Phase portrait of 1-dim conservative system.

- Physical problem. Point $P$ of mass $m > 0$ in $\mathbb{R}^3$ subjected to a positional force field. The point $P$ is constrained on a curve. On the reference $\mathcal{O}xyz$, the position of the point $P$ is given by the vector $\vec{OP}$. The positional force field is given by the function $F : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{OP} \mapsto \vec{F}(\vec{OP})$.

The dynamic of the point $P$ is given by the Newton's law:

$$m \ddot{\vec{OP}} = \vec{F}(\vec{OP}) + \vec{\Phi}$$

Forces (only positional)

- Example.

$$\begin{align*}
&\text{Constrain: } x \text{-axis.} \\
&\text{Ideal constrain: } \vec{\Phi} = (\phi_1, \phi_2, \phi_3) \\
&\text{such that } \vec{e}_1 \cdot \vec{\Phi} = 0 \Leftrightarrow \phi_1 = 0.
\end{align*}$$

Project $m \ddot{\vec{OP}} = \vec{F}(\vec{OP}) + \vec{\Phi}$ on the coordinate axis,

$$\begin{align*}
\{ m \ddot{x} &= -Kx \\
0 &= \phi_2 \\
0 &= -mg + \phi_3
\end{align*}$$

and the dynamic is given by the equation $m \ddot{x} = -Kx$ of type

$$m \ddot{x} = f(x) = \vec{F}(\vec{OP}) \cdot \vec{e}_2$$

second order differential equation

- The force field $\vec{OP} \mapsto \vec{F}(\vec{OP})$ is called conservative if there exists a function $V : \mathbb{R}^3 \to \mathbb{R}$, $\vec{OP} \mapsto V(\vec{OP})$ s.t.$\vec{F}(\vec{OP}) = -\nabla V(\vec{OP})$

- In the sequel, we will study the case $m = 1$ with $f : \mathbb{R} \to \mathbb{R}$ continuous.
Then \( V(x) = -\int_a^x f(t) \, dt \) \( \forall x \in \mathbb{R} \),
Fundamental theorem
of calculus.

This means that in the sequel we will study in detail equations of type:

\[ \text{max} = f(x) \equiv -V''(x) \]

**Remarks**

1. Configuration space: \( \mathbb{R} \times \mathbb{R} \)
   Phase space: \( \mathbb{R}^2 = \mathbb{R}_x \times \mathbb{R}_v \)
2. The potential energy \( V: \mathbb{R} \to \mathbb{R} \) is defined up to an additive constant.
3. First order
   \[
   \begin{cases}
   \dot{x} = V \\
   \dot{v} = -\frac{V'(x)}{m}
   \end{cases}
   \] (\#)

4. Let \( t \mapsto (x(t), v(t)) \) be a curve solution of (\#), then \( \dot{x}(t) = v(t) \) \( \forall t \in \mathbb{R} \).

**What does it mean vertical tangent?**

It is a vector of type \((0, \dot{v})\) \( \iff \dot{x} = 0 \)

\[ \iff \dot{v} = 0 \]

Then: Orbit of (\#) cannot have vertical tangent. This phenomenon is possible only when orbit crosses the \( x \)-axis (where \( v = 0 \)).

5. The dynamical system (\#) admits
   the energy function
   \( E(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + V(x) \) as first integral.

**Proof**

\[
L_x E(x, \dot{x}) = \nabla E(x, \dot{x}) \cdot \dot{x} = m \ddot{x} \left( -\frac{V'(x)}{m} \right) + V'(x) v \equiv 0
\]

\[ \square \]
\[ x = \text{distance from P to earth} \]
\[ F(x) = -mg = \frac{d}{dx} m\ddot{x} - mg = 0 \quad \begin{cases} \dot{x} = v \\ \ddot{x} = -g \end{cases} \]
\[ V(x) = mgx \]
\[ E(x, v) = \frac{1}{2}mv^2 + mgx \]

Keplerian gravitational force.

