\[ \begin{align*}
\hat{m}\hat{\alpha} &= \vec{F} + \vec{\phi} \\
\dot{\phi} &= (\phi_1, \phi_2, \phi_3) = \dot{\vec{\phi}}(\vec{O}_P, \vec{v})
\end{align*} \]

is an unknown!

Therefore: 3 equations and \( m = 1 \), 3 unknowns, for the dynamics \( \vec{v} \) for \( \vec{\phi} \)

**Lesson 16 from here**

**Example 3**

\[ \begin{align*}
\vec{F} &= -m\vec{g} \\
\vec{p} &= m\vec{v}
\end{align*} \]

If \( \vec{P} \) is not in movement:

\[ \begin{align*}
\hat{m}\hat{\alpha} &= 0 = \vec{F} + \vec{\phi} \\
\vec{F} &= -m\vec{g}
\end{align*} \]

**Example 4**

\[ \begin{align*}
\vec{\phi} \text{ may depend on } \vec{v} \parallel \vec{\phi} \\
P \text{ in uniform movement} \\
(\vec{\Theta} = \text{const}) \Rightarrow \hat{a} = 0.
\end{align*} \]

In such a case:

\[ \begin{align*}
\hat{a}_e &= -\vec{e}_2 \omega \hat{e}_2 \\
\vec{\phi} &= m\hat{\alpha}_e - \vec{F} = -m\hat{\omega} \hat{e}_2 + mg\hat{e}_2
\end{align*} \]

**Ideal Constraints**

The constraint \( \vec{\phi} \) is called ideal if the surface or the curve are

\[ \begin{align*}
\vec{\phi} &= T_{\vec{\phi}} \omega \\
\hat{\phi} &= \vec{0} \\
\n\hat{\phi} &\in T_{\vec{\phi}} Q \\
\rho \cdot \hat{\omega} &= 0 \\
\forall \hat{\omega} \in T_{\vec{\phi}} Q \\
\hat{\rho} \cdot \hat{\omega} &= 0 \\
\rho &\cdot \hat{\omega} = 0 \\
\rho &\cdot \hat{\omega} = 0 \\
\rho &\cdot \hat{\omega} = 0
\end{align*} \]

Therefore, with this hypothesis:

\[ \begin{align*}
\hat{\omega} = \hat{\phi} + \hat{\phi} \\
\rho \cdot \hat{\omega} = \hat{\phi} - \hat{\phi} = 0 \quad \forall \ h
\end{align*} \]
Constrained systems of $N$ points.

$\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_N$ points.

$\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N, \mathbf{w}_{N+1}, \mathbf{w}_{N+2}, \ldots, \mathbf{w}_{3N-2}, \mathbf{w}_{3N-1}, \mathbf{w}_{3N})$

$\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N, \mathbf{t})$

$m = 1, 2$

$N$ points constrained in an $m$-dimensional manifold such that

$\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_N, \mathbf{t})$

$3N \times m$ matrix

$m$ vectors $(n=1, 2)$ with $

\text{length } 3N.$

Then the $m$ vectors $\mathbf{w}_i$, $i = 1 \ldots m$, are linearly independent and therefore a basis for $T_{\mathbf{W}} \mathbf{W}$.

$S_{\mathbf{W}} = \sum_{h=1}^{m} \frac{\mathbf{w}_h}{\mathbf{w}_h}$

$m = 1 \ldots m$

$S_{\mathbf{W}} = \sum_{h=1}^{m} \frac{\mathbf{w}_h}{\mathbf{w}_h}$

$= \mathbf{0}$

$\mathbf{O}_{\mathbf{W}} = \sum_{h=1}^{m} \frac{\mathbf{w}_h}{\mathbf{w}_h}$

virtual displacement of the point $\mathbf{P}_i$.

