EX 1

Let \( f \in C^\infty (\mathbb{R}, \mathbb{R}) \) be defined as

\[
    f(x) = \frac{2}{\pi} \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right).
\]

(a) Study the graph of \( f \).

(b) Let consider the first order equation \( x' = f(x) \). Draw the phase-portrait and discuss the attractivity / repulsivity of equilibria.

(c) Let consider the second order equation \( \ddot{x} = -f'(x) \). Draw the phase-portrait and discuss the stability / instability of equilibria.

(d) Referring to equation in (c). Establish the subset \( E \) of \( \mathbb{R} \) of energy values for which do not correspond a periodic orbit.

(e) Referring to equation in (c), let \( E = \frac{4}{\pi} \arctg \left( \frac{1}{6} \right) \).

Is every orbit of energy \( E \) periodic?

Write the formula (without solving the integral) for the period of the corresponding orbit.

SOL.

(a) Graph of \( f(x) \).

\[
    \lim_{x \to +\infty} \frac{2}{\pi} \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right) = \frac{2}{\pi} \left( -\frac{\pi}{2} \right) = -1
\]

\[
    \lim_{x \to -\infty} \frac{2}{\pi} \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right) = +1
\]

\[
    f'(x) = \frac{2}{\pi} \cdot \frac{1}{1 + \left( \frac{x^2}{2} - \frac{x^3}{3} \right)^2} \left( x - x^2 \right) = 0 \iff x = 0, x = 1
\]

Therefore, considering the sign of \( f'(x) \) (or the limit for \( x \to \pm \infty \))

\[
    V(1) = \frac{2}{\pi} \arctg \left( \frac{1}{6} \right)
\]

\[
    V(x) = 0 \iff x = 0, x = \frac{3}{2}
\]

\[
    V(x) = \frac{2}{\pi} \arctg \left( \frac{1}{6} \right)
\]
Let $E > 1$. Consider:

\[
\frac{1}{2} u^2 + 2/\pi \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right) = E. \quad \text{Then} \quad \lim_{x \to +\infty} E - 2/\pi \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right) = E + 1
\]

\[
\lim_{x \to -\infty} E - 2/\pi \arctg \left( \frac{x^2}{2} - \frac{x^3}{3} \right) = E - 1
\]

(d) $E = [R \setminus [0, v(1)] = R \setminus [0, 2/\pi \arctg (1/6)]$.

(Recall that fixed points are periodic orbits!)

e) $E = 1/\pi \arctg (1/6) = 1/2 v(2) \in [0, 2/\pi \arctg (1/6)] = 0$. At this value, there exists a periodic orbit. But not every orbit of energy $E$ is periodic! See the phase portrait. Finally:

\[
T = \frac{1}{2} \int_{x_-}^{x_+} \frac{1}{\sqrt{2(E - V(x))}} \, dx.
\]
Let \( X \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \) such that
\[
\begin{align*}
\dot{x} &= 4y(y^2-1) \\
\dot{y} &= 4x(x^2-1)
\end{align*}
\]
Verify that there exist a first integral for \( X \) and explicitly determine this first integral.

**Sol.**

\( f \) is a first integral for \( X \) iff \( \nabla f \cdot X(x, y) = 0 \) (identically zero).

That is
\[

\nabla f(x, y) \cdot X(x, y) = 0 \iff \frac{\partial f}{\partial x}(x, y)(4y(y^2-1)) + \frac{\partial f}{\partial y}(x, y)(4x(x^2-1)) = 0.

\]

Therefore, we search \( f \) such that
\[
\begin{align*}
\frac{\partial f}{\partial x} &= +4x(x^2-1) \\
\frac{\partial f}{\partial y} &= -4y(y^2-1)
\end{align*}
\]
\[
\Rightarrow f(x, y) = (x^2-1)^2 - (y^2-1)^2 + C \quad (C \in \mathbb{R})
\]

**EX 3**

Let consider the 1-dim. mechanical system for a point \( P \) of mass \( m = 1 \) subjected to the potential energy:
\[
V(x) = e^{-2x^2}(e^{-2x^2} - 1)
\]

a) Study the graph of \( V(x) \).

b) Determine equilibria for \( \ddot{x} = -V'(x) \).

c) Discuss stability of equilibria. Draw the phase portrait.

d) Determine the set of initial values which generate periodic trajectories.

e) Verify that the trajectories with \( E = -3/16 \) are periodic and write the corresponding period (without solving the integral).

