Analytical Methods — Exam Simulation

Exercise 1. Let $f_n(x) := \frac{1}{1+x^n}$, $x \in [0,+\infty[$, $n \in \mathbb{N}$, $n \geqslant 2$. Plot quickly the graph of f_n. Is $f_n \in L^1([0,+\infty[)$? Is (f_n) convergent (and, in the case, to what) in $L^1([0,+\infty[)$? Justify your answer.

Exercise 2. Let $H = L^2([0,\pi])$. Solve

$$\min_{a,b\in\mathbb{R}} \|x - (a\cos x + b\sin x)\|_2 .$$

Exercise 3. State precisely the differentiation under integral sign theorem. Let now

$$F(t) := \int_0^{+\infty} e^{-tx}\frac{1-\cos x}{x}\,dx.$$

i) Determine the domain of definition of F, that is the set of $t \in \mathbb{R}$ such that $F(t)$ is well defined.
ii) Is F continuous on its domain? Justify carefully your answer.
iii) Determine for which t is well defined $F'(t)$ and compute it.
iv) Determine F explicitly.

Exercise 4. The goal is to compute the FT of $f(x) = \frac{1}{1+x^4}$.

i) Does $\widehat{f}$ exists? If yes, which of the following statements are true/false and why: $\widehat{f} \in L^1(\mathbb{R})$; $\widehat{f} \in L^2(\mathbb{R})$; $\widehat{f} \in \mathscr{C}^1(\mathbb{R})$; $\widehat{f} \in \mathscr{S}(\mathbb{R})$.
ii) By reducing to suitable Cauchy distributions, compute FT of

$$\frac{1}{x^2 \pm \sqrt{2}x + 1}.$$

iii) Noticed that $1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$, express $\frac{1}{1+x^4}$ in terms of $\frac{1}{x^2\pm\sqrt{2}x+1}$. Use this to determine $\widehat{f}$.

Exercise 5. Let

$$f_n(x) := \frac{1}{\sqrt{x+\frac{1}{n}}},\ x \in [0,1],\ n \in \mathbb{N}.$$

i) Plot quickly the graph of f_n. Is $(f_n) \subset L^1([0,1])$? Is $(f_n) \subset L^2([0,1])$?
ii) Is (f_n) convergent in $L^1([0,1])$ and, in the case, to what? Is (f_n) convergent in $L^2([0,1])$ and, in the case, to what?

Exercise 6. Let $H := L^2(\mathbb{R})$ with usual real scalar product. Consider

$$U := \{f \in H\ :\ f(-x) = f(x),\ \text{a.e. } x \in \mathbb{R}\}.$$

i) Accept U closed. In general, what is the chatacteristic property of $\Pi_U f$? A natural guess is $\Pi_U f(x) = \frac{1}{2}(f(x) + f(-x))$, $f \in H$, $x \in \mathbb{R}$. Check that this guess is correct.
ii) Show that U is indeed closed, that is: if $(f_n) \subset U$ is such that $f_n \xrightarrow{L^2} f$ then $f \in U$. (hint: recall that $f_n \xrightarrow{L^2} f$ does not imply that (f_n) converges pointwise but...).

Exercise 7. Let

$$g_{a,b}(\xi) := \frac{e^{-a|\xi|} - e^{-b|\xi|}}{\xi},\ \xi \in \mathbb{R}\backslash\{0\}.$$

Here $a, b > 0$ are fixed.

i) Check that $g_{a,b} \in L^1(\mathbb{R})$ for every a, b. Is also $g_{a,b} \in L^2(\mathbb{R})$? Justify carefully.
ii) Without doing calculations, show that $g_{a,b}$ has a Fourier original, that is a function $f_{a,b}$ such that $\widehat{f_{a,b}} = g_{a,b}$.
iii) Determine $f_{a,b}$.

Exercise 8. Let

$$F(\lambda) := \int_0^{+\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx.$$

i) Check that F is well defined for every $\lambda \geqslant 0$.
ii) Discuss differentiability of F on $[0, +\infty[$ and compute $\partial_\lambda F$. It may be helpful to know $\int e^{\alpha x} \sin(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x} \left(\sin(\beta x) - \frac{\beta}{\alpha}\cos(\beta x)\right)$ for $\alpha \neq 0$.
iii) By ii), determine $F(\lambda)$ explicitely.
iv) (facultative, 6 extra marks) How can you use previous calculations to determine

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx\ ?$$

Justify your answer.

Exercise 9. Let
$$f_n(x) := n \log\left(1 + \frac{1}{n\sqrt{x}}\right),\ x \in [0,1].$$

i) Check that $(f_n) \subset L^1([0,1])$.
ii) Discuss convergence of (f_n) in $L^1([0,1])$.

It might be useful to know that $\log(1+t) \leqslant t$, $\forall t > -1$.

Exercise 10. Let
$$g(\xi) = (1-\xi^2)1_{[-1,1]}(\xi),\ \xi \in \mathbb{R}.$$

i) Show that there exists $f \in L^2$ such that $g = \widehat{f}$.
ii) Determine f explicitly. Is $f \in L^1$?

Exercise 11. Let H be a real Hilbert space, $\phi, \psi \in H$ two linearly independent unit vectors (that is $\|\phi\| = \|\psi\| = 1$). Let also $U := \{\alpha\phi \ : \ \alpha \in \mathbb{R}\}$, $V := \{\beta\psi \ : \ \beta \in \mathbb{R}\}$ and $U + V = \{u + v \ : \ u \in U,\ v \in V\}$. Clearly, U and V are closed. We accept $U + V$ is closed as well.

i) Determine the orthogonal projections Π_U and Π_V.
ii) Determine Π_{U+V}.
iii) Under which condition on ϕ, ψ is it true that $\Pi_{U+V} = \Pi_U + \Pi_V$?

Exercise 12. Let
$$F(\xi) := \int_{\mathbb{R}} \frac{\sin(\xi x)}{x(x^2+2x+2)}\, dx.$$

i) Check that F is well defined for every $\xi \in \mathbb{R}$.
ii) Show that F is differentiable on $\mathbb{R}$ and compute $\partial_\xi F$.
iii) Reducing $\partial_\xi F$ to suitable Fourier Transforms, determine F exactly. It may be useful to know $\int e^{\alpha x}\cos(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x}\left(\cos(\beta x) + \frac{\beta}{\alpha}\sin(\beta x)\right)$.

Exercise 13. On

$$H := \left\{ f \in \mathscr{C}^1([-1,1]) \ : \ f(0) = 0, \int_{-1}^{1} |f'(x)|^2 \, dx < +\infty \right\}$$

define

$$\langle f, g \rangle := \int_{-1}^{1} f'(x) g'(x) \, dx.$$

i) Check that $\langle \cdot, \cdot \rangle$ is a well defined inner product on H.
ii) Determine an orthonormal base for the linear space generated by x, x^2, x^3.
iii) What is the best approximation of $f(x) := \sin x$ on the subspace $\mathrm{Span}(x, x^2, x^3)$?