\[ F(\mathbf{z}) = -\frac{GMm}{r^2} \quad \begin{cases} \dot{r} = v \\ \dot{v} = -\frac{GM}{r^2} \end{cases} \]
\[ V(\mathbf{z}) = -\frac{GMm}{r} \]
\[ E(\mathbf{r}, \mathbf{v}) = \frac{1}{2}mv^2 - \frac{GMm}{r} \]

Spring.

\[ F(x) = -kx = 0 \quad m\ddot{x} = -kx \quad \text{(harmonic oscillator)} \]
\[ V(x) = \frac{1}{2}kx^2 = E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \]

Penobulum.

\[ S = \ell \Theta \]

Project on \( \hat{t} \):
\[ m\dddot{\Theta} = -mg \cos \left( \frac{\pi}{2} - \Theta \right) + \phi \]
\[ = -mg \sin \Theta + 0 = 0 \]
\[ m\Theta = -mg \sin \Theta \]

\[ E(\Theta, \dot{\Theta}) = \frac{1}{2}m\dot{\Theta}^2 - mg \cos \Theta \]
\[ E(s, \dot{s}) = \frac{1}{2}m\dot{s}^2 - mg \cos \left( \frac{s}{\ell} \right) \]
\[ E(s, \dot{s}) = \frac{1}{2}m\dot{s}^2 - mg \ell \cos \left( \frac{s}{\ell} \right) \]
Let $N_C = \{ (x, \dot{x}) \in \mathbb{R}^2 : E(x, \dot{x}) = c \}$ be the level set for $E(x, \dot{x})$.

We want to draw on the phase plane $(x, \dot{x})$ (phase-space) the set $N_C$, with $c \in \mathbb{R}$.

Recall that the total energy is conserved. So, if

$$ t \mapsto (x_t, \dot{x}_t) = (x_0, \dot{x}_0) $$

is a solution passing through $(x_0, \dot{x}_0) = (x_0, \dot{x}_0)$ then:

$$ E(x_0, \dot{x}_0) = \frac{1}{2} m \dot{x}_0^2 + V(x_0) = c $$

Then:

$$ E(x_t, \dot{x}_t) = \frac{1}{2} m \dot{x}_t^2 + V(x_t) = c = \frac{1}{2} m \dot{x}_t^2 = c - V(x_t) \quad \text{that is} $$

$$ c - V(x_t) \geq 0 \quad \Rightarrow \quad V(x_t) \leq c \quad \forall t \in \mathbb{R}.$$

with a draw:

![Phase Portrait](image)

The phase portrait is symmetric with respect to $x$-axis.

The phase portrait is regular outside equilibrium. In other words, unique singular points of $N_C$ (isolated point, auto-intersection, angle point) are equilibrium.
Exercise 1

Let consider the 2-dim system:
\[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -x_1 - x_2 + g(x_2)
\end{align*}
\]
where \(g \in C^0(\mathbb{R}; \mathbb{R})\), \(g(0) = 0\).

(a) Determine an hypothesis on \(g\) so that \((0,0)\) is a stable equilibrium.

(b) Determine an hypothesis on \(g\) so that \((0,0)\) is an asymptotically stable equilibrium.

Solution

We use the candidate Lyapunov function \(E(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)\).

(a) \(\dot{E}(x_1, x_2) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 + x_2) + x_2(-x_1 - x_2 + g(x_2)) = -x_1^2 + x_1 x_2 - x_1 x_2 - x_2^2 + x_2 g(x_2) \leq 0\)

If there exists a neighborhood \(U\) of \(0\) such that \(\forall x_2 \in U\) such that \(\text{sgn} x_2 = -\text{sgn} g(x_2)\)

For example:

![Diagram]

But \(g\) can also be \(0\) in \(U\).