**Sol.**

\[
V(x) = e^{-2x^2}(e^{-2x^2} - 1).
\]
\[
V'(x) = ??
\]
Let \( t = e^{-2x^2} \). Then \( V(x) = v(t) = t(t-1) \).
\[ \frac{dx}{dt} = u'(t) \frac{dt}{dx} \quad \text{where} \quad u'(t) = 2t - 1 \]

\[ \frac{dt}{dx} = e^{-2x^2} (-4x) = -4xt \]

Therefore \( V'(x) = (2t - 1) (-4xt) = 0 \) \( \Rightarrow \) \( t = \frac{1}{2} \sqrt{\log \frac{x}{x}^2} \)

That is \[ e^{-2x^2} = \frac{1}{2} \sqrt{\log^2 \frac{x}{x}} \]
\[ x = 0 \]
\[ x = 0 \]
\[ \log \frac{x}{x} = \frac{1}{2} \]
\[ x = \pm \sqrt{\frac{\log^2 \frac{x}{x}}{2}} \]

\[ V''(x) = u''(t) \left( \frac{dt}{dx} \right)^2 + u'(t) \left( -4t - 4x \frac{dt}{dx} \right) = \]
\[ = 2 \left( 4xt \right)^2 + (2t - 1) (-4t - 4x (-4xt)) = \]
\[ = 32x^2 t^2 + (2t - 1) (-8t + 16x^2 t) = 32x^2 t^2 - 8t^2 + 32x^2 t^2 + 4t - 16x^2 t = \]
\[ = 64x^2 t^2 - 16x^2 t - 8t^2 + 4t \]

\[ V''(0) = 0 - 0 - 8 + 4 = -4 < 0 \quad \text{max., local} \]
\[ V''(\pm \sqrt{\frac{\log^2 \frac{x}{x}}{2}}) > 0 \]

\[ \text{min., local} \]

Moreover

\[ \text{If } V(x) = 0, \text{ } x \to \pm \infty \]

(b) Equilibria: \((0, 0), (\pm \sqrt{\frac{\log^2 \frac{x}{x}}{2}}, 0)\)

\[ \text{at } (x_1, 0) \text{ and } (x_2, 0) \]

(c) \((x_1, 0), (x_2, 0)\) are stable while \((0, 0)\) is unstable.

(d) \(E \in [V(x_0), 0]\) generate periodic (or fixed) trajectory.
\[ E = 0 : \quad \frac{1}{2} \dot{x}^2 + e^{-2x^2} (e^{-2x^2} - 1) = 0 \]

\[ \lim_{x \to \pm \infty} - e^{-2x^2} (e^{-2x^2} - 1) = 0 \quad \text{if} \quad E > 0 \]

\[ \lim_{x \to \pm \infty} E - e^{-2x^2} (e^{-2x^2} - 1) = E > 0 \]

(e) \quad V(x_1) = e^{-2x_1^2} (e^{-2x_1^2} - 1) = e^{-2x_1^2} \left( e^{-2x_1^2/2} - 1 \right) =

\[ \frac{d}{dx} \left( \frac{d}{dx} - 1 \right) = \frac{1}{2} \left( -\frac{1}{2} \right) = -\frac{1}{4} \]

\[-3/16 > -1/4 \quad (\Rightarrow \quad 3/4 < 1) \quad \text{implies} \quad E = -3/16 \quad \text{gives} \quad 2 \text{ periodic orbits.} \]

\[ T = 2 \int_{x_3}^{x_4} \frac{dx}{\sqrt{2(E-V(x))}} \]

In particular, we can find \( x_3 \) and \( x_4 \) by solving \( V(x) = -3/16 \Rightarrow V(t) = t(t-1) = t^2 - t = -3/16 \Rightarrow \begin{cases} \frac{t}{t} = 1/4 \text{ or} \frac{t}{t} = 3/4 \end{cases} \]
\[
\begin{align*}
  x_3 &= \sqrt{\log \frac{1}{2}} \\
  x_4 &= \frac{\log(4/3)}{2} \\
  T &= 2 \int \frac{\log(4/3)}{2} \, dx \left/ \sqrt{2 \left( -\frac{1}{16} x^2 \right)} \right.
\end{align*}
\]

**EX 1.4**

1. Let consider the differential equation \( \dot{x} = k x e^{-x^2/2} \) \( \text{K} \in \mathbb{R}, x \in \mathbb{R} \).
   a) Draw the phase portrait \( \forall \text{K} \in \mathbb{R} \).
   b) Determine equilibria and their nature \( \forall \text{K} \in \mathbb{R} \).
   c) Linearize \( \dot{x} = k x e^{-x^2/2} \) around the origin.