Exercise 14. On $X := \{ f \in \mathscr{C}^1([0,1]) \ : \ f(0) = 0 \}$ we define

$$\|f\| := \max_{t \in [0,1]} t^{1/2} |f'(t)|.$$

i) Check that $\| \cdot \|$ is a norm on X.
ii) Show that $\|f\|$ is stronger than $\|f\|_\infty$ on X.
iii) Define $(f_n) \subset X$ as

$$f_n(t) := \begin{cases} t^{1/4}, & t \in [\frac{1}{n}, 1], \\ \\ \frac{n^{3/4}}{4} t, & t \in [0, \frac{1}{n}[. \end{cases}$$

Compute $\|f_n\|$ and $\|f_n\|_\infty$. What can be deduced about equivalence of $\| \cdot \|$ and $\| \cdot \|_\infty$?

Exercise 15. Define the 1-dim. Fourier transform for a function $f \in L^1$.

i) Give and prove a sufficient condition on f in order $\widehat{f}$ be derivable.
ii) You know $\widehat{f}(\xi) = \frac{\xi}{1+\xi^4}$. Compute

$$\int_{-\infty}^{\infty} x f(x) \, dx.$$

What about $f(0)$?

Solutions

Exercise 1. Notice that
$$f_n(x) \longrightarrow \begin{cases} 1, & 0 \leqslant x < 1, \\ 1/2, & x = 1, \\ 0, & x > 1. \end{cases}$$
A reasonable guess is $f_n \xrightarrow{L^1} 1_{[0,1]}$. To check this, we need to prove that
$$\begin{aligned} \|f_n - 1_{[0,1]}\|_1 &= \int_{\mathbb{R}} |f_n(x) - 1_{[0,1]}|\,dx = \int_0^1 \left|\tfrac{1}{1+x^n} - 1\right|\,dx + \int_1^{+\infty} \left|\tfrac{1}{1+x^n}\right|\,dx \\ &= \int_0^1 \tfrac{x^n}{1+x^n}\,dx + \int_1^{+\infty} \tfrac{1}{1+x^n}\,dx \longrightarrow 0. \end{aligned}$$
To compute these limits we apply dominated convergence. In the first case
$$0 \leqslant \frac{x^n}{1+x^n} \leqslant 1 \in L^1([0,1]) \text{ and } \frac{x^n}{1+x^n} \longrightarrow \begin{cases} 0, & 0 \leqslant x < 1, \\ 1/2, & x = 1. \end{cases}$$
Thus
$$\lim_n \int_0^1 \frac{x^n}{1+x^n}\,dx = \int_0^1 \lim_n \frac{x^n}{1+x^n}\,dx = \int_0^1 0 = 0.$$
In the second case, for $n \geqslant 2$,
$$0 \leqslant \frac{1}{1+x^n} \leqslant \frac{1}{1+x^2} \in L^1([1,+\infty[), \text{ and } \frac{1}{1+x^n} \longrightarrow 0,\ \forall x > 1.$$
Again,
$$\lim_n \int_1^{+\infty} \frac{1}{1+x^n}\,dx = \int_1^{+\infty} \lim_n \frac{1}{1+x^n}\,dx = \int_1^{+\infty} 0 = 0. \quad \blacksquare$$

Exercise 2. Let $U = \mathrm{Span}\langle \cos x, \sin x\rangle$. Being evident that $\cos x, \sin x$ are linearly independent (otherwise: $a\cos x + b\sin x \equiv 0$ would mean $\cos x \equiv \lambda \sin x$ that is $\tan x \equiv \lambda$ on $[0,\pi]$, which is impossible), U is a two dimensional (hence closed) space. An orthonormal base may be determined in the following way:
$$e_0 = \frac{\cos}{\|\cos\|_2},\ e_1 = \frac{\sin - \langle \sin, e_0\rangle e_0}{\|\sin - \langle \sin, e_0\rangle e_0\|_2}.$$
Now,
$$\begin{aligned} \|\cos\|_2^2 &= \int_0^\pi |\cos x|^2\,dx = \int_0^\pi \cos x(\sin x)'\,dx = [\cos x \sin x]_{x=0}^{x=\pi} + \int_0^\pi (\sin x)^2\,dx \\ &= \int_0^\pi (1 - (\cos x)^2)\,dx = \pi - \|\cos\|_2^2, \end{aligned}$$

by which $\|\cos\|_2^2 = \frac{\pi}{2}$, thus

$$e_0(x) = \sqrt{\frac{2}{\pi}}\cos x.$$

Notice that

$$\langle \sin, e_0\rangle = \sqrt{\frac{2}{\pi}}\int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\sqrt{2\pi}}\int_0^{\pi} \sin(2x)\, dx = 0,$$

and because

$$\|\sin\|_2^2 = \int_0^{\pi} (\sin x)^2 = \pi - \int_0^{\pi} (\cos x)^2 \, dx = \frac{\pi}{2},$$

we deduce

$$e_1(x) = \sqrt{\frac{2}{\pi}}\sin x.$$

According to projection theorem, the element of U at minimum distance to x is

$$\Pi_U x = \langle x, e_0\rangle e_0 + \langle x, e_1\rangle e_1.$$

Now,

$$\langle x, e_0\rangle = \sqrt{\frac{2}{\pi}}\int_0^{\pi} x\cos x \, dx = \sqrt{\frac{2}{\pi}}\left([x\sin x]_{x=0}^{x=\pi} - \int_0^{\pi} \sin x \, dx\right) = \sqrt{\frac{2}{\pi}}[\cos x]_{x=0}^{x=\pi} = -2\sqrt{\frac{2}{\pi}},$$

and

$$\langle x, e_1\rangle = \sqrt{\frac{2}{\pi}}\int_0^{\pi} x\sin x \, dx = \sqrt{\frac{2}{\pi}}\left([-x\cos x]_{x=0}^{x=\pi} + \int_0^{\pi} \cos x \, dx\right) = -\pi\sqrt{\frac{2}{\pi}},$$

hence, in conclusion,

$$\Pi_U x = -\frac{4}{\pi}\cos x - 2\sin x.$$

Therefore, $a = -\frac{4}{\pi}$, $b = -2$. ■

Exercise 3. i) Domain of F is

$$\left\{ t \in \mathbb{R} \ : \ \int_0^{+\infty} \left| e^{-tx}\frac{1-\cos x}{x}\right| \, dx < +\infty \right\}.$$

It is easy to check that if $t \leqslant 0$ the integral diverge while it is convergent for $t > 0$. First, notice that at $x = 0$, $\frac{1-\cos x}{x} \longrightarrow 0$, thus there's no problem with integrability at $x = 0$ for every $t \in \mathbb{R}$. The unique problem is at $x = +\infty$. If $t = 0$ we have $\frac{1-\cos x}{x}$ which sounds like $\frac{\sin x}{x}$, not integrable. If $t > 0$,

$$\left| e^{-tx}\frac{1-\cos x}{x}\right| \leqslant e^{-tx}\frac{2}{x} \leqslant 2e^{-tx}, \ x \geqslant 1,$$

integrable at $+\infty$.

ii) Let $f(t,x) = e^{-tx}\frac{1-\cos x}{x}$. We apply the continuity theorem. Clearly $f(\sharp, x) \in \mathscr{C}(]0,+\infty[)$ and

$$|f(t,x)| \le e^{-\varepsilon x}\frac{1-\cos x}{x} \in L^1([0,+\infty[),\ \forall t \geqslant \varepsilon.$$

By this it follows $F \in \mathscr{C}([\varepsilon,+\infty[)$ and because $\varepsilon > 0$ is arbitrary, we conclude $F \in \mathscr{C}(]0,+\infty[)$.

iii) We apply the differentiability theorem. Clearly,

$$\exists \partial_t f(t,x) = -xe^{-tx}\frac{1-\cos x}{x} = e^{-tx}(\cos x - 1),\ \forall x \in [0,\infty[,$$

and because

$$|\partial_t f(t,x)| \leqslant 2e^{-tx} \leqslant 2e^{-\varepsilon x},\ \forall t \geqslant \varepsilon.$$

We may apply differentiation theorem on $[\varepsilon,+\infty[$ and conclude

$$\partial_t F(t) = \int_0^{+\infty} e^{-tx}(\cos x - 1)\,dx,\ \forall t \geqslant \varepsilon,$$

and because $\varepsilon > 0$ is arbitrary, previous formula holds true for every $t > 0$. Easily,

$$\int_0^{+\infty} e^{-tx}\,dx = \left[\frac{e^{-tx}}{-t}\right]_{x=0}^{x=+\infty} = \frac{1}{t},$$

while

$$\begin{aligned}\int_0^{+\infty} e^{-tx}\cos x\,dx &= \left[\frac{e^{-tx}}{-t}\cos x\right]_{x=0}^{x=+\infty} - \frac{1}{t}\int_0^{+\infty} e^{-tx}\sin x\,dx \\ &= \frac{1}{t} - \frac{1}{t}\left(\left[\frac{e^{-tx}}{-t}\sin x\right]_{x=0}^{x=+\infty} + \frac{1}{t}\int_0^{+\infty} e^{-tx}\cos x\,dx\right) \\ &= \frac{1}{t} - \frac{1}{t^2}\int_0^{+\infty} e^{-tx}\cos x\,dx,\end{aligned}$$

by which

$$\int_0^{+\infty} e^{-tx}\cos x\,dx = \frac{t}{1+t^2}.$$

In conclusion

$$\partial_t F = \frac{t}{1+t^2} - \frac{1}{t},$$

iv) By iii)

$$F(t) = \frac{1}{2}\log(1+t^2) - \log t + c,$$

where c is a constant. To determine the value of c we need some more information on F. For instance, we may notice that, by Lebesgue dominated convergence (whose hypotheses are fulfilled by initial bound obtained in ii)),

$$\lim_{t\to+\infty} F(t) = 0,$$

and because

$$\frac{1}{2}\log(1+t^2)-\log t=\log\frac{\sqrt{1+t^2}}{t}=\log\sqrt{1+\frac{1}{t}}\longrightarrow\log 1=0,$$

we deduce $c=0$. ∎

Exercise 4. i) Clearly $f\in L^1$ thus $\widehat{f}$ is well defined. To check $\widehat{f}\in L^1$, we apply the well known result: if $f,f',f''\in L^1$ then $\widehat{f}\in L^1$. We already said $f\in L^1$. About f',

$$f'(x)=-\frac{4x^3}{(1+x^4)^2}\in\mathscr{C}(\mathbb{R}),\ f'(x)\sim_{\pm\infty}-4\frac{x^3}{x^8}=\frac{C}{x^5},$$

which is integrable at $\pm\infty$. Similarly for f'':

$$f''(x)=-4\frac{3x^2(1+x^4)^2-2x^3(1+x^4)4x^3}{(1+x^4)^4}\in\mathscr{C}(\mathbb{R}),\ f''(x)\sim_{\pm\infty}-4\frac{-5x^{10}}{x^{16}}=\frac{C}{x^6},$$

which is integrable at $\pm\infty$. Is $\widehat{f}\in L^2$? Yes, this because $f\in L^2$ (yet, $f\in\mathscr{C}(\mathbb{R})$ and $|f(x)|^2\sim_{\pm\infty}\frac{1}{x^8}$ is integrable, thus $\int_{\mathbb{R}}|f|^2<+\infty$) and the FT maps L^2 into itself. Last: is $\widehat{f}\in\mathscr{S}(\mathbb{R})$? No, this because FT maps the Schwarz space $\mathscr{S}(\mathbb{R})$ into itself, thus $\widehat{f}\in\mathscr{S}(\mathbb{R})$ iff $f\in\mathscr{S}(\mathbb{R})$. Clearly, $f\in\mathscr{C}^\infty$ but, for instance,

$$x^4f(x)\not\longrightarrow 0,\ |x|\longrightarrow\pm\infty.$$

ii) We may notice that

$$\frac{1}{x^2\pm\sqrt{2}x+1}=\frac{1}{\left(x\pm\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}},$$

thus, recalling that $\widehat{g(\sharp+c)}=e^{-i2\pi c}\widehat{g}$,

$$\widehat{\frac{1}{\sharp^2\pm\sqrt{2}\sharp+1}}(\xi)=\widehat{\frac{1}{\left(\sharp\pm\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}}(\xi)=e^{\mp i\sqrt{2}\pi\xi}\widehat{\frac{1}{\sharp^2+\left(\frac{1}{\sqrt{2}}\right)^2}}(\xi)=e^{\mp i\sqrt{2}\pi\xi}\sqrt{2}\pi e^{-\sqrt{2}\pi|\xi|}$$

iii) Because $(1+x^4)=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$ we have

$$\frac{1}{1+x^4}=\frac{1}{(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)}=-\frac{1}{2\sqrt{2}x}\left(\frac{1}{x^2+\sqrt{2}x+1}-\frac{1}{x^2-\sqrt{2}x+1}\right)$$

thus

$$-2\sqrt{2}x\frac{1}{1+x^4}=\frac{1}{x^2+\sqrt{2}x+1}-\frac{1}{x^2-\sqrt{2}x+1}$$

hence

$$-2\sqrt{2}\widehat{(\sharp f)}(\xi)=e^{-i\sqrt{2}\pi\xi}\sqrt{2}\pi e^{-\sqrt{2}\pi|\xi|}-e^{+i\sqrt{2}\pi\xi}\sqrt{2}\pi e^{-\sqrt{2}\pi|\xi|}=-2i\sqrt{2}\pi e^{-\sqrt{2}\pi|\xi|}\sin(\sqrt{2}\pi\xi).$$

Recalling that

$$\widehat{(i2\sharp)f)} = \partial_\xi \widehat{f},$$

we have

$$\partial_\xi \widehat{f}(\xi) = -2\pi e^{-\sqrt{2}\pi|\xi|}\sin(\sqrt{2}\pi\xi).$$

To compute $\widehat{f}$ let first compute

$$\begin{aligned}\int e^{\alpha\xi}\sin(\beta\xi)\,d\xi &= \tfrac{e^{\alpha\xi}}{\alpha}\sin(\beta\xi) - \int \tfrac{e^{\alpha\xi}}{\alpha}\beta\cos(\beta\xi)\,d\xi \\ &= \tfrac{e^{\alpha\xi}}{\alpha}\sin(\beta\xi) - \tfrac{\beta}{\alpha}\left[\tfrac{e^{\alpha\xi}}{\alpha}\cos(\beta\xi) + \int \tfrac{e^{\alpha\xi}}{\alpha}\beta\sin(\beta\xi)\,d\xi\right],\end{aligned}$$

by which, easily,

$$\int e^{\alpha\xi}\sin(\beta\xi)\,d\xi = \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha\xi}\left(\sin(\beta\xi) - \frac{\beta}{\alpha}\cos(\beta\xi)\right).$$

Now,

$$\widehat{f}(\xi) = \begin{cases} \xi \geqslant 0, & -2\pi\int e^{-\sqrt{2}\pi\xi}\sin(\sqrt{2}\pi\xi)\,d\xi + c_1 = \tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\pi\xi}\left(\sin(\sqrt{2}\pi\xi) + \cos(\sqrt{2}\pi\xi)\right) + c_1, \\ \\ \xi < 0, & -2\pi\int e^{\sqrt{2}\pi\xi}\sin(\sqrt{2}\pi\xi)\,d\xi + c_2 = -\tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\pi\xi}\left(\sin(\sqrt{2}\pi\xi) - \cos(\sqrt{2}\pi\xi)\right) + c_2. \end{cases}$$

Because easily $\widehat{f}(\pm\infty) = c_1, c_2$ and $\widehat{\in} L^1$, necessarily $c_1 = c_2 = 0$. Thus

$$\widehat{f}(\xi) = \begin{cases} \xi \geqslant 0, & \tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\pi\xi}\left(\sin(\sqrt{2}\pi\xi) + \cos(\sqrt{2}\pi\xi)\right), \\ \\ \xi < 0, & -\tfrac{1}{\sqrt{2}}e^{-\sqrt{2}\pi\xi}\left(\sin(\sqrt{2}\pi\xi) - \cos(\sqrt{2}\pi\xi)\right) \end{cases} = \frac{1}{\sqrt{2}}e^{-\sqrt{2}\pi|\xi|}\left(\sin(\sqrt{2}\pi|\xi|) + \cos(\sqrt{2}\pi\xi)\right). \ \blacksquare$$

December 2019

Exercise 5. i) The plot is quite easy. Clearly, being $f_n \in \mathscr{C}([0,1])$ it is $f_n \in L^1([0,1]) \cap L^2([0,1])$ for every n.

ii) Notice that

$$f_n(x) \overset{n\to+\infty}{\longrightarrow} \frac{1}{\sqrt{x}} =: f(x),$$

and because $\frac{1}{\sqrt{x}} \in L^1([0,1])$ ($\int_0^1 \frac{1}{\sqrt{x}}\,dx = \left[2x^{1/2}\right]_{x=0}^{x=1} = 2$) but $\frac{1}{\sqrt{x}} \notin L^2([0,1])$ (because $\int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx = \int_0^1 \frac{1}{x}\,dx = +\infty$), this suggests the following guess:

$$f_n \overset{L^1}{\longrightarrow} f, \text{ while } (f_n) \text{ does not converge in } L^2([0,1]).$$

To check the first, we have

$$\|f_n - f\|_1 = \int_0^1 \left| \frac{1}{\sqrt{x+\frac{1}{n}}} - \frac{1}{\sqrt{x}} \right| dx = \int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+\frac{1}{n}}} \right) dx$$

$$= 2 - \left[2\left(x + \frac{1}{n}\right)^{1/2} \right]_{x=0}^{x=1} = 2 - 2\left(\sqrt{1+\frac{1}{n}} - \sqrt{\frac{1}{n}} \right) \longrightarrow 0.$$

To check the second we notice first that being $f \notin L^2$, the statement $f_n \xrightarrow{L^2} f$ does not make any sense. However, this does not exclude a priori that (f_n) may be convergent in L^2 to some other limit. So we need to justify more our ansatz. The key point is to notice that

$$\|f_n\|_2^2 = \int_0^1 \left(\frac{1}{\sqrt{x+\frac{1}{n}}} \right)^2 dx = \int_0^1 \frac{1}{x+\frac{1}{n}}\, dx = \left[\log\left(x + \frac{1}{n}\right) \right]_{x=0}^{x=1} = \log\left(1 + \frac{1}{n}\right) - \log\frac{1}{n} \longrightarrow +\infty,$$

that is $(\|f_n\|_2)$ is unbounded, therefore (f_n) cannot be convergent in L^2. ∎

Exercise 6. i) The characteristic property of $\Pi_U f$ is the unique element of U such that

$$\langle f - \Pi_U f, u \rangle = 0,\ \forall u \in U.$$

To check that $\Pi_U f(x) = \frac{1}{2}(f(x) + f(-x))$ we first notice that $\frac{1}{2}(f(x) + f(-x)) \in U$. Therefore, to be the orthogonal projection of f on U we have to checl that

$$\int_{\mathbb{R}} \left(f(x) - \frac{1}{2}(f(x) + f(-x)) \right) u(x)\, dx = 0,\ \forall u \in U.$$

We notice that

$$\int_{\mathbb{R}} \left(f(x) - \tfrac{1}{2}(f(x) + f(-x)) \right) u(x)\, dx = \frac{1}{2} \int_{\mathbb{R}} (f(x) - f(-x))\, u(x)\, dx$$

$$= \frac{1}{2} \left(\int_{\mathbb{R}} f(x)u(x)\, dx - \int_{\mathbb{R}} f(-x)u(x)\, dx \right).$$

Because

$$\int_{\mathbb{R}} f(-x)u(x)\, dx \overset{y=-x}{=} \int_{\mathbb{R}} f(y)u(-y)\, dy \overset{u\in U}{=} \int_{\mathbb{R}} f(y)u(y)\, dy \equiv \int_{\mathbb{R}} f(x)u(x)\, dx,$$

the conclusion follows.

ii) Let $(f_n) \subset U$ be such that $f_n \xrightarrow{L^2} f$. The goal is to prove $f \in U$ that is $f(-x) = f(x)$ a.e. x. We know $f_n \in U$, that is

$$f_n(-x) = f_n(x),\ \text{a.e. } x.$$

Now, because $f_n \xrightarrow{L^2} f$, there exists $(f_{n_k}) \subset (f_n)$ such that $f_{n_k} \longrightarrow f$ a.e. x. Thus,

$$f(-x) \longleftarrow f_{n_k}(-x) = f_{n_k}(x) \longrightarrow f(x), \text{ a.e. } x, \implies f(-x) = f(x), \text{ a.e. } x,$$

that is $f \in U$. ∎

Exercise 7. i) Let

$$g_{a,b}(\xi) := \frac{e^{-a|\xi|} - e^{-b|\xi|}}{\xi}, \ \xi \in \mathbb{R}\backslash\{0\}.$$

Here $a, b > 0$ are fixed. We notice that, because $e^t = 1 + t + o(t)$,

$$g_{a,b}(\xi) = \frac{1 - a|\xi| + o(\xi) - (1 - b|\xi| + o(\xi)}{\xi} = \frac{(b-a)|\xi| + o(\xi)}{\xi} \sim_0 (b-a)\mathrm{sgn}(\xi),$$

which is integrable at $\xi = 0$. At $\pm\infty$ we could say that

$$|g_{a,b}(\xi)| \leqslant \left(e^{-a|\xi|} + e^{-b|\xi|}\right), \ \forall |\xi| \geqslant 1,$$

thus $g_{a,b}$ is integrable at $\pm\infty$. In conclusion $g_{a,b} \in L^1(\mathbb{R})$. Similarly, $|g_{a,b}|^2 \sim_0 (b-a)^2$ is integrable at $\xi = 0$ while, as above,

$$|g_{a,b}(\xi)|^2 \leqslant \left(e^{-a|\xi|} + e^{-b|\xi|}\right)^2, \ \forall |\xi| \geqslant 1,$$

thus easily $|g_{a,b}|^2$ is integrable at $\pm\infty$. In conclusion, $g_{a,b} \in L^2(\mathbb{R})$.

ii) Because $g_{a,b} \in L^2(\mathbb{R})$, $g_{a,b}$ has a Fourier original in $L^2(\mathbb{R})$. The same does not apply for an L^1 original. Indeed, is $g_{a,b} = \widehat{f_{a,b}}$ for some $f_{a,b} \in L^1$ then $g_{a,b} \in \mathscr{C}(\mathbb{R})$. However, as in i), $g_{a,b}(\xi) \sim_0 (b-a)\mathrm{sgn}(\xi)$ which is not continuous at $\xi = 0$.

iii) If $f_{a,b} \in L^2$ is such that $\widehat{f_{a,b}} = g_{a,b}$ then, according to inversion formula,

$$\widehat{g_{a,b}}(x) = \widehat{\widehat{f_{a,b}}}(x) = f_{a,b}(-x),$$

that is $f_{a,b}(x) = \widehat{g_{a,b}}(-x)$. We compute then $\widehat{g_{a,b}}$. To this aim notice that

$$\widehat{\sharp g_{a,b}}(x) = \widehat{e^{-a|\sharp|} - e^{-b|\sharp|}}(x) = \frac{2a}{a^2 + 4\pi^2 x^2} - \frac{2b}{b^2 + 4\pi^2 x^2},$$

and because $\widehat{(i2\pi\sharp) g_{a,b}} = \partial_x \widehat{g_{a,b}}$ we deduce

$$\partial_x \widehat{g}_{a,b} = i\pi \left(\frac{2a}{a^2 + 4\pi^2 x^2} - \frac{2b}{b^2 + 4\pi^2 x^2} \right).$$

Thus,

$$\begin{aligned} \widehat{g_{a,b}}(x) &= i\pi \left(\tfrac{2}{a} \int \tfrac{1}{1 + (\frac{2\pi}{a} x)^2} \, dx - \tfrac{2}{b} \int \tfrac{1}{1 + (\frac{2\pi}{b} x)^2} \, dx \right) + c \\ &= i \left(\arctan\left(\tfrac{2\pi}{a} x\right) - \arctan\left(\tfrac{2\pi}{b} x\right) \right) + c, \end{aligned}$$

where c is a suitable constant. Finally, to determine the value of c, we may notice that letting $x \longrightarrow +\infty$,

$$\widehat{g_{a,b}}(x) \longrightarrow i\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + c = c,$$

and because we already know that $\widehat{g_{a,b}} \in L^2$, this can be possible only if $c = 0$. By this we finally obtain that the original of $g_{a,b}$ is

$$f_{a,b}(x) = \widehat{g_{a,b}}(-x) = i\left(\arctan\left(\frac{2\pi}{b}x\right) - \arctan\left(\frac{2\pi}{a}x\right)\right). \blacksquare$$

Exercise 8. i) Let $f(t,x) := e^{-\lambda x}\frac{\sin x}{x}$. Because $\sin x \sim_0 x$, we may consider f well defined and continuous at $x = 0$, thus $f(\lambda, \sharp)$ is integrable at $x = 0$ for every $\lambda \in \mathbb{R}$. At $x = +\infty$, because $|\sin x| \leqslant |x|$, we have

$$|f(\lambda,x)| \leqslant e^{-\lambda x} \in L^1([0,+\infty[),\ \forall\lambda > 0.$$

For $\lambda = 0$,

$$F(0) = \int_0^{+\infty} \frac{\sin x}{x}\, dx$$

exists (as generalized integral but not in L^1 sense). Thus, we may still consider F well defined at $\lambda = 0$.

ii) We wish to apply differentiation under integral, that is

$$\partial_\lambda F = \int_0^{+\infty} \partial_\lambda f(\lambda,x)\, dx.$$

To ensure this for every $\lambda \in \Lambda$ we need to check a) $f(\lambda,\sharp) \in L^1([0,+\infty[)$ for every $\lambda \in \Lambda$. This is true with $\Lambda =]0,+\infty[$. b) $\exists \partial_\lambda f(\lambda,x) = -xe^{-\lambda x}\frac{\sin x}{x} = -e^{-\lambda x}\sin x$, for every $\lambda \in]0,+\infty[$, a.e. $x \in [0,+\infty[$. c) there exists $g \in L^1([0,+\infty[)$ such that

$$|\partial_\lambda f(\lambda,x)| \leqslant g(x), \forall\lambda \in \Lambda,\ \text{a.e. } x \in [0,+\infty[.$$

Now,

$$|\partial_\lambda f(\lambda,x)| \leqslant e^{-\lambda x} \leqslant e^{-\lambda_0 x} \in L^1([0,+\infty[),\ \forall\lambda \in [\lambda_0,+\infty[.$$

Thus, on $\Lambda = [\lambda_0,+\infty[$ with $\lambda_0 > 0$, we can conclude

$$\partial_\lambda F(\lambda) = \int_0^{+\infty} -e^{-\lambda x}\sin x\, dx,\ \forall\lambda \geqslant \lambda_0,$$

and because λ_0 can be chosen arbitrarily > 0, we conclude the previous holds true for every $\lambda > 0$. Recalling that

$$\int e^{\alpha x}\sin(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2}e^{\alpha x}\left(\sin(\beta x) - \frac{\beta}{\alpha}\cos(\beta x)\right)$$

we have

$$\partial_\lambda F(\lambda) = \frac{\lambda}{\lambda^2+1}\left[e^{-\lambda x}\left(\sin x + \frac{1}{\lambda}\cos x\right)\right]_{x=0}^{x=+\infty} = -\frac{\lambda}{\lambda^2+1}\frac{1}{\lambda} = -\frac{1}{1+\lambda^2}.$$

iii) By last calculation,

$$F(\lambda) = -\arctan\lambda + c,$$

where c is a constant. The value of c can be determined letting $\lambda \longrightarrow +\infty$ and computing

$$\lim_{\lambda\to+\infty} F(\lambda) = \lim_{\lambda\to+\infty}\int_0^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx.$$

We can invert limit with integral applying the Dominated Convergence noticing that

- $\lim_{\lambda\to+\infty} f(\lambda,x) = 0$, for all $x > 0$;
- $|f(\lambda,x)| \leqslant e^{-\lambda x} \leqslant e^{-x}$ for every $\lambda \geqslant 1$, a.e. $x \in [0,+\infty[$.

Therefore

$$\lim_{\lambda\to+\infty} F(\lambda) = \lim_{\lambda\to+\infty}\int_0^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx = \int_0^{+\infty}\lim_{\lambda\to+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx = \int_0^{+\infty} 0\,dx = 0.$$

On the other hand,

$$\lim_{\lambda\to+\infty} F(\lambda) = \lim_{\lambda\to+\infty}(-\arctan\lambda + c) = -\frac{\pi}{2} + c,$$

thus $c = \frac{\pi}{2}$ and

$$F(\lambda) = \frac{\pi}{2} - \arctan\lambda.$$

iv) This part is much more delicate. Ideally, we would like to compute

$$\lim_{\lambda\to0+} F(\lambda) = \lim_{\lambda\to0+}\left(\frac{\pi}{2} - \arctan\lambda\right) = \frac{\pi}{2},$$

while, on the other side,

$$\lim_{\lambda\to0+} F(\lambda) = \lim_{\lambda\to0+}\int_0^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx \overset{?}{=} \int_0^{+\infty}\lim_{\lambda\to0+} e^{-\lambda x}\frac{\sin x}{x}\,dx = \int_0^{+\infty}\frac{\sin x}{x}\,dx.$$

If $\overset{?}{=}$ were correct we would have

$$\int_0^{+\infty}\frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Unfortunately, it is not easy ti justify $\overset{?}{=}$ this because, on one side, we cannot use monotone convergence (the function in the integral having variable sign), nor we can use Dominated Convergence (this because the unique possible bound on $[0,+\infty[$ independent by λ is

$$\left|e^{-\lambda x}\frac{\sin x}{x}\right| \leqslant \left|\frac{\sin x}{x}\right| \notin L^1([0,+\infty[).$$

To circumvent this difficulty we may proceed as follows: first we divide $\int_0^{+\infty} = \int_0^R + \int_R^{+\infty}$ where $R > 0$ will be fixed later. On $[0, R]$, $\frac{\sin x}{x} \in L^1([0, R])$, thus

$$\int_0^R e^{-\lambda x}\frac{\sin x}{x}\,dx \longrightarrow \int_0^R \frac{\sin x}{x}\,dx.$$

About the remaining integral on $[R, +\infty[$, suppose we are able to prove that

$$\left|\int_R^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx\right| \leqslant \varepsilon(R),\ \forall \lambda > 0,\ (\star)$$

where $\varepsilon(R)$ does not depend on λ and $\varepsilon(R) \longrightarrow 0$ as $R \longrightarrow +\infty$. Then we may write

$$F(\lambda) = \int_0^R e^{-\lambda x}\frac{\sin x}{x}\,dx + \int_R^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx,$$

that is

$$\left|\frac{\pi}{2} - \int_0^R \frac{\sin x}{x}\,dx\right| \overset{\lambda\to 0+}{\longleftarrow} \left|F(\lambda) - \int_0^R e^{-\lambda x}\frac{\sin x}{x}\,dx\right| \leqslant \varepsilon(R),$$

thus

$$\left|\frac{\pi}{2} - \int_0^R \frac{\sin x}{x}\,dx\right| \leqslant \varepsilon(R),$$

and letting $R \longrightarrow +\infty$ we finally would obtain

$$\left|\frac{\pi}{2} - \int_0^{+\infty} \frac{\sin x}{x}\,dx\right| \leqslant 0,$$

that is the conclusion. To prove $(\star)$ we first notice that

$$\int_R^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx = \int_R^{+\infty} \frac{1}{x}\left(e^{-\lambda x}\sin x\right)\,dx = \int_R^{+\infty} \frac{1}{x}\varphi'(x)\,dx$$

where

$$\varphi(x) = \int e^{-\lambda x}\sin x\,dx = -\frac{\lambda}{\lambda^2+1}e^{-\lambda x}\left(\sin x + \frac{1}{\lambda}\cos x\right).$$

Integrating by parts,

$$\int_R^{+\infty} \frac{1}{x}\varphi'(x)\,dx = \left[\frac{1}{x}\varphi(x)\right]_{x=R}^{x=+\infty} + \int_R^{+\infty} \frac{1}{x^2}\varphi(x)\,dx = \frac{\varphi(R)}{R^2} + \int_R^{+\infty} \frac{\varphi(x)}{x^2}\,dx,$$

being easily $\varphi(+\infty) = 0$ for all $\lambda > 0$. Now,

$$|\varphi(x)| \leqslant \frac{\lambda}{\lambda^2+1}\cdot 1 \cdot \left(1 + \frac{1}{\lambda}\cdot 1\right) = \frac{\lambda+1}{\lambda^2+1} \leqslant C,\ \forall \lambda > 0,$$

thus

$$\left|\int_R^{+\infty} e^{-\lambda x}\frac{\sin x}{x}\,dx\right| \leqslant \frac{C}{R^2} + \int_R^{+\infty} \frac{C}{x^2}\,dx = \frac{C}{R^2} + \frac{C}{R} \leqslant \frac{2C}{R},$$

assuming $R > 1$. Clearly, $\varepsilon(R) = \frac{2C}{R}$ fulfills the desired conditions. ■

Exercise 9. i) Let

$$f_n(x) := n \log\left(1 + \frac{1}{n\sqrt{x}}\right),\ x \in [0,1].$$

Clearly $f_n \in \mathscr{C}(]0,1])$, thus f_n is measurable. To check if $f_n \in L^1(]0,1])$ we have to show

$$\int_0^1 |f_n(x)|\ dx = n \int_0^1 \log\left(1 + \frac{1}{n\sqrt{x}}\right)\ dx < +\infty.$$

Because $\log(1+t) \leqslant t$ we have

$$\log\left(1 + \frac{1}{n\sqrt{x}}\right) \leqslant \frac{1}{n\sqrt{x}},$$

thus

$$\int_0^1 |f_n(x)|\ dx \leqslant n \int_0^1 \frac{1}{n\sqrt{x}}\ dx = \left[2\sqrt{x}\right]_{x=0}^{x=1} = 2.$$

ii) To discuss convergence of (f_n) we first notice that

$$f_n(x) = \log\left(1 + \frac{1}{n\sqrt{x}}\right)^n \longrightarrow \log e^{1/\sqrt{x}} = \frac{1}{\sqrt{x}} =: f(x),\ \forall x \in]0,1].$$

This f is a natural candidate to be a limit in L^1 for (f_n). As well known, pointwise convergence is not sufficient for L^1 convergence. We have to prove that

$$0 \longleftarrow \|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\ dx.$$

By previous remark we already know

$$|f_n(x) - f(x)| \longrightarrow 0,\ \forall x \in]0,1],$$

hence a.e. $x \in [0,1]$. By i) we have also $f_n(x) \leqslant f(x)$ a.e. x, and because $f_n \geqslant 0$, we have

$$|f_n(x) - f(x)| = f(x) - f_n(x) \leqslant f(x),\ \text{a.e. } x \in [0,1].$$

Finally, because $f \in L^1$ we may apply dominated convergence to conclude. ■

Exercise 10. i) Let

$$g(\xi) = (1 - \xi^2)1_{[-1,1]}(\xi),\ \xi \in \mathbb{R}.$$

Because $g \in L^2$ and FT is a bijection on L^2, g has an L^2 original f.

ii) According to inversion formula, if $g = \widehat{f}$ then $\widehat{g} = \widehat{\widehat{f}} = f(-\sharp)$, thus $f(x) = \widehat{g}(-x)$. In particular

$$\begin{aligned} f(x) &= \widehat{(1-\sharp^2)\mathrm{rect}_2}(-x) = \frac{\sin(-2\pi x)}{-\pi x} + \frac{1}{4\pi^2}\widehat{(-i2\pi\sharp)^2\mathrm{rect}_2}(-x) \\ &= \frac{\sin(2\pi x)}{\pi x} + \frac{1}{4\pi^2}\partial^2\widehat{\mathrm{rect}_2}(-x). \end{aligned}$$

Now, because $\widehat{\mathrm{rect}_2}(x) = 2\mathrm{sinc}\,(2\pi x)$ and

$$\begin{aligned} \partial_y^2 \mathrm{sinc}\; y = &= \partial_y^2 \frac{\sin y}{y} = \partial_y \frac{y\cos y - \sin y}{y^2} \\ &= \frac{(\cos y - y\sin y - \cos y)y^2 - (y\cos y - \sin y)2y}{y^4} \\ &= \frac{-y^2\sin y - 2y\cos y + 2\sin y}{y^3}, \end{aligned}$$

we have

$$\begin{aligned} \partial_x^2\widehat{\mathrm{rect}_2}(-x) &= 2\cdot 4\pi^2 \frac{-(-2\pi x)^2\sin(-2\pi x) + 4\pi x\cos(-2\pi x) + 2\sin(-2\pi x)}{(-2\pi x)^3} \\ &= -\frac{4\pi^2x^2\sin(2\pi x) + 4\pi x\cos(2\pi x) - 2\sin(2\pi x)}{\pi x^3}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \frac{\sin(2\pi x)}{\pi x} - \frac{4\pi^2x^2\sin(2\pi x) + 4\pi x\cos(2\pi x) - 2\sin(2\pi x)}{4\pi^3x^3} \\ &= \frac{4\pi^2x^2\sin(2\pi x) - 4\pi^2x^2\sin(2\pi x) - 4\pi x\cos(2\pi x) + 2\sin(2\pi x)}{4\pi^3x^3} \\ &= \frac{\sin(2\pi x) - 2\pi x\cos(2\pi x)}{2\pi^3x^3}. \end{aligned}$$

To conclude, we need to say if $f \in L^1$. Of course, being $|f| \leqslant 2\pi\frac{1+|x|}{|x|^3} \sim_{+\infty} \frac{8\pi}{|x|^2}$, f is integrable at $\pm\infty$. Because of singularity at $x = 0$ we have to check also integrability at $x = 0$. We may notice that

$$f(x) = \frac{2\pi x - \frac{(2\pi x)^3}{6} + o(x^3) - 2\pi x\left(1 - \frac{(2\pi x)^2}{2} + o(x^2)\right)}{2\pi^3x^3} = \frac{8\pi^3\frac{x^3}{3} + o(x^3)}{2\pi^3x^3} \sim_0 \frac{4}{3}$$

by which f is integrable also at $x = 0$. We conclude $f \in L^1$. ∎

Exercise 11. i) Because ϕ, ψ are unit vectors,

$$\Pi_U f = \langle f, \phi\rangle\phi, \quad \Pi_V f = \langle f, \psi\rangle\psi.$$

ii) Notice that $U+V=\{\alpha\phi+\beta\psi\ :\ \alpha,\beta\in\mathbb{R}\}$, thus $\{\phi,\psi\}$ is a basis for $U+V$. In general, this is not an orthonormal basis. However, according to Gram-Schmidt algorithm,

$$\phi,\ \frac{\psi-\langle\psi,\phi\rangle\phi}{\|\psi-\langle\psi,\phi\rangle\phi\|}$$

it is. Therefore

$$\Pi_{U+V}f = \langle f,\phi\rangle\phi+\langle f,\frac{\psi-\langle\psi,\phi\rangle\phi}{\|\psi-\langle\psi,\phi\rangle\phi\|}\rangle\frac{\psi-\langle\psi,\phi\rangle\phi}{\|\psi-\langle\psi,\phi\rangle\phi\|},$$

iii) By ii), denoting for brevity $\delta:=\|\psi-\langle\psi,\phi\rangle\phi\|>0$, we may write

$$\begin{aligned}\Pi_{U+V}f &= \langle f,\phi\rangle\phi+\tfrac{1}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle(\psi-\langle\psi,\phi\rangle\phi)\\ &= \left(\langle f,\phi\rangle+\tfrac{\langle\psi,\phi\rangle}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle\right)\phi+\tfrac{1}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle\psi,\end{aligned}$$

while

$$(\Pi_U+\Pi_V)f=\Pi_U f+\Pi_V f=\langle f,\phi\rangle\phi+\langle f,\psi\rangle\psi.$$

Now, being ϕ,ψ linearly independent, identity $\Pi_{U+V}=\Pi_U+\Pi_V$ holds true iff

$$\begin{cases}\langle f,\phi\rangle=\langle f,\phi\rangle+\frac{\langle\psi,\phi\rangle}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle,\\ \\ \langle f,\psi\rangle=\frac{1}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle\end{cases}$$

that is

$$\begin{cases}\frac{\langle\psi,\phi\rangle}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle=0,\\ \\ \langle f,\psi\rangle=\frac{1}{\delta^2}\langle f,\psi-\langle\psi,\phi\rangle\phi\rangle\end{cases}$$

First equation gives two alternatives: either $\langle\psi,\phi\rangle=0$ (and then second equation is automatically fulfilled as easily checked), or $\langle f,\psi-\langle\psi,\phi\rangle\psi\rangle=0$ (and then, by second equation, it follows $\langle f,\psi\rangle=0$; plugging this back into the first one, we get $-\frac{\langle\psi,\phi\rangle^2}{\delta^2}=0$, that is again $\langle\psi,\phi\rangle=0$). Thus, in any case, identity $\Pi_{U+V}=\Pi_U+\Pi_V$ holds true iff $\langle\phi,\psi\rangle=0$, that is iff ϕ and ψ are perpendicular. ■

Exercise 12. i) Let $f(x,\xi):=\frac{\sin(\xi x}{x(x^2+2x+2)}$. Clearly $f(\sharp,\xi)\in\mathscr{C}(\mathbb{R}\backslash\{0\})$ for every $\xi\in\mathbb{R}$. Because $\sin(\xi x)\sim_{x\to 0}\xi x$, we may extend by continuity $f(\sharp,\xi)$ also at $x=0$, thus we may consider $f(\sharp,\xi)\in\mathscr{C}(\mathbb{R})$ for any $\xi\in\mathbb{R}$. Therefore, in order $F(\xi)$ be well defined, we need just to discuss integrability of $f(\sharp,\xi)$ at $\pm\infty$. Easily,

$$|f(x,\xi)|\leqslant\frac{1}{|x|(x^2+2x+2)}\sim_{\pm\infty}\frac{1}{|x|^3},$$

which is integrable at $\pm\infty$. We conclude $f(\sharp,\xi)\in L^1(\mathbb{R})$ for any $\xi\in\mathbb{R}$, that is F is well defined for any $\xi\in\mathbb{R}$.

ii) To show differentiability, we use the differentiation thm. We have to check:

- $f(\sharp, \xi) \in L^1(\mathbb{R})$ for every $\xi \in \mathbb{R}$: already checked in i);
- $\exists \partial_\xi f(x,\xi) = \frac{x\cos(\xi x)}{x(x^2+2x+2)} = \frac{\cos(\xi x)}{x^2+2x+2}$, $\forall \xi \in \mathbb{R}$, a.e. $x \in \mathbb{R}$;
- there exists $g = g(x) \in L^1(\mathbb{R})$ such that $|\partial_\xi f(x,\xi)| \leqslant g(x)$, a.e. $x \in \mathbb{R}$: noticed that

$$|\partial_\xi f(x,\xi)| \leqslant \frac{1}{x^2+2x+2} =: g(x) \in L^1(\mathbb{R})$$

  conclusion follows.

We conclude that

$$\partial_\xi F(\xi) = \int_{\mathbb{R}} \frac{\cos(\xi x)}{x^2+2x+2}\, dx.$$

iii) By Euler formulas, $\cos(\xi x) = \frac{e^{i\xi x}+e^{-i\xi x}}{2}$, thus

$$\partial_\xi F = \frac{1}{2}\left(\int_{\mathbb{R}} \frac{1}{x^2+2x+2} e^{i\xi x}\, dx + \int_{\mathbb{R}} \frac{1}{x^2+2x+2} e^{-i\xi x}\, dx\right).$$

Denoting by $g(\xi)$ the FT of $\frac{1}{x^2+2x+2}$, the previous says

$$\partial_\xi F(\xi) = \frac{1}{2}\left(g\left(-\frac{\xi}{2\pi}\right) + g\left(\frac{\xi}{2\pi}\right)\right).$$

Let's compute g: notice that

$$g(\xi) = \widehat{\frac{1}{\sharp^2+2\sharp+2}}(\xi) = \widehat{\frac{1}{(\sharp+1)^2+1}}(\xi) = e^{i2\pi\xi}\widehat{\frac{1}{1+\sharp^2}}(\xi) = \pi e^{i2\pi\xi} e^{-2\pi|\xi|}.$$

Therefore

$$\partial_\xi F(\xi) = \frac{\pi}{2}\left(e^{-i\xi}e^{-|\xi|} + e^{i\xi}e^{-|\xi|}\right) = \pi e^{-|\xi|}\cos\xi.$$

To determine F we separate case $\xi > 0$ from $\xi < 0$:

$$F(\xi) = \pi \begin{cases} (\xi \geqslant 0) & \int e^{-\xi}\cos\xi\, d\xi + c_1, \\ \\ (\xi \leqslant 0) & \int e^{\xi}\cos\xi\, d\xi + c_2. \end{cases}$$

By $\int e^{\alpha x}\cos(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x}\left(\cos(\beta x) + \frac{\beta}{\alpha}\sin(\beta x)\right)$ we have

$$F(\xi) = \pi \begin{cases} (\xi \geqslant 0), & -\frac{1}{2}e^{-\xi}(\cos\xi - \sin\xi) + c_1, \\ \\ (\xi \leqslant 0), & \frac{1}{2}e^{\xi}(\cos\xi + \sin\xi) + c_2. \end{cases}$$

Finally, to determine c_1, c_2 just notice that $F(0) = 0$, thus $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$. ∎

Exercise 13. i) Let $f, g \in H$. Since $f, g \in \mathscr{C}^1([-1,1])$, $f', g' \in \mathscr{C}([-1,1])$ thus integral $\int_{-1}^1 f'g'$ is well defined. To show it is an inner product on H we have to check:

- positivity: $\langle f, f\rangle = \int_{-1}^1 f'(x)^2\,dx \geqslant 0$, trivial;
- vanishing: $\langle f, f\rangle = 0$ means $\int_{-1}^1 f'(x)^2\,dx = 0$. By a well known general fact, because $f' \in \mathscr{C}$ we have $f'(x)^2 \equiv 0$ on $[-1,1]$, that is f is constant, and because $f(0) = 0$ we deduce $f \equiv 0$.
- symmetry: $\langle f, g\rangle = \int_{-1}^1 f'g' = \int_{-1}^1 g'f' = \langle g', f'\rangle$, trivial;
- linearity: $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$, trivial.

ii) We apply the Gram-Schmidt orthogonalization. First $e_0 = \frac{x}{\|x\|}$. Warning: here $\|x\|$ is the norm induced by inner product defined in the text, thus

$$\|x\|^2 = \langle x, x\rangle = \int_{-1}^1 1\,dx = 2, \implies \|x\| = \sqrt{2},$$

thus $e_0\frac{x}{\sqrt{2}}$. About $e_1 = \frac{x^2 - \langle x, e_0\rangle e_0}{\|x^2 - \langle x, e_0\rangle e_0\|}$. Notice that

$$\langle x^2, e_0\rangle = \frac{1}{\sqrt{2}}\int_{-1}^1 2x \cdot 1\,dx = 0.$$

Therefore $\|x^2 - \langle x, e_0\rangle e_0\|^2 = \|x^2\|^2 = \int_{-1}^1 (2x)^2\,dx = 4\left[\frac{x^3}{3}\right]_{-1}^1 = \frac{8}{3}$ thus

$$e_1 = \sqrt{\frac{3}{8}}x^2.$$

Finally $e_2 = \frac{x^3 - \langle x^3, e_0\rangle e_0 - \langle x^3, e_1\rangle e_1}{\|x^3 - \langle x^3, e_0\rangle e_0 - \langle x^3, e_1\rangle e_1\|}$. We have

$$\langle x^3, e_0\rangle = \frac{1}{\sqrt{2}}\int_{-1}^1 3x^2 \cdot 1\,dx = \sqrt{2}, \quad \langle x^3, e_1\rangle = \sqrt{\frac{3}{8}}\int_{-1}^1 (3x^2)(2x)\,dx = 0.$$

Moreover

$$\|x^3 - \langle x^3, e_0\rangle e_0 - \langle x^3, e_1\rangle e_1\|^2 = \|x^3 - x\|^2 = \int_{-1}^1 (3x^2 - 1)^2\,dx = \frac{18}{5} - \frac{12}{3} + 2 = \frac{54 - 60 + 30}{15} = \frac{8}{5},$$

thus $e_2 = \sqrt{\frac{5}{8}}(x^3 - x)$.

iii) The best approximation of $\sin x$ on $\mathrm{Span}(x, x^2, x^3)$ is the orthogonal projection of $\sin x$, that is

$$\Pi \sin x = \sum_{j=0}^{2}\langle \sin x, e_j\rangle e_j.$$

We have

$$\begin{aligned}
\langle \sin x, e_0\rangle &= \tfrac{1}{\sqrt{2}}\textstyle\int_{-1}^{1} \cos x\, dx = \sqrt{2}\sin 1,\\
\langle \sin x, e_1\rangle &= \sqrt{\tfrac{3}{8}}\textstyle\int_{-1}^{1} \cos x(2x)\, dx = 0,\\
\langle \sin x, e_2\rangle &= \sqrt{\