{u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))ds\},$$

coincides –as a function of (t,x)– with the representation formula (3.14).

Proof. Up to the Legendre transformation, the representation formula (3.14) is given by

$$S(t,x) = \inf_{\substack{\gamma \in H^1([0,t],N),\\ \gamma(t)=x}} \{u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))ds\}. \tag{3.24}$$

Now $H^1((0,t),N) \hookrightarrow C^0([0,t],N)$ compactly, then the infimum in (3.24) coincides with the infimum in the class of C^0-curves (with $\gamma(t) = x$).
As a consequence that the minimum is an extremal curve with the same regularity of the Lagrangian –therefore at least C^2– we obtain that in (3.24) it is sufficient to consider the class of C^1-curves (with $\gamma(t) = x$). □

The strict convexity in the fibers of the Lagrangian L implies the strict convexity in the p-variables of the related Hamiltonian H; therefore, theorem 3.1.10 works, assuring that the time-dependent function (3.23) solves the Cauchy problem:

$$\begin{cases} H(x, DS(t,x)) = c \\ S(0,x) = u_-(x) \end{cases} \tag{3.25}$$

both in the viscous and minimax sense.

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