Now let assume that the point $\mathbf{P}_i$ is subjected to the force

$\mathbf{F}_i = \mathbf{F}_i (\mathbf{O}_{\mathbf{P}_1}, \ldots, \mathbf{O}_{\mathbf{P}_N}, \mathbf{w}_i, \mathbf{t})$

$m$ unknowns

Newton equation for the point $\mathbf{P}_i$: $\mathbf{F}_i + \mathbf{F}_i = \mathbf{0}$

let assume that $\mathbf{F}_i = (\mathbf{F}_i, \ldots, \mathbf{F}_i)$

is an ideal constraint, that is

$\sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{O}_{\mathbf{P}_i} = \mathbf{0} \iff \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{O}_{\mathbf{P}_i} = \mathbf{0}$

$m$ unknowns

$\sum_{h=1}^{m} \mathbf{w}_h = \mathbf{0}$

$\forall h = 1 \ldots m$
Newton equation + ideal constraint for the point $P_i$:

$$m \ddot{a}_i = F_i + \phi_i$$

$$\sum_{i=1}^N \phi_i \cdot \frac{\partial \phi}{\partial \dot{q}_i} = 0 \quad h = 1 - m$$

$3 + m$ equations \quad and \quad $3 + m$ unknowns

---

UN ESERCIZIO SU CALCOLO DI REAZIONI VINCOLARI:

Problema 11 pag. 23 (Benettin)

\[ \phi = \phi(\theta) ? \]

\[ \Theta = "Lagrangian coordinate" \]

\[ (\Theta_o, \dot{\Theta}_o) = (\frac{\pi}{2}, 0) \]

\[ \rightarrow \rightarrow \phi(\Theta) ? \]

\[ \overrightarrow{\text{pendulum}} \]

\[ m, l \]

\[ \Theta = "Lagrangian coordinate" \]

\[ (\Theta_o, \dot{\Theta}_o) = (\frac{\pi}{2}, 0) \]

\[ \phi = \phi(\Theta) ? \]

We know the force: \[ F = -mg \overrightarrow{e_z} \]

\[ \ddot{a} = \ddot{\Theta} \overrightarrow{e_t} - \frac{l}{2} \dot{\Theta}^2 \overrightarrow{e_z} = \frac{l}{2} \dot{\Theta}^2 \overrightarrow{e_z} - l \dot{\Theta} \overrightarrow{e_t} \]

We need to determine $\dot{\theta}$ and $\ddot{\theta}$.

Then we use the equation of motion (for $\dot{\theta}$) and the conservation of energy (for $\ddot{\theta}$), we obtain:

\[ ml \dot{\theta} = -mg \sin \theta \Rightarrow \dot{\theta} = \frac{-g \sin \theta}{l} \]

And

\[ \frac{1}{2} ml (\dot{\theta}^2) - mgl \cos \theta = 0 \quad -mgl \cos (\frac{\pi}{2}) = 0 \]

\[ \Rightarrow \frac{1}{2} ml \dot{\theta}^2 = mgl \cos \theta \Rightarrow \dot{\theta}^2 = \frac{2g \cos \theta}{l} \]

\[ \Rightarrow \phi = ml \dot{\theta} \overrightarrow{e_t} - ml \ddot{\theta} \overrightarrow{e_t} + mge_z = \]

\[ = ml \left(-\frac{g \sin \theta}{l}\right) \overrightarrow{e_t} - ml \frac{2g \cos \theta}{l} \overrightarrow{e_z} - mge_z = \]

\[ = -mg \sin \theta \overrightarrow{e_t} - 2mg \cos \theta \overrightarrow{e_z} + mg \overrightarrow{e_z} \]
But
\[ \vec{e}_2 = \cos\left(\frac{\pi}{2} - \Theta\right) \vec{e}_t - \sin\left(\frac{\pi}{2} - \Theta\right) \vec{e}_z \]
\[ = \sin \Theta \vec{e}_t - \cos \Theta \vec{e}_z \]

Finally
\[ \phi = -mg \sin \Theta \vec{e}_t - 2mg \cos \Theta \vec{e}_x + mg R \dot{\Theta} \vec{e}_t - mg \cos \Theta \vec{e}_2 \]
\[ = -3mg \cos \Theta \vec{e}_x \] → The constraint is ideal. \(\phi \perp T_p \vec{S}\)

→ On example 4...

\[ \vec{a} = \vec{S} \dot{e}_t - \frac{R^2}{S} \vec{e}_z \]

