G. Parmeggiani, 9/12/2019 **Algebra Lineare, a.a. 2019/2020,**

Scuola di Scienze - Corsi di laurea: **Statistica per l'economia e l'impresa**
Statistica per le tecnologie e le scienze

Studenti: **numero di MATRICOLA PARI**

ESERCIZIO TIPO 14

Si trovi una base ortonormale del sottospazio di $\mathbb{C}^4$

$$V = \left\langle \begin{pmatrix}1\\0\\i\\0\end{pmatrix};\begin{pmatrix}2i\\2\\0\\-2\end{pmatrix};\begin{pmatrix}-6\\6i\\0\\-6i\end{pmatrix};\begin{pmatrix}0\\0\\2\\1\end{pmatrix}\right\rangle.$$

$\boxed{1^0\textbf{MODO}}$

$\boxed{1}$ Troviamo una base $\mathcal{B}_1$ di V.

Poniamo

$$\mathbf{w_1} = \begin{pmatrix}1\\0\\i\\0\end{pmatrix}, \quad \mathbf{w_2} = \begin{pmatrix}2i\\2\\0\\-2\end{pmatrix}, \quad \mathbf{w_3} = \begin{pmatrix}-6\\6i\\0\\-6i\end{pmatrix}, \quad \mathbf{w_4} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix}$$

e costruiamo la matrice $\mathbf{A} = \begin{pmatrix}\mathbf{w_1} & \mathbf{w_2} & \mathbf{w_3} & \mathbf{w_4}\end{pmatrix}$, ossia una matrice tale che $C(\mathbf{A}) = V$.

$$\mathbf{A} = \begin{pmatrix}1 & 2i & -6 & 0\\0 & 2 & 6i & 0\\i & 0 & 0 & 2\\0 & -2 & -6i & 1\end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix}1 & 2i & -6 & 0\\0 & 2 & 6i & 0\\0 & 2 & 6i & 2\\0 & -2 & -6i & 1\end{pmatrix} \xrightarrow{E_{42}(2)E_{32}(-2)E_2(\frac{1}{2})}$$

$$\to \begin{pmatrix}1 & 2i & -6 & 0\\0 & 1 & 3i & 0\\0 & 0 & 0 & 2\\0 & 0 & 0 & 1\end{pmatrix} \xrightarrow{E_{43}(-1)E_3(\frac{1}{2})} \begin{pmatrix}1 & 2i & -6 & 0\\0 & 1 & 3i & 0\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{pmatrix} = \mathbf{U}$$

Dunque $\mathcal{B}_1 = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_4}\}$ è una base di $C(\mathbf{A}) = V$.

$\boxed{2}$ Troviamo una base ortogonale $\mathcal{B}_2$ di V: poniamo $\mathbf{v_1} = \mathbf{w_1}, \mathbf{v_2} = \mathbf{w_2}$ e $\mathbf{v_3} = \mathbf{w_4}$, e applichiamo l'algoritmo di Gram-Schmidt a $\{\mathbf{v_1}; \mathbf{v_2}; \mathbf{v_3}\}$.

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1}, \qquad \mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{12} = \frac{(\mathbf{u_1}|\mathbf{v_2})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_2}) = \mathbf{u_1}^H \mathbf{v_2} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 2i \\ 2 \\ 0 \\ -2 \end{pmatrix} = 2i$$

$$(\mathbf{u_1}|\mathbf{u_1}) = \mathbf{u_1}^H \mathbf{u_1} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = 2$$

$$\Longrightarrow \quad \alpha_{12} = 2i/2 = i$$

$$\begin{aligned} \mathbf{u_2} &= \mathbf{v_2} - \alpha_{12}\mathbf{u_1} = \\ &= \mathbf{v_2} - i\mathbf{u_1} = \end{aligned}$$

$$= \begin{pmatrix} 2i \\ 2 \\ 0 \\ -2 \end{pmatrix} - i \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2},$$

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{13} = \frac{(\mathbf{u_1}|\mathbf{v_3})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_3}) = \mathbf{u_1}^H \mathbf{v_3} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -2i$$

$$\Longrightarrow \quad \alpha_{13} = -\frac{2i}{2} = -i$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{23} = \frac{(\mathbf{u_2}|\mathbf{v_3})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_3}) = \mathbf{u_2}^H \mathbf{v_3} = \begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = 2 - 2 = 0$$

$$\Longrightarrow \quad \alpha_{23} = 0$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2} = \mathbf{v_3} + i\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

$\boldsymbol{\mathcal{B}}_2 = \{\mathbf{u_1}; \mathbf{u_2}; \mathbf{u_3}\}$, dove

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

è una base ortogonale di V.