(b) Analogously, for the asymptotic stability:

If there exists a neighborhood \(U\) of \(0\) such that
\[
\begin{align*}
g(x_2) \neq 0 \forall x_2 \in U \setminus \{0\} \\
\text{sgn} x_2 = -\text{sgn} g(x_2) \forall x_2 \in U \setminus \{0\}.
\end{align*}
\]

Exercise 2

Let consider the 2-dim system:
\[
\begin{align*}
\dot{x} &= -x^3 + y^3 \\
\dot{y} &= -x^3 - y^3
\end{align*}
\]

By using one of these function:
\[ f(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \frac{1}{4} (x^4 + y^4) \]
\[ g(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (x^4 + y^4) \]
\[ h(x, y, \dot{x}, \dot{y}) = \frac{1}{2} (\dot{x}^2 - \dot{y}^2)^2 + \frac{1}{4} (x^4 - y^4)^2 \]

Prove that \((0, 0)\) is a stable equilibrium.

**Solution**

We can only use \(g(x, y, \dot{x}, \dot{y})\) and we obtain:

\[ L_x g(x, y) = x^3 \dot{x} + y^3 \dot{y} = x^3 (-x^3 + y^3) + y^3 (-x^3 - y^3) \]
\[ = -x^6 - y^6 < 0 \quad \forall (x, y) \neq (0, 0) \]

\(0, 0)\) is asymptotically stable.

**Exercise 3**

Let consider the second order differential equation:

\[ \ddot{x} = -\sin x - x^4 \dot{x}, \quad x \in \mathbb{R}. \]

Discuss the stability of \(x = 0\) with both Lyapunov's methods.

**Solution**

1. Spectral method. No conclusion, since \(JX(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)

\[ \begin{cases} \dot{x} = y \\ \dot{y} = -\sin x - x^4 y \end{cases} \]

with \(\lambda_{1,2} = \pm i\).

2. With an appropriate Lyapunov function (the system is a perturbed pendulum...). We use the candidate Lyapunov function:

\[ E(x, y) = \frac{1}{2} y^2 + 1 - \cos x \]

\[ L_x E(x, y) = \sin x (\dot{x}) + y (-\sin x - x^4 y) = -x^4 y^2 < 0 \quad \Rightarrow \]

\(0\) is stable.

**Exercise 4**

Draw the phase portrait of the HARMONIC OSCILLATOR and the HARMONIC REPELER.
Harmonic oscillator \( m \ddot{x} = -Kx \Rightarrow E(x, \sigma) = \frac{1}{2} mv^2 + \frac{1}{2} Kx^2 \quad (K > 0) \)

Harmonic repeller \( m \ddot{x} = Kx = 0 \Rightarrow E(x, \sigma) = \frac{1}{2} mv^2 - \frac{1}{2} Kx^2 \quad (K > 0) \)

Note on Ex. 4, Lesson 9.

\[ W(x) = \prod_{i=1}^{n} |f_i(x)|^2 = \prod_{i=1}^{n} |f_i(x)|^2 + f_2^2 + \ldots + f_n^2. \]

\[ \nabla W(x) = 2(f_1(x) \partial_x f_1(x) + f_2(x) \partial_x f_2(x) + \ldots + f_n(x) \partial_x f_n(x)) \]

\[ \begin{align*}
    &f_1(x) \partial_x f_1(x) + f_2(x) \partial_x f_2(x) + \ldots + f_n(x) \partial_x f_n(x) \\
    &f_1(x) \partial_x f_1(x) \partial_x f_1(x) + f_2(x) \partial_x f_2(x) \partial_x f_2(x) + \ldots + f_n(x) \partial_x f_n(x) \partial_x f_n(x)
\end{align*} \]

\[ = 2 \left( \begin{array}{c}
    f_1(x) \\
    \vdots \\
    f_n(x)
\end{array} \right) \left( \begin{array}{c}
    \partial_x f_1(x) \\
    \partial_x f_2(x) \\
    \vdots \\
    \partial_x f_n(x)
\end{array} \right) = 2f(x)f'(x) \]
LESSON 11

Other phase portraits

- Gravitational force near earth.
  \[ V(x) = mgx \]

- Keplerian gravitational force.
  \[ V(r) = -\frac{GMm}{r} \]

- Pendulum.
  \[ V(\theta) = -\frac{mg \cos \theta}{l} \]
EXERCISE 1

Solve the Cauchy problem

\[
\begin{align*}
\ddot{x} &= 4x^3 + 4x \\
\dot{x}(0) &= x_0 = 0 \\
x(0) &= \dot{x}_0 = \sqrt{2}
\end{align*}
\]