2. Let consider the differential equation \( \dot{x} = -V'(x) \) where \( V(x) = x e^{-x^2/2} \).
   a) Draw the phase portrait.
   b) Determine equilibria and their nature.
   c) Determine for which value of \( \text{K} \in \mathbb{R} \) the solution with initial datum \( (x_0, y) \) is periodic.
   d) By using an appropriate Lyapunov function, discuss the stability of equilibria \( (-1, 0) \) for
      \[
      \dot{x} = -V'(x) + F_x(x, y)
      \]
      where \( F_x(x, y) = -2\mu (x+1)^2 y \), \( \mu > 0 \).

**SOL**

\[
\begin{align*}
  \text{K} > 0 \\
  \text{K} = 0
\end{align*}
\]

\( K e^{-x^2/2} \) has derivative: \( K e^{-x^2/2} + Kx (-x) e^{-x^2/2} = K e^{-x^2/2} (1-x^2) \)

\( K = 0 \) \hspace{1cm} \text{Every point is a (stable) fixed point.}
Linearization around \( x = 0 \):
\[
x = \lambda x.
\]

The corresponding phase portrait:

Let \( E \geq 0 \)

\[
\frac{1}{2} \dot{u}^2 + x e^{-x^2/2} = E \quad x \Rightarrow \frac{1}{2} \dot{u}^2 = E - x e^{-x^2/2}
\]

\[
\lim_{x \to +\infty} e^{-1/2} - x e^{-x^2/2} = E \quad \lim_{x \to -\infty} e^{-1/2} - x e^{-x^2/2} = E
\]

Equilibria: \((-1,0)\) no Stable
\((+1,0)\) no Unstable

The conditions so that the solution starting from \((-\frac{1}{2}, v)\) is periodic is
\[-\left(-\frac{1}{2}, \nu\right) = \frac{1}{2} \sigma^2 - \frac{1}{2} e^{-\frac{1}{4}} \quad (0 < \sigma < e^{-\frac{1}{4}}) \quad \nu^2 < e^{-\frac{1}{4}} \quad \Rightarrow -\frac{1}{2} e^{-\frac{1}{8}} \sigma < \nu < e^{-\frac{1}{8}}.\]

Finally, for the differential equation
\[\dot{x} = -V'(x) + F_\mu(x, \nu) \quad (\mu > 0),\]
we use the "Lyapunov function" as the energy of the system for \(\mu = 0\):
\[W(x, \nu) = \frac{1}{2} \sigma^2 + V(x) - V(-\nu).\]

Moreover:
\[\Delta W(x, \nu) = \Delta x \cdot \nabla W(x, \nu) = \sigma \nabla V(x) + (\nabla V(x) - 2\mu (x + 1)^2 \nu) \nabla V(x) - \nabla V(x) \nabla \nu - 2\mu (x + 1)^2 \nu^2 \leq 0 \quad \text{around} \quad (-1, 0),\]
\[= 0 \quad (-1, 0) \quad \text{is stable}.\]

**ATTRACTION FIXED POINT**

**IMPORTANT REMARK!** ATTRACTIONS are IMPOSSIBLE for CONSERVATIVE SYSTEMS!!! Let \(x = X(x)\) on \(\mathbb{R}^n\).

- A first integral is a function \(E \in \mathbb{E}^2(\mathbb{R}^n, \mathbb{R})\) such that \(L_x E(x) = 0\). To avoid trivial examples, we also require that \(E(x)\) be nonconstant on every open set. Otherwise, a constant function \(E(x) \equiv C (C \in \mathbb{R})\) would qualify as a conserved quantity for the system, and so every system would be conservative! As a consequence:

- A CONSERVATIVE SYSTEM CANNOT HAVE ANY ATTRACTION FIXED POINT.

Proof: Suppose - on the contrary - that \(x^* \in \mathbb{R}^n\) is an attracting fixed point. Then all points in its basin of attraction \(V\) would have to be the same "energy" \(E(x^*) = 0\), \(E(x) \equiv E(x^*)\) in \(V\), which is a contradiction.
LESSON 14

- The limit cycle phenomenon.
- The mechanical clock.

Introduction

In this lesson we intend to construct a mathematical model reproducing the phenomenology of the mechanical clock. We first notice that it differs from conservative systems like harmonic oscillator or pendulum for the following facts:

1) In real phenomena—like the mechanical clock—there is always dissipation. Therefore, a real model for a mechanical clock must contain a dissipative term. A mechanism able to release energy to the system in order to offset the dissipation.

2) The harmonic oscillator and the pendulum have infinitely many periodic motions of amplitude depending on the initial datum. The mechanical clock has a unique periodic motion, of fixed amplitude. The motion of the mechanical clock reaches the periodic motion asymptotically by:

- increasing the oscillations if the original amplitude is small.
- decreasing the oscillations if the original amplitude is large

This system has a unique closed trajectory and the other trajectories approach this one asymptotically. This dynamical phenomenon is called limit cycle.

The term "limit cycle" come from H. Poincaré 1854 - 1912.
DEF: A limit cycle is an isolated closed trajectory. Isolated means that neighboring trajectories are not closed; they spiral either toward or away from the limit cycle (see figure in the next page). Stable & Unstable limit cycle.

Limit cycles can't occur in linear systems \( \dot{x} = Ax \). In fact, if \( x(t) \) is a periodic solution of \( \dot{x} = Ax \), then also \( cx(t) \) \( (c \neq 0) \) is a periodic solution of \( \dot{x} = Ax \). Hence \( x(t) \) is surrounded by a 1-parameter family of closed orbits (center).

First (simple) example

Consider the system
\[
\begin{align*}
\dot{z} &= z (1 - z^2) \\
\dot{\theta} &= 1
\end{align*}
\]
\( z \geq 0 \)
\( \theta \in [0, 2\pi) \)

Radial and angular dynamics are uncoupled and so can be analyzed separately. And the motion in the \( \theta \)-direction is simply a rotation at constant angular velocity:
\[ \theta(t) = \theta_0 + t \]
Model.

We start from the equation of the harmonic oscillator with dissipation.

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -\omega^2 x - 2\mu v \\
(\omega^2 &= k/m) \\
\end{align*}
\]

Phase space \( \mathbb{R} \times \mathbb{R} = (x, v) \). Take an initial point \((0, v_0)\), \(v_0 > 0\).

The solution \((x(t), v(t))\) with \((x(0), v(0)) = (0, v_0)\) is

\[
\begin{align*}
x(t) &= \frac{v_0}{\sigma^*} e^{-\mu t} \sin(\sigma t) \\
v(t) &= v_0 e^{-\mu t} \left( -\frac{\mu}{\sigma^*} \sin(\sigma t) + \cos(\sigma t) \right) \\
\end{align*}
\]

where \(\sigma^* = \sqrt{\omega^2 - \mu^2}\).

\(x(t)\) crosses the \(v > 0\) axis periodically, with period \(T = 2\pi/\sigma^*\).

The corresponding velocities are:

\[
\begin{align*}
v_0 \\
v_1 &= v_0 e^{-2\pi \mu / \sigma^*} = v_0 e^{-\mu T} \\
v_2 &= v_0 e^{-4\pi \mu / \sigma^*} = v_1 e^{-\mu T} \\
v_3 &= v_0 e^{-6\pi \mu / \sigma^*} = v_2 e^{-\mu T} \quad \text{and so on...}
\end{align*}
\]

That is

\[
\begin{align*}
v_{k+1} &= a_k v_k, \quad \text{with} \quad a_k = e^{-\mu T} < 1. \\
v_k &= a_k v_0 
\end{align*}
\]

This model is fated to stop...

Therefore, we add an external effect of the dissipation.

\[\text{CENTRO DI RICERCA MATEMATICA ENNIO DE GIORGI}\]

force as follows: when the point $P$ of mass $m > 0$ passes through the $x > 0$ axis, it receives a positive impulse (external force) which increases the velocity, of a fixed quantity $b > 0$. In formula:

$au_0 \implies au_0 + b = u_1$
$au_1 \implies au_1 + b = u_2$
$au_2 \implies au_2 + b = u_3$ and so on...

That is

$u_{k+1} = au_k + b \quad \forall k = \{0, 1, 2, \ldots\}, \quad a = e^{-\mu T} < 1$ fixed

$b > 0$

$f(u) = u$

$f(u) = au + b$

**Dynamics?**

All solutions approach asymptotically the limit cycle (periodic motion...)

$f(u^*) = au^* + b = u^*$

$\iff au^* - u^* = -b$

$\Rightarrow u^* = \frac{b}{1-a}$
\[ \ddot{x} + 2\mu \dot{x} + \omega^2 x = 0 \]

\[ \lambda^2 + 2\mu \lambda + \omega^2 = 0 \]

\[ \lambda_{1,2} = -\mu \pm \sqrt{\mu^2 - \omega^2} = -\mu \pm i \sqrt{\omega^2 - \mu^2} \]

\[ \mu > 0 \text{ small} \]

\[ \sigma = \sqrt{\omega^2 - \mu^2} \]

\[ x(t) = Ae^{-\mu t} \cos(\sigma t) + Be^{-\mu t} \sin(\sigma t) \]

\[ x(0) = 0 = A + 0 \implies A = 0 \implies x(t) = Be^{-\mu t} \sin(\sigma t) \]

\[ \sigma(t) = -\mu Be^{-\mu t} \sin(\sigma t) + Be^{-\mu t} \sigma \cos(\sigma t) \]

\[ \sigma(0) = \sigma_0 = 0 + B\sigma_0 \implies B = \frac{\sigma_0}{\sigma} \]

Therefore the required solution is:

\[ x(t) = \frac{\sigma_0}{\sigma} e^{-\mu t} \sin(\sigma t) \]

And

\[ \sigma(t) = \frac{\sigma_0}{\sigma} e^{-\mu t} \left( -\frac{\mu}{\sigma} \sin(\sigma t) + \cos(\sigma t) \right) \]

\[ T = \frac{2\pi}{\sigma} \]
One-dimensional maps

New class of dynamical systems in which time is discrete, rather than continuous. Maps arise in various ways:

- As tools for analyzing differential equations (discretization)
- As models of natural phenomena (growth population)
- As simple examples of chaos

\[ f^0(x) \]
\[ f : \mathbb{R} \rightarrow \mathbb{R}, \ x \in \mathbb{R} \]
\[ \text{Orb} = \{ f^k(x), \ k \in \mathbb{Z} \} = \{ \ldots, f^{-2}(x), f^{-1}(x), x, f'(x), f^2(x), \ldots \} \]
\[ f^0(x) \]

orbit of \( x \) with respect to \( f \)

Give the discrete dynamical \( x_{n+1} = f(x_n) \)
\( (n \in \mathbb{N} \cup \{0\} \text{ if we only study the future}) \)

Suppose \( x^* \) satisfies \( f(x^*) = x^* \) then \( x^* \) is a fixed point, which means that
\[ \text{Orb}^+ = \{ f^n(x^*), \ n \in \mathbb{N} \cup \{0\} \} = \{ x^* \} \]

Stability of \( x^* \)? We consider \( x^* + \gamma \) (\( \gamma \) small)
\[ f(x^* + \gamma) = f(x^*) + f'(x^*) \gamma + o(\gamma) = x^* + f'(x^*) \gamma + o(\gamma) \]
\[ \downarrow \text{Taylor} \]
\[ x^* \text{ is fixed} \]

that is
\[ x^* + \gamma \mapsto x^* + f'(x^*) \gamma + o(\gamma) \]
which means (neglecting \( o(\gamma) \) - terms)
\[ \gamma \mapsto f'(x^*) \gamma \]

Let \( \lambda = f'(x^*) \)
Then \( \text{Orb}^+(\gamma) = \{ \gamma, \lambda \gamma, \lambda^2 \gamma, \ldots \} = \{ \lambda^n \gamma, \ n \in \mathbb{N} \cup \{0\} \} \)

So
- if \( |f'(x^*)| = |\lambda| < 1 \) then \( \lim_{n \rightarrow \infty} \lambda^n \gamma = 0 \) and \( x^* \) is linearly stable.
- if \( |f'(x^*)| = |\lambda| > 1 \) then \( x^* \) is linearly unstable.
The linearization tells us nothing in the marginal case \( f'(x^*) = 1 \).

**Remark** All of these results have parallels for differential equations...

**Example** \( f(x) = x^2 \)

\[
x \mapsto x^2 \mapsto (x^2)^2 = x^4 \mapsto (x^4)^2 = x^8 \text{ etc.}
\]

\[
f(x) = x \quad \Rightarrow \quad x^2 = x \quad \Rightarrow \quad x = 0 \text{ or } x = 1.
\]

\[
f'(x) = 2x, \quad f'(0) = 0 < 1 = 0 \quad \Rightarrow \quad 0 \text{ is a (linearly) stable fixed point.} \quad f'(1) = 2 > 1 \quad \Rightarrow \quad 1 \text{ is an unstable fixed point.}
\]

**Dynamics**

\[
y = x^2 \quad y = x
\]

\[
\text{diverging to } +\infty \quad \text{o is unstable}
\]

\[
\text{converging to } 0 \quad \text{o is stable}
\]

**Example** (Logistic map)

\[f : [0, 1] \rightarrow [0, 1], \quad f(x) = 2x(1-x), \quad x \geq 0\]

In order that \( 2x(1-x) \in [0, 1] \) we need \( 0 \leq x \leq 1 \). In fact:

\[
f'(x) = 2(1-x) + 2x(-1) = 2 - 2x - 2x = -2x + 2 = 0 \quad \Rightarrow \quad x = \frac{1}{2}
\]

and \( f\left(\frac{1}{2}\right) = \frac{1}{4} = 0 \). The graph of \( f \) is a parabola with a max value of \( \frac{1}{4} \) at \( x = \frac{1}{2} \).
fixed point: $2x(1-x) = x \Rightarrow 2x - 2x^2 = x \Leftrightarrow x = 0$ or $2 - 2x = 1 \Leftrightarrow 2x = 2 - 1 \Leftrightarrow x = \frac{1 - 1}{2} = 0$ (if $x > 0$)

Then:
The origin is a fixed point for all $x \in [0, 1]$. Whereas $x^k = 1 - \frac{1}{2}^k$ is in the range of allowable $x$ only if $1 - \frac{1}{2} > 0$ that is $1/2 < 1 \Rightarrow 2 > 1$. (For $x = 1$ we obtain $0$ again...)

Moreover, stability depends on $f'$. In particular:
$f'(0) = 2 > 0$ stable for $0 < 2 < 1$
$\n$unstable for $2 > 1$

Let now $x > 1$
$f'(x^k) = -2x \left( \frac{2 - 1}{x} \right) + x = -2x^2 + 2x = x - 2x^2$
stable for $-1 < x < 1 \Rightarrow x < 2 < 3$
unstable for $x > 3$

These previous results are clarified by the next graphical analysis.
SEE ALSO: Cobweb plot for the logistic map.
Bifurcation diagram for the logistic map.
Cobweb plot for the logistic map

A Logistic Map Cobweb Plot, $r=1.0$

B Logistic Map Cobweb Plot, $r=2.7$

C Logistic Map Cobweb Plot, $r=3.5$

D Logistic Map Cobweb Plot, $r=3.9$
Bifurcation diagram for the logistic map.
REMARK: See also Problem 2 in G. Benetti, "Eserciziario".

Ex 1
\[ \ddot{x} = -\omega^2 \sin x + \Omega^2 \sin x \cos x \]

1. Determine and classify the equilibria. Draw the bifurcation diagram and the phase portrait for different \( \omega, \Omega > 0 \).
2. Let consider \( \ddot{x} = -\omega^2 \sin x + \Omega^2 \sin x \cos x - 2\mu \dot{x}, \mu > 0 \).
   Determine and classify the equilibria.

Sol

First order
\[
\begin{align*}
\dot{x} &= V \\
\dot{V} &= -V'(x) = -\omega^2 \sin x + \Omega^2 \sin x \cos x
\end{align*}
\]

Therefore

\[ V(x) = -\omega^2 \cos x + \frac{1}{2} \Omega^2 \cos^2 x \]

\[ V_{gr}(x) = -\omega^2 \cos x = 0 \]

\[ \frac{dv}{dx} = -\omega^2 \sin x \]

\[ v_{ef} = -\frac{1}{2} \Omega^2 \sin^2 x \]

Equilibria

\[ -\omega^2 \sin x + \Omega^2 \sin x \cos x = 0 \implies \sin x \left( -\omega^2 + \Omega^2 \cos x \right) = 0 \implies \]

\[ x = 0, \pi \text{ or } -\omega^2 + \Omega^2 \cos x = 0 \implies \cos x = \frac{\omega^2}{\Omega^2} \implies x = \pm \arccos \left( \frac{\omega^2}{\Omega^2} \right) \]

(If \( \frac{\omega^2}{\Omega^2} < 1 \).

Then \( x = 0 \).

\[ J(x, \dot{x}) = \begin{pmatrix} 0 & 1 \\ -V''(x) & 0 \end{pmatrix} \]

- \[ V''(x) = -\omega^2 \cos x + \Omega^2 \cos^2 x - \Omega^2 \sin^2 x \]

For \( x = 0 \), \( -V''(0) = -\omega^2 + \Omega^2 \). Trace \( \equiv 0 \)

\[ \det = V''(x) > 0 \implies \omega^2 - \Omega^2 > 0 \implies \Omega^2 < \omega^2 \] (center)

\[ \det = V''(x) < 0 \implies \omega^2 - \Omega^2 < 0 \implies \Omega^2 > \omega^2 \] (saddle)
For $x = \pi$, $-V''(\pi) = \omega^2 + \Omega^2 > 0$. Then $\text{Trace} = 0$ and $\text{det} = V''(\pi) < 0 \Rightarrow (\pi, 0)$ is a saddle.

For $\Omega^2 > \omega^2$, we have only the equilibria $(\pm \arccos(\omega^2/\Omega^2), 0)$.

$-V''(\pm \arccos(\omega^2/\Omega^2)) = -\omega^2 \cos x^k + \Omega^2 \cos^2 x^k - \omega^2 + \Omega^2 \cos^2 x^k = \frac{\omega^2 - \Omega^2}{\Omega^2} < 0 \Rightarrow \omega^2 - \Omega^2 < 0$

$\Rightarrow \omega^2 - \Omega^2 < 0 \Rightarrow \Omega^2 > \omega^2.$

$\Rightarrow \text{det} = V''(x^k) > 0$, $\text{tr} = 0 \Rightarrow (-\pm x^k, 0)$ are center.