[3] Troviamo una base ortonormale $\boldsymbol{\mathcal{B}}$ di V, normalizzando gli elementi di $\boldsymbol{\mathcal{B}}_2$.

$$\|\mathbf{u_1}\|_2 = \sqrt{(\mathbf{u_1}|\mathbf{u_1})} = \sqrt{2}$$

$$\|\mathbf{w_2}\|_2 = \sqrt{(\mathbf{u_2}|\mathbf{u_2})} = \sqrt{\mathbf{u_2}^H\mathbf{u_2}} = \sqrt{\begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}} = \sqrt{10}$$

$$\|\mathbf{u_3}\|_2 = \sqrt{(\mathbf{u_3}|\mathbf{u_3})} = \sqrt{\mathbf{u_3}^H\mathbf{u_3}} = \sqrt{\begin{pmatrix} -i & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}} = \sqrt{3}$$

Concludendo: $\boldsymbol{\mathcal{B}} = \{\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|_2}; \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|_2}; \frac{\mathbf{u_3}}{\|\mathbf{u_3}\|_2}\}$, dove

$$\frac{\mathbf{u_1}}{\|\mathbf{u_1}\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{u_3}}{\|\mathbf{u_3}\|_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

è una base ortonormale di V.

2^0MODO

[1] Prima costruiamo **un insieme di generatori ortogonale di V**: poniamo

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{v_2} = \begin{pmatrix} i \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v_3} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{v_4} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

e applichiamo l'algoritmo di Gram-Schimdt a $\{\mathbf{v_1};\mathbf{v_2};\mathbf{v_3};\mathbf{v_4}\}$. Otterremo 4 vettori, $\mathbf{u_1},\mathbf{u_2},\mathbf{u_3},\mathbf{u_4}$, e l'insieme $\{\mathbf{u_1};\mathbf{u_2};\mathbf{u_3};\mathbf{u_4}\}$ sarà un insieme di generatori ortogonale di V.

Per sapere se alcuni degli $\mathbf{u_i}$ saranno nulli, e in tal caso quali, troviamo innanzitutto una forma ridotta di Gauss $\mathbf{U}$ della matrice $\mathbf{A}$ che ha come colnne $\mathbf{v_1},\mathbf{v_2},\mathbf{v_3},\mathbf{v_4}$: le eventuali colonne libere di U corrisponderanno agli $\mathbf{u_i}$ nulli.

$$\mathbf{A} = \begin{pmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4}\end{pmatrix} = \begin{pmatrix}1 & 2i & -6 & 0\\ 0 & 2 & 6i & 0\\ i & 0 & 0 & 2\\ 0 & -2 & -6i & 1\end{pmatrix} \xrightarrow{E_{31}(-i)} \begin{pmatrix}1 & 2i & -6 & 0\\ 0 & 2 & 6i & 0\\ 0 & 2 & 6i & 2\\ 0 & -2 & -6i & 1\end{pmatrix} \rightarrow$$

$$\xrightarrow{E_{42}(2)E_{32}(-2)E_2(\frac{1}{2})} \begin{pmatrix}1 & 2i & -6 & 0\\ 0 & 1 & 3i & 0\\ 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 1\end{pmatrix} \xrightarrow{E_{43}(-1)E_3(\frac{1}{2})} \begin{pmatrix}1 & 2i & -6 & 0\\ 0 & 1 & 3i & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} = \mathbf{U}$$

Poichè $\mathbf{U}$ ha come unica colonna libera la 3^a, allora applicando l'algoritmo di Gram-Schimdt a $\{\mathbf{v_1};\mathbf{v_2};\mathbf{v_3};\mathbf{v_4}\}$ otterremo $\mathbf{u_3} = \mathbf{0}$.

$$\mathbf{u_1} = \mathbf{v_1} = \begin{pmatrix}1\\ 0\\ i\\ 0\end{pmatrix}$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1}, \qquad \mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{12} = \frac{(\mathbf{u_1}|\mathbf{v_2})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_2}) = \mathbf{u_1}^H\mathbf{v_2} = \begin{pmatrix}1 & 0 & -i & 0\end{pmatrix}\begin{pmatrix}2i\\ 2\\ 0\\ -2\end{pmatrix} = 2i$$

$$(\mathbf{u_1}|\mathbf{u_1}) = \mathbf{u_1}^H\mathbf{u_1} = \begin{pmatrix}1 & 0 & -i & 0\end{pmatrix}\begin{pmatrix}1\\ 0\\ i\\ 0\end{pmatrix} = 2$$

$$\Longrightarrow \quad \alpha_{12} = 2i/2 = i$$

$$\mathbf{u_2} = \mathbf{v_2} - \alpha_{12}\mathbf{u_1} = \mathbf{v_2} - i\mathbf{u_1} = \begin{pmatrix}2i\\ 2\\ 0\\ -2\end{pmatrix} - i\begin{pmatrix}1\\ 0\\ i\\ 0\end{pmatrix} = \begin{pmatrix}i\\ 2\\ 1\\ -2\end{pmatrix}.$$