SOLUTION

Use the first integral \( E(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + V(x) \). In such a case:

\[
\begin{align*}
4x^3 + 4x &= -V'(x) = 0 \\
V(x) &= -\int_{x_0}^{x} (4s^3 + 4s) \, ds = \\
&= -[s^4 + 2s^2] \Big|_{x_0}^{x} = -[s^4 + 2s^2]_0^x = \\
&= -x^4 - 2x^2.
\end{align*}
\]

Therefore:

\[
E(x, \dot{x}) = \frac{1}{2} \dot{x}^2 - x^4 - 2x^2 = E(x_0, \dot{x}_0) = E(0, \sqrt{2}) = 1
\]

\[
\dot{x} = \operatorname{sgn}(\dot{x}_0) \sqrt{2 \left( 1 + x^4 + 2x^2 \right)}
\]

the equality hold until the first instant such that \( \dot{x} = 0 \).

\[
\sqrt{2 \left( 1 + x^4 + 2x^2 \right)} = \sqrt{2} \left( 1 + x^2 \right)
\]

In our case \( \operatorname{sgn}(\dot{x}_0) = \operatorname{sgn}(\sqrt{2}) > 0 \).

Let integrate by separation of variables,

\[
\frac{dx}{\sqrt{2(x^2 + 1)}} = \int_{\dot{x}_0}^{x} \frac{1}{y^2 + 1} \, dy
\]

\[
\begin{align*}
\arctg y \Big|_{x_0}^{x} &= \arctg y \Big|_{\dot{x}_0}^{x} \\
= \arctg x - \arctg \dot{x}_0
\end{align*}
\]

\[
= 0 \quad \Rightarrow \quad x(t; 0, \sqrt{2}) = \tan(\sqrt{2} \cdot t).
\]

Phase portrait.

Recall that \( V(x) = -x^4 - 2x^2 \), hence:

Unique equilibrium \((0, 0)\), clearly

(by the phase portrait) unstable. In fact

(other check...)

\[
\begin{align*}
&\dot{x} = \nu \\
&\nu = 4x^3 + 4x
\end{align*}
\]

\[
J(x, \nu) = \begin{pmatrix}
0 & 12x^2 + 4 \\
1 & 0
\end{pmatrix}
\]
And $J_K(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ with $\lambda_{1,2} = \pm 2 = 0$ instability!

**EXERCISE 2**

Describe qualitatively the motion of a point $P$ of mass $m > 0$ subjected to the conservative force with potential energy:

$$V(x) = \frac{x^3}{3} - \frac{x^2}{2} - x.$$ 

**Solution**

We first draw the graph of $V(x) = x\left(\frac{x^2}{3} - \frac{x}{2} - 1\right)$

$$V'(x) = x^2 - x - 1 = 0 \quad x_{1,2} = \frac{1 \pm \sqrt{1 + 4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$V''(x) = 2x - 1$$

$$V''(x_1) = V''\left(\frac{1 + \sqrt{5}}{2}\right) = 4 + \sqrt{5} - 1 = \sqrt{5} > 0 \quad \text{so } x_1 \text{ is a local minimum.}$$

$$V''(x_2) = V''\left(\frac{1 - \sqrt{5}}{2}\right) = -\sqrt{5} < 0 \quad \text{so } x_2 \text{ is a local maximum.}$$

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**Phase portrait**

Separatrix corresponding to $E = V(x_2)$. 

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EXERCISE 3

Calculate the period of the motion given by
\[ \begin{cases} \ddot{m} x = -V'(x) \\ x_0 = 2 \\ \dot{x}_0 = 0 \end{cases} \]

with \( V(x) = x^2 - 4 \).