**Bifurcation Diagram** (for the linearized system)

For $\frac{\Omega^2}{\omega^2} < 1 \rightarrow$ Only 2 equilibria $(0, 0)$ and $(\pi, 0)$.

For $\frac{\Omega^2}{\omega^2} > 1 \rightarrow$ Four equilibria.

By using the potential energy: $V(x) = -\omega^2 \cos x + \frac{1}{2} \Omega^2 \cos^2 x$
If $\Omega^2/\omega^2 < 1$

$y = V(x)$ for $\Omega^2/\omega^2 < 1$

If $\Omega^2/\omega^2 > 1$

$y = V(x)$ for $\Omega^2/\omega^2 > 1$

WITH THE DISSIPATIVE TERM...

First order

\[
\begin{align*}
\dot{x} &= v \\
\dot{v} &= -v'(x) - 2\mu v
\end{align*}
\]

so that

\[
J(x, \sigma) = \begin{pmatrix}
0 & 1 \\
-v''(x) & -2\mu
\end{pmatrix}
\]

det $= v''(x)$ as in the previous case, $tr = -2\mu < 0$.

For $x = 0$,

\[
det = v''(0) = +\omega^2 - \Omega^2
\]

and $tr = -2\mu$.

Therefore:
\[(tr)^2 - 4 \det = 4 \mu^2 - 4(\omega^2 - \Omega^2) = 4 \left[ \mu^2 - \omega^2 + \Omega^2 \right]\]

- If \(\mu^2 - \omega^2 + \Omega^2 > 0\) that is \(\omega^2 - \mu^2 < \Omega^2 < \omega^2\)
- \((0,0)\) is a stable node.

- If \(\mu^2 - \omega^2 + \Omega^2 = 0\) that is \(\mu^2 = \omega^2 - \Omega^2 > 0\)
- \((0,0)\) is a stable spiral.

- If \(\omega^2 - \Omega^2 < 0 \Rightarrow \omega^2 < \Omega^2\) then \((0,0)\) is a saddle.

2. For \(x = \pi\), \(-v''(\pi) = \omega^2 + \Omega^2 > 0\) then \((\pi,0)\) is a saddle.

3. Let \(\Omega^2 > \omega^2\), then \(x^* = \pm \arccos \left( \frac{\omega^2}{\Omega^2} \right) \) are such that \(
\det = v''(x^*) > 0\). In particular:
\[(tr)^2 - 4 \det = 4 \mu^2 - 4 \left[ \omega^2 - \Omega^2 \right] = 4 \mu^2 - 4 \Omega^2 + 4 \omega^4 = 4 \left[ \mu^2 - \Omega^2 + \frac{\omega^4}{\Omega^2} \right]\]

- If \(\mu^2 - \Omega^2 + \frac{\omega^4}{\Omega^2} > 0\) that is \(\omega^2 - \Omega^2 < \omega^4 + \mu^2\)
- \((x^*,0)\) are stable nodes.

- If \(\mu^2 - \Omega^2 + \frac{\omega^4}{\Omega^2} = 0\) that is \(\mu^2 = \Omega^2 - \omega^4\)
- \((x^*,0)\) are stable spiral node.

- If \(\mu^2 - \Omega^2 + \frac{\omega^4}{\Omega^2} < 0\)
- \((x^*,0)\) are stable spiral.
Constrained dynamical systems

1. Surface $Q \subset \mathbb{R}^3$ (dim = 2)

\[
\begin{align*}
\omega &= \omega(q_1, q_2, q_3, q_4) \\
\text{can be described in two ways:}
\end{align*}
\]

- Implicitly, by $F(x, y, z) = 0$ with $F \in C^\infty(\mathbb{R}^3; \mathbb{R})$ and
  \[
  \nabla F(x, y, z) \bigg|_{Q} = \begin{pmatrix} 2x F \\ 0 \\ 2y F \end{pmatrix}(x, y, z) \neq 0.
  \]

- By a local parameterization:
  \[
  x = x(q_1, q_2), \quad y = y(q_1, q_2), \quad z = z(q_1, q_2) \quad \text{with } (q_1, q_2) \in \mathbb{R}^2
  \]
  \[
  \omega = (x, y, z) = \omega(q_1, q_2).
  \]

In particular, $Q$ admits a tangent plane at every point $\omega$ and the Jacobian matrix

\[
\frac{\partial(\omega)}{\partial(q_1, q_2)} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix}
\]

has maximum rank $2$.

Consequently, the pair of vectors $\frac{\partial x}{\partial q_1}, \frac{\partial y}{\partial q_1}$ is in any point $\omega \in Q$ a basis for the local tangent plane $T_{\omega}Q$.

\[
\sum_{h=1}^{n} \frac{\partial \omega}{\partial q_h} \frac{\partial \omega}{\partial q_h}
\]

EXAMPLE 1. Sphere $S^2$

- $x^2 + y^2 + z^2 - R^2 = 0$. The condition $\nabla F(x, y, z) = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ is satisfied.

- By a local parameterization:
  \[
  x = q_1, \quad y = q_2, \quad z = \sqrt{R^2 - q_1^2 - q_2^2},
  \]
  \[
  \text{and } x = q_1, \quad y = q_2, \quad z = -\sqrt{R^2 - q_1^2 - q_2^2}.
  \]
Nord emisphere \( \frac{d\mathbf{w}}{dq} = \begin{pmatrix} 4 & 0 \\ -2q_1 & -4z \\ \frac{z}{\sqrt{R^2 - q_1^2}} & \frac{\sqrt{R^2 - q_1^2}q_1}{\sqrt{R^2 - q_1^2}^2} \end{pmatrix} \) has \( rk = 2 \).

(In \( \Theta = 0 \) we don't have the differentiability.)

- By spherical coordinate
  \[
  \begin{align*}
  x &= R \sin \Theta \cos \phi \\
  y &= R \sin \Theta \sin \phi \\
  z &= R \cos \Theta 
  \end{align*}
  \]

\( \frac{d\mathbf{w}}{dq} = \begin{pmatrix} R \cos \Theta \cos \phi & -R \sin \Theta \sin \phi & 0 \\ R \cos \Theta \sin \phi & R \sin \Theta \cos \phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \) has \( rk = 2 \) \( \iff (\Theta \neq 0, \pi) \) (north and south pole)

2. Curve \( Q \subset \mathbb{R}^3 \) (clim = 1)

\(- \frac{\partial G}{\partial x}(x, y, z) = 0 \quad \text{and} \quad \frac{\partial G}{\partial y}(x, y, z) = 0 \)

\(- x \cdot x(q_1), y = y(q_1), z = z(q_1) \) and \( \frac{d\mathbf{w}}{dq_1} = \begin{pmatrix} 2x \partial x/\partial q_1 \\ 2y \partial y/\partial q_1 \\ 2z \partial z/\partial q_1 \end{pmatrix} \neq 0. \)

**Example 2. Pendulum**

\[
\begin{align*}
\dot{x}^2 + \dot{y}^2 + \dot{z}^2 &= 0 \\
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} l \cos \Theta \\ t \sin \Theta \\ 0 \end{pmatrix} \\
\dot{z} &= 0
\end{align*}
\]

**Dynamic on constrained system??**

In the unconstrained case the Newton equation does not present problem. \( m \ddot{q} = F \) gives:
\[ m\ddot{a} = F + \phi \]

In this case, \[ \phi = (\phi_1, \phi_2, \phi_3) = \phi (\vec{r}, \vec{v}, t) \]

is an unknown.

Therefore, 3 equations and \( m = 1, 2 + 3 \) unknowns.

For the dynamics \( \vec{v} \) for \( \vec{F} \),

**Example 3**

If \( P \) is not in movement:

\[ \vec{a} = 0 = \vec{F} + \phi \Rightarrow \phi = +m\vec{g} \]

\[ \vec{F} = -mg e_3 \]

**Example 4**

\( \phi \) may depend on \( v \) !

\( \vec{v} \) in uniform movement

\( \dot{\theta} = \text{const} \) \( \Rightarrow \dot{\vec{a}} = 0 \).

In such a case, \[ \vec{a}_c = -c \dot{\theta}^2 e_3 \Rightarrow \phi = m\vec{a}_c = \vec{F} = -m\ddot{\theta} e_3 - mg e_3 \]

**Ideal Constraints**

The constraint \( \phi \) is called ideal if the surface or the curve are smooth.

\( \phi \perp T_n Q \)

\[ \phi \cdot \delta \vec{w} = 0 \]

\( \forall \delta \vec{w} \in T_n Q \)

\[ \phi \cdot \delta \vec{w} = 0 \quad \text{or} \quad \phi \cdot \delta \vec{w} = 0 \forall \delta \vec{w} \]

Therefore, with this hypothesis,

\[ m\ddot{a} = \vec{F} + \phi \Rightarrow (m\ddot{a} - \vec{F}) \cdot \frac{\partial \phi}{\partial q_n} = 0 \quad \forall \delta \vec{w} \]

\[ \frac{\partial \phi}{\partial q_n} \]

\[ \text{h equation} \quad \text{and} \quad \text{unknown} \]