$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2}$,

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{13} = \frac{(\mathbf{u_1}|\mathbf{v_3})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_3}) = \mathbf{u_1}^H \mathbf{v_3} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} -6 \\ 6i \\ 0 \\ -6i \end{pmatrix} = -6$$

$$(\mathbf{u_1}|\mathbf{u_1}) = 2$$

$$\Longrightarrow \quad \alpha_{13} = -\frac{6}{2} = -3$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{23} = \frac{(\mathbf{u_2}|\mathbf{v_3})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_3}) = \mathbf{u_2}^H \mathbf{v_3} = \begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} -6 \\ 6i \\ 0 \\ -6i \end{pmatrix} =$$

$$= 6i + 12i + 12i = 30i$$

$$(\mathbf{u_2}|\mathbf{u_2}) = \mathbf{u_2}^H \mathbf{u_2} = \begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix} =$$

$$= 1 + 4 + 1 + 4 = 10$$

$$\Longrightarrow \quad \alpha_{23} = \frac{30i}{10} = 3i$$

$$\mathbf{u_3} = \mathbf{v_3} - \alpha_{13}\mathbf{u_1} - \alpha_{23}\mathbf{u_2} = \mathbf{v_3} + 3\mathbf{u_1} - 3i\mathbf{u_2} =$$

$$= \begin{pmatrix} -6 \\ 6i \\ 0 \\ -6i \end{pmatrix} + 3 \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} - 3i \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\mathbf{u_4} = \mathbf{v_4} - \alpha_{14}\mathbf{u_1} - \alpha_{24}\mathbf{u_2} - \alpha_{34}\mathbf{u_3}$

$$\mathbf{u_1} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{14} = \frac{(\mathbf{u_1}|\mathbf{v_4})}{(\mathbf{u_1}|\mathbf{u_1})}$$

$$(\mathbf{u_1}|\mathbf{v_4}) = \mathbf{u_1}^H\mathbf{v_4} = \begin{pmatrix} 1 & 0 & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = -2i$$

$$\Longrightarrow \quad \alpha_{14} = -2i/2 = -i$$

$$\mathbf{u_2} \neq \mathbf{0} \quad \Longrightarrow \quad \alpha_{24} = \frac{(\mathbf{u_2}|\mathbf{v_4})}{(\mathbf{u_2}|\mathbf{u_2})}$$

$$(\mathbf{u_2}|\mathbf{v_4}) = \mathbf{u_2}^H\mathbf{v_4} = \begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\Longrightarrow \quad \alpha_{24} = 0$$

$$\mathbf{u_3} = \mathbf{0} \quad \Longrightarrow \quad \alpha_{34} = 0 \quad \text{per def.}$$

$$\mathbf{u_4} = \mathbf{v_4} - \alpha_{14}\mathbf{u_1} = \mathbf{v_4} + 2i\mathbf{u_1} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Dunque

$$\left\{ \mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{u_2} = \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}; \mathbf{u_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \mathbf{u_4} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

è un insieme di generatori ortogonale di V.

$\boxed{2}$ Costruiamo **una base ortogonale di V** togliendo dall'insieme di generatori ortogonale di V trovato al punto $\boxed{1}$ gli eventuali $\mathbf{u_i}$ nulli. In questo caso poniamo:

$$\mathbf{w_1} = \mathbf{u_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \mathbf{w_2} = \mathbf{u_2} = \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{w_3} = \mathbf{u_4} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

L'insieme

$$\left\{ \mathbf{w_1} = \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}; \mathbf{w_2} = \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}; \mathbf{w_3} = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

è una base ortogonale di V.

3 Costruiamo **base ortonormale di V** normalizzando la base ortogonale trovata al punto 2, ossia dividendo ciascun elemento della base ortogonale trovata in 2 per la propria norma euclidea.

Cominciamo con il calcolare la norma euclidea di $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$:

$$\|\mathbf{w_1}\|_2 = \sqrt{(\mathbf{u_1}|\mathbf{u_1})} = \sqrt{2}$$

$$\|\mathbf{w_2}\|_2 = \sqrt{(\mathbf{u_2}|\mathbf{u_2})} = \sqrt{\mathbf{u_2}^H \mathbf{u_2}} = \sqrt{\begin{pmatrix} -i & 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}} = \sqrt{10}$$

$$\|\mathbf{w_3}\|_2 = \sqrt{(\mathbf{u_4}|\mathbf{u_4})} = \sqrt{\mathbf{u_4}^H \mathbf{u_4}} = \sqrt{\begin{pmatrix} -i & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}} = \sqrt{3}$$

Allora

$$\mathcal{B} = \{ \frac{\mathbf{w_1}}{\|\mathbf{w_1}\|_2}; \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|_2}; \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|_2} \},$$

dove

$$\frac{\mathbf{w_1}}{\|\mathbf{w_1}\|_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix}, \quad \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} i \\ 2 \\ 1 \\ -2 \end{pmatrix}, \quad \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|_2} = \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

è una base ortonormale di V.