SOLUTION

\[ y = x^2 - 4 = V(x) \]

By using the conservation of energy: \( E(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 + V(x) = \)
\[ = \frac{1}{2} m \dot{x}^2 + x^2 - 4 = E(x_0, \dot{x}_0) = 0 + V(2) = 0. \]
\[ = \frac{1}{2} m \dot{x}^2 + x^2 - 4 = 0 \]

And \( \frac{1}{2} m \dot{x}^2 + x^2 - 4 = 0 \) for \( \dot{x} = 0 \) (x-axis) \iff \( x = \pm 2 \)

Therefore \( m \dot{x}^2 = 2(4-x^2) \iff \dot{x}^2 = \frac{2}{m} (4-x^2) \)
\[ x = \text{sgn}(\dot{x}_0) \sqrt{\frac{2}{m} (4-x^2)} \iff \frac{dx}{dt} = \text{sgn}(\dot{x}_0) \sqrt{\frac{2}{m} (4-x^2)} \]

\[ = 0 \quad T = \text{period} = 2 \int_{-2}^{2} \frac{dy}{\sqrt{\frac{2}{m} \sqrt{4-y^2}}} = 2 \sqrt{\frac{2}{m}} \int_{-2}^{2} \frac{dy}{\sqrt{4-y^2}} = \]
\[ = \sqrt{2m} \left( \arcsin \left( \frac{y}{2} \right) \right) \bigg|_{-2}^{2} = \sqrt{2m} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \sqrt{2m} \pi \]

EXERCISE 4

Let consider the differential equation \( \ddot{x} = -V'_K(x) \), \( x \in \mathbb{R} \).

Where \( V_K(x) = Kx(x^2 - k) \).
(a) Draw the phase portrait for every \( k \in \mathbb{R} \).

(b) Let now \( k = 1 \). Establish for which values \( k \in \mathbb{R} \) the solution with initial datum \( (x(0), v(0)) = (0, 0) \) is periodic.

**Solution**

1. \( k < 0 \). \( V_k(x) = kx(x^2 - k) = 0 \) if \( x = 0 \).

2. \( k = 0 \). \( V_0(x) = 0 \) (free particle...)

3. \( k > 0 \). \( V_k(x) = kx(x^2 - k) = 0 \) \( \Rightarrow \) \( x = 0 \)
   
   or \( x = \pm \sqrt{k} \)

   \[ V_k'(x) = k(x^2 - k) + 2kx^2 = kx^2 - k^2 + 2kx^2 = 3kx^2 - k^2 = 0 \]

   \( \Rightarrow \) \( x^2 = k/3 \) \( \Rightarrow \) \( x_{1,2} = \pm \sqrt{k/3} \) \( \Rightarrow \) \( x_1 = -\sqrt{k/3} \) is the max. local.

   \( x_2 = +\sqrt{k/3} \) is the min. local.
\[ K = 1 \] We obtain \( V_a(x) = x(x^2 - 1) \). The conditions are:

\[ E(0, \sigma) < V_a( -\frac{2}{\sqrt{3}}, -\frac{2}{3} ) = -\frac{2}{\sqrt{3}} \cdot (-\frac{2}{3}) = \frac{2}{3\sqrt{3}} \]

That is:

\[ \frac{1}{2} \sigma^2 + V_a(0) < 2/3\sqrt{3} \]

\[ \Rightarrow \frac{1}{2} \sigma^2 + 0 < 2/3\sqrt{3} \]

\[ \Rightarrow \sigma^2 < \frac{4}{3\sqrt{3}} \]

\[ \sigma \in \left( -\frac{2}{\sqrt{3\sqrt{3}}}, \frac{2}{\sqrt{3\sqrt{3}}} \right) \]

The other condition:

\[ E(0, \sigma) = \frac{1}{2} \sigma^2 + V_a(0) \geq V_a\left( \frac{1}{\sqrt{3}} \right) \]

is automatically satisfied.

Since \( V_a\left( \frac{1}{\sqrt{3}} \right) = \frac{1}{\sqrt{3}} \cdot (-\frac{2}{3}) = -\frac{2}{3\sqrt{3}} < 0 \)

Therefore, the solution is

\[ \sigma \in \left( -\frac{2}{\sqrt{3\sqrt{3}}}, \frac{2}{\sqrt{3\sqrt{3}}} \right) \]