Uniform circular motion means: \(S = R \dot{\Theta}\) and \(\ddot{S} = 0\).
Therefore
\[ \vec{a} = \vec{0} - \frac{R^2}{S} \vec{e}_z = -R \dot{\Theta} \vec{e}_x \]

\[ m \ddot{r} = \vec{F} + \phi \] → \(\phi = -m R \dot{\Theta} \vec{e}_x - \vec{F} = -mR \dot{\Theta} \vec{e}_x + mg \vec{e}_2\)
Lesson 17

Kinetic energy in terms of \( \vec{q} = (q_1, \ldots, q_N) \)

Without constraint: \( k_i = \frac{1}{2} m_i \dot{v}_i^2 = \text{Kinetic energy of the point } P_i \)

\[
\vec{\ddot{v}}_i = \vec{\ddot{r}}_i (\vec{q}, t) = \frac{\partial}{\partial t} \left( \sum_{h=1}^{N} \frac{2 \vec{a}_{pq}}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_h} \right) \dot{q}_h + \frac{2 \sum_{h=1}^{N} \vec{a}_{pq} \dot{q}_h}{\partial q_h} \dot{q}_i/t
\]

Therefore \( K = \text{total kinetic energy of the system } P_1 \ldots P_N \) of point

\[
K = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{v}_i^2 = \frac{1}{2} \sum_{h=1}^{N} \sum_{k=1}^{N} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \left( \sum_{i=1}^{N} m_i \vec{a}_{pq} \frac{\partial \vec{r}_i}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_h \dot{q}_k
\]

\[
+ \frac{2 \sum_{h=1}^{N} \vec{a}_{pq} \dot{q}_h}{\partial q_h} \dot{q}_i/t
\]

\[
= \frac{1}{2} \sum_{h=1}^{N} \sum_{k=1}^{N} \left( \sum_{i=1}^{N} m_i \vec{a}_{pq} \frac{\partial \vec{r}_i}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_h \dot{q}_k
\]

\[
= \frac{1}{2} \sum_{h=1}^{N} \sum_{k=1}^{N} \left( \sum_{i=1}^{N} m_i \vec{a}_{pq} \frac{\partial \vec{r}_i}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_h \dot{q}_k
\]

\[
+ \frac{1}{2} \sum_{h=1}^{N} \sum_{k=1}^{N} \frac{\partial}{\partial t} \frac{\partial}{\partial t} \left( \sum_{i=1}^{N} m_i \vec{a}_{pq} \frac{\partial \vec{r}_i}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_h \dot{q}_k
\]

\[
= \frac{1}{2} \sum_{h=1}^{N} \sum_{k=1}^{N} \left( \sum_{i=1}^{N} m_i \vec{a}_{pq} \frac{\partial \vec{r}_i}{\partial q_h} \frac{\partial \vec{r}_i}{\partial q_k} \right) \dot{q}_h \dot{q}_k
\]

Remark

The matrix \( \{a_{hk}\} \) is symmetric and positive definite.

- \( \{a_{hk}\} \) is clearly symmetric \( \forall h, k = 1 \ldots N \)

- \( \{a_{hk}\} \) is positive definite. To this purpose, let \( \vec{v}_i' = \sum_{h=1}^{N} \vec{a}_{pq} \dot{q}_h \)

Then \( \sum_{h=1}^{N} a_{hk} \dot{q}_h \dot{q}_k = \sum_{h=1}^{N} \sum_{i=1}^{N} m_i \dot{v}_i^2 \)

If \( \vec{q} \neq 0 \) then \( \vec{v}'_i \neq 0 \Rightarrow \dot{v}'_i > 0 \)

If \( \vec{q} \neq 0 \) then \( \vec{v}'_i \neq 0 \Rightarrow \dot{v}'_i > 0 \)
EXAMPLE 1
Kinetic energy of a point in polar and spherical coordinates.

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]

\[ K = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right) \]

EXAMPLE 2
Pendulum

EXAMPLE 3 (Problem 5 pg. 12)

\[ \mathbf{OP} = (s, 0, 0) \]
\[ \mathbf{OQ} = (s + l \sin \theta, -l \cos \theta, 0) \]

\[ \dot{V}_{Q} \cdot \dot{\mathbf{V}}_{Q} = \left( \dot{s} + l \dot{\theta} \cos \theta \right)^2 + l^2 \dot{\theta}^2 \sin^2 \theta \]

\[ K = \frac{1}{2} m_p \dot{s}^2 + \frac{1}{2} m_a \left[ \dot{s}^2 + l^2 \dot{\theta}^2 + 2l \dot{s} \dot{\theta} \cos \theta \right] \]

\[ Q = \left( \begin{array}{c} m_p + m_a \sin^2 \theta \cos \theta \\ m_a \cos \theta \end{array} \right) \]
EXAMPLE 4 (Problema 6 pag 14) DOPOPIO PENDOLO - DOUBLE PENDULUM -

\[ K_p = \frac{1}{2} m_1 l_1^2 \dot{x}_1^2 \]

\[ \dot{x}_1 = (l \sin \theta + l \cos \phi \dot{\theta} - l \cos \theta - l \cos \phi \dot{\phi}) \]

\[ \dot{\theta} = (l \dot{\cos} \theta + l \dot{\cos} \phi \dot{\phi}) \]

\[ V = \frac{1}{2} M l^2 \dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\phi}^2 + 2 l^2 \dot{\theta} \cos \phi \dot{\phi} \cos \theta \]

\[ K_q = \frac{1}{2} \dot{2} m_2 l^2 \left( \dot{\theta}_2^2 + \dot{\phi}_2^2 + 2 \cos (\theta - \phi) \dot{\theta}_2 \dot{\phi}_2 \right) \]

\[ K = K_p + K_q = \frac{1}{2} m_1 l_1^2 \dot{x}_1^2 + \frac{1}{2} m_2 l^2 \left( \dot{\theta}_2^2 + \dot{\phi}_2^2 + 2 \cos (\theta - \phi) \dot{\theta}_2 \dot{\phi}_2 \right) \]

\[ \Rightarrow Q = m_2 l^2 \left( \frac{2}{\cos (\theta - \phi)} \right) \]

Toward Lagrange equations...

\[ \sum_{i=1}^{N} m_i \ddot{\mathbf{a}}_i = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{\Phi}_i) \Rightarrow \sum_{i=1}^{N} m_i \ddot{\mathbf{a}}_i \cdot \mathbf{\overrightarrow{OP}_i} = \sum_{i=1}^{N} (\mathbf{F}_i + \mathbf{\Phi}_i) \cdot \mathbf{\overrightarrow{OP}_i} \]

\[ \Rightarrow \sum_{i=1}^{N} m_i \ddot{\mathbf{a}}_i \cdot \mathbf{\overrightarrow{OP}_i} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{\overrightarrow{OP}_i} \quad \forall h = 1 \ldots N \]

**Ideal Coupling**

\[ \Rightarrow \sum_{i=1}^{N} m_i \ddot{\mathbf{a}}_i \cdot \mathbf{\overrightarrow{OP}_i} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{\overrightarrow{OP}_i} \]

**Definition**

\[ Q_h = \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{\overrightarrow{OP}_i} \quad \forall h = 1 \ldots N \]

\[ \Rightarrow \sum_{i=1}^{N} m_i \ddot{\mathbf{a}}_i \cdot \mathbf{\overrightarrow{OP}_i} = Q_h \quad \forall h = 1 \ldots N \]

As a consequence:

\[ \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{\overrightarrow{OP}_i} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{h=1}^{N} \mathbf{\overrightarrow{OP}_i} \delta_{q_h} = \sum_{h=1}^{N} \sum_{i=1}^{N} \mathbf{F}_i \cdot \mathbf{\overrightarrow{OP}_i} \delta_{q_h} = \sum_{h=1}^{N} Q_h \delta_{q_h} \]

Work of forces corresponding to virtual displacements of the point $P_1 \ldots P_N$
\textbf{CONSERVATIVE CASE} \quad \vec{F}_i = -\nabla V(P_1, \ldots, P_N, t)

With respect to coordinates \( \vec{q} : \dot{\hat{V}}(\vec{q}, t) = \dot{V}(\vec{q}, \vec{\omega}(\vec{q}, t), t) \), we obtain:

\[
Q_n = \sum_{i=1}^{N} -\nabla V \cdot \vec{a}P_i = -\frac{\partial V}{\partial q_n} \quad \forall n = 1 \ldots m.
\]

by the rule of derivation of the composition of functions.

\textbf{THEOREM (Lagrange equation)}

N point P_1, P_N, \( m \) degree of freedom \( q_1 \ldots q_m \).
\( F_i \), \( i = 1 \ldots N \) forces and ideal constraint.

Then, it holds:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = Q_n, \quad n = 1 \ldots m. \quad \text{(Lagrange equation)}
\]

\textbf{PROOF} \quad \text{Next lesson!}

\textbf{REMARK} \quad \text{In the conservative case,} \quad \vec{F}_i = -\nabla V(P_1 \ldots P_N, t), \text{ the Lagrange equation becomes:}

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = -\frac{\partial \hat{V}}{\partial q_n} \quad \Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = \nabla \hat{V} = 0
\]

\[
\Rightarrow \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} = \frac{\partial L}{\partial q_n} = 0, \quad \forall n = 1 \ldots m
\]

\( L = K - \hat{V} \quad \text{does not depend on} \quad q_n, \ldots \)

\quad \text{In the mixed case:} \quad Q_n = Q'_n + Q''_n = -\frac{\partial \hat{V}}{\partial q_n} + Q''_n, \text{ we clearly obtain:}

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = Q'_n \quad \forall n = 1 \ldots n
\]

for \( L = K - \hat{V}, \)
In un piano verticale, due aste di lunghezza \( l \) e massa trascurabile sono appese nei punti \( O(0,0) \) e \( A(d,0) \). Alle estremità le due aste recano due punti materiali \( P \) e \( Q \) di massa \( m \), i quali sono collegati da una molla di costante elastica \( k > 0 \). Siano \( \theta \) e \( \varphi \) gli angoli che le due aste formano con la verticale.

1) Determinare la Lagrangiana del problema;

2) scrivere le equazioni del moto.

1) Le coordinate dei due punti sono:

\[
P(x_P, y_P) = (l \sin \theta, l \cos \theta), \quad Q(x_Q, y_Q) = (d - l \sin \varphi, l \cos \varphi)
\]
e quindi le rispettive velocità sono:

\[
y_P(l \dot{\theta} \cos \theta, -l \dot{\theta} \sin \theta), \quad y_Q(-l \dot{\varphi} \cos \varphi, -l \dot{\varphi} \sin \varphi)
\]

ovvero \( v_P^2 = l^2 \dot{\theta}^2 \), \( v_Q^2 = l^2 \dot{\varphi}^2 \). Pertanto l'energia cinetica assume la forma:

\[
T = \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\varphi}^2).
\]

Poiché la distanza tra i punti \( P \) e \( Q \) vale:

\[
|PQ|^2 = d^2 + 2l^2 - 2l^2 \cos(\theta - \varphi) - 2dl(\sin \theta + \sin \varphi),
\]

l'energia potenziale è data dalla seguente espressione:

\[
V = -mg y_P - mg y_Q + \frac{1}{2} k |PQ|^2
= -mgl(\cos \theta + \cos \varphi) - kl[l \cos(\theta + \varphi) + d(\sin \theta + \sin \varphi)] + \text{cost.}
\]

Infine, la Lagrangiana completa è:

\[
L(\dot{\theta}, \dot{\varphi}, \varphi) = \frac{1}{2} ml^2 (\dot{\theta}^2 + \dot{\varphi}^2) + mgl(\cos \theta + \cos \varphi) + kl[l \cos(\theta + \varphi) + d(\sin \theta + \sin \varphi)].
\]

2) Le equazioni di Lagrange sono:

\[
\begin{align*}
\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= \frac{\partial L}{\partial \theta} \\
\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} &= \frac{\partial L}{\partial \varphi}
\end{align*}
\]

ovvero

\[
\begin{align*}
ml^2 \ddot{\theta} &= -mgl \sin \theta - kl^2 \sin(\theta + \varphi) + kdl \cos \theta \\
ml^2 \ddot{\varphi} &= -mgl \sin \varphi - kl^2 \sin(\theta + \varphi) + kdl \cos \varphi.
\end{align*}
\]
Let consider the differential equation in $\mathbb{R}^2 \times (x, y)$
\[
\begin{align*}
\dot{x} &= 2x(1+y) - \sin(x+y) \\
\dot{y} &= xy - y
\end{align*}
\] (1)

a) Determine equilibria (Recall that $2t = \sin t$ $x = y$ $t = 0$)
b) Linearize around equilibria
c) Draw the phase portrait of the linearized systems.
d) What about the stability of equilibria of (1)?
e) Determine the hyperbolic and elliptic equilibria of (1). In the hyperbolic case, determine the dimension of the stable/unstable manifold.

---

Solve:
\[
x - y = 0 \iff y(x-1) = 0 \iff y = 0 \text{ or } x = 1
\]

If $y = 0$:
\[
\begin{align*}
\dot{x} &= 2x - \sin x = 0 \iff x = 0 \implies P_0 = (0, 0)
\end{align*}
\]

If $x = 1$:
\[
\begin{align*}
2(1+y) - \sin(1+y) = 0 \iff 1+y = 0 \implies P_a = (1, -1)
\end{align*}
\]

\[
JX(x, y) = \begin{pmatrix}
2(1+y) - \cos(x+y) & 2x - \cos(x+y) \\
y & x-1
\end{pmatrix}
\]

\[
JX(0,0) = \begin{pmatrix}
1 & -1 \\
0 & -1
\end{pmatrix} = 0 \implies \begin{cases}
\dot{x} = x - y \\
\dot{y} = -y
\end{cases}
\]

\[
JX(1,-1) = \begin{pmatrix}
-2 & 1 \\
-1 & 0
\end{pmatrix} = 0 \implies \begin{cases}
\dot{x} = -(x-1) + (y+1) \\
\dot{y} = -(x-2)
\end{cases}
\]

(0,0) no eigenvalue:
\[
\begin{cases}
1 \text{ with eigenvector } (\begin{pmatrix} 1 \end{pmatrix}) \\
-1 \text{ with eigenvector } (\begin{pmatrix} 1 \\ 2 \end{pmatrix})
\end{cases}
\]

Therefore:
\[(1, -1): \det \begin{pmatrix} -1 - \lambda & 1 \\ -1 & -2 \end{pmatrix} = \lambda + \lambda^2 + 1 = 0 \quad \lambda = \gamma\]

\[\lambda_\pm = \frac{-1 \pm \sqrt{3}}{2} \quad \text{stable spiral.}\]

By Lyapunov spectral theorem, \((0,0)\) is unstable and \((1,-1)\) is asymptotically stable.

\((0,0)\) is hyperbolic: stable and unstable manifolds have
\[\text{dim} = 1\]

\((1,-1)\) is hyperbolic: stable manifold has dimension 2.

Draw the phase portrait for
\[\begin{align*}
&x' = x^3 + x^2 \\
&\text{sol:} \\
&x^3 + x^2 = \frac{d}{dx} \left( -\frac{1}{4} x^4 - \frac{1}{3} x^3 \right) = \frac{d}{dx} \left[ -\frac{x^3}{4} \left( \frac{4}{3} + x \right) \right]
\end{align*}\]

Therefore:
\[\lim_{x \to \pm \infty} V(x) = -\infty\]

\[V'(x) = -x^3 - x^2 = -x^2(x+1) = 0 \iff x = -1\]

\[V''(x) = -3x^2 - 2x = 0 \quad V''(0) = 0 \quad \text{lessen a tg ontroval}\]

\[V''(-1) = -3 + 2 = -1 < 0 \implies \text{max local.}\]

\[V(x) = 0 \iff x = 0 \text{ or } x = -4/3 < -1\]
Let consider the diff. equation in $\mathbb{R}^2: (x, y)$
\[
\begin{align*}
  \dot{x} &= \alpha x - \beta xy - \varepsilon x^2 \\
  \dot{y} &= -\gamma y + \delta xy
\end{align*}
\]

"modified Lotka-Volterra" ($\alpha, \beta, \gamma, \delta, \varepsilon > 0$, small $\varepsilon > 0$)

Determine and classify equilibria, draw the phase portrait.

Solution:
\[
\begin{align*}
  \dot{x} &= 0 \iff x (\alpha - \beta y - \varepsilon x) = 0 \iff x = 0 \text{ or } \alpha - \beta y - \varepsilon x = 0 \\
  \dot{y} &= 0 \iff y (-\gamma + \delta x) = 0 \iff y = 0 \text{ or } x = \delta / \gamma
\end{align*}
\]
On $x = 0$: $y = -xy < 0$ for $y > 0$

On $y = 0$: $\dot{x} = \alpha x - \varepsilon x^2 = x(\alpha - \varepsilon x) = 0 \Rightarrow x < \alpha / \varepsilon$

\[
JX(x, y) = \begin{pmatrix}
\alpha - \beta y - 2\varepsilon x & -\beta x \\
\delta y & -\delta + \delta x
\end{pmatrix}
\]

For $x > 0$

\[JX(0, 0) = \begin{pmatrix}
\alpha & 0 \\
0 & -\gamma
\end{pmatrix}, \quad \text{det} < 0 \Rightarrow \text{saddle.}
\]

\[JX(\frac{x}{\delta}, \frac{y}{\delta}, \alpha - \varepsilon x) = \begin{pmatrix}
\frac{\delta - \alpha - \varepsilon x}{\delta} & -\frac{\beta \delta}{\delta} \\
\frac{\delta \alpha - \varepsilon x}{\beta} & 0
\end{pmatrix}
\]

\[\text{tr} = -\frac{\varepsilon \delta}{\delta} < 0
\]

\[\text{det} = \frac{\beta \delta}{\delta} \left( \frac{\delta \alpha - \varepsilon x}{\beta} \right) = \frac{\delta \alpha - \varepsilon \delta^2}{\delta} > 0 \quad (\delta > 0 \text{ small!})
\]

Moreover, $(\text{tr})^2 - 4\text{det} = \frac{\varepsilon^2 \delta^2}{\delta^2} - 4 \delta \alpha + 4 \delta \varepsilon^2 < 0 \quad (\delta > 0 \text{ small!})$

$= 0 \Rightarrow C_2$ is a stable spiral.
\[ \mathbf{J}(x, 0) = \begin{pmatrix} -\alpha & -\frac{\beta x}{\epsilon} \\ 0 & -\gamma + \frac{\delta x}{\epsilon} \end{pmatrix} \]

\[ \text{det} = \alpha \gamma - \frac{\delta \gamma}{\epsilon} < 0 \quad (\epsilon > 0 \text{ small}) \implies \text{Saddle} \]

Phase portrait

---

4. \( m = 2 \).

\[ V(x) = x^2 (1-x)(3-x). \]

(a) Phase portrait for \( \dot{x} = -V'(x) \).

(b) Period for \( x(0) = 1 \) and \( \dot{x}(0) = 0 \).

(c) Estimate the period (Recall that \( \int_{a}^{b} \frac{dx}{\sqrt{-(a-x)(b-x)}} = \pi \)).

---

\[ \lim_{x \to \pm \infty} V(x) = +\infty. \]

\[ V'(x) = 2x(1-x)(3-x) - x^2(3-x) - x^2(1-x) \]

\[ = 2x(1-x)(3-x) - x^2(3-x+1-x) \]

\[ = 2x(1-x)(3-x) - x^2(-2x+4) \]

\[ = 2x[ (1-x)(3-x) + x^2 - 2x ] \]

\[ = 2x [ 2x^2 - 6x + 3 ] \]

\[ = 0 \quad V'(x) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x_{1,2} = \frac{3 \pm \sqrt{3}}{2} \]
Moreover
\[ V''(x) = 2(2x^2 - 6x + 3) + 2(4x - 6) \]

\[ V''(0) = 6 \quad \text{local minimum} \]

\[ V''(x_1) = V''\left(\frac{3 + \sqrt{3}}{2}\right) > 0 \quad \text{local minimum} \left( \approx 16.39 \right) \]

\[ V''(x_2) = V''\left(\frac{3 - \sqrt{3}}{2}\right) < 0 \quad \text{local maximum} \left( \approx -9.39 \right) \]

\[ T = \frac{1}{2} \int_{-a}^{a} \frac{dx}{\sqrt{\frac{2}{m} \left[ E - V(x) \right]}} = \frac{1}{2} \int_{1}^{3} \frac{dx}{\sqrt{-x^2(1-x)(3-x)}} \]

**In order to obtain an estimate for the period, we can use:***

\[ m = 2, E = 0 \]

So that:

Moreover, for \( E = 0 \):

\[ x^2 (1-x)(3-x) = 0 \]

\[ x = 0 \text{ or } x = 1, 3. \]

This periodic orbit.
\[ \frac{2}{3} \int_{2}^{3} \frac{dx}{\sqrt{(1-x)(3-x)}} \leq T \leq 2 \int_{1}^{3} \frac{dx}{\sqrt{(1-x)(3-x)}} \]

\[ \frac{2\pi}{3} \leq T \leq 2\pi \]

**Esercizio**. Si consideri il sistema meccanico monodimensionale che descrive un punto materiale di massa \( m = 1 \), soggetto alla forza di energia potenziale

\[ V(x) = \frac{x^3}{|x^2 - 1|} \]

Si richiede di
(i) Determinare eventuali costanti del moto (non banali).
(ii) Determinare eventuali equilibri e studiarne la natura.
(iii) Studiare qualitativamente l'andamento delle orbite nello spazio delle fasi.
(iv) Determinare i dati iniziali che originano orbite periodiche.
ESERCIZIO 5

Il sistema è del tipo \( \dot{x} = -V'(x) \), quindi \( E = \frac{1}{2} x^2 + V(x) \) è costante del moto.

Equilibri:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -V'(x)
\end{align*}
\]

Quindi

\[
\frac{3x^2(x^2-1) - 2x^4}{(x^2-1)x^2-1} = \frac{x^4 - 3x^2}{(x^2-1)x^2-1} = \frac{x^2(x^2-3)}{(x^2-1)x^2-1} = V'(x)
\]

\( P_0 = (0,0) \), \( P_1 = (\sqrt{3},0) \), \( P_2 = (-\sqrt{3},0) \)

Segno di \( V'(x) \) (NB \( V(x) \) non è definita in \( \pm 2 \))

\( P_0 = (0,0) \) plesso a tangente orizzontale (eq. instabile)
\( P_1 = (\sqrt{3},0) \) minimo locale (eq. stabile)
\( P_2 = (-\sqrt{3},0) \) massimo locale (eq. instabile)

Grafico di \( V(x) \) e orbite nello spazio delle fasi nella pagina seguente.

In particolare:

- \( E_1 : E < V(-\sqrt{3}) \) 3 orbite
- \( E_2 : E = V(-\sqrt{3}) \) 6 orbite
- \( E_3 : V(-\sqrt{3}) < E < 0 \) 3 orbite
- \( E_4 : E = 0 \) 5 orbite
- \( E_5 : 0 < E < V(\sqrt{3}) \) 3 orbite
- \( E_6 : E = V(\sqrt{3}) \) 4 orbite
- \( E_7 : E > V(\sqrt{3}) \) 4 orbite

Orbita periodica?

- Gli equilibri

- Dati iniziali che soddisfano

\[ x > 1, \quad E = \frac{1}{2} x^2 + V(x) \geq V(\sqrt{3}) \]

sempre vera per \( x > 1 \)!
$P_0 : (\phi, \phi)$ \textbf{FLESSO TG. ORIZZONTALE} $\Rightarrow$ \textbf{EQUILIBRIO INSTABILE}

$P_1 : (\sqrt{3}, \phi)$ \textbf{MINIMO LOCALE} $\Rightarrow$ \textbf{STABILE} (E cost. moto di Lyapunov)

$P_2 : (-\sqrt{3}, \phi)$ \textbf{Massimo locale} $\Rightarrow$ \textbf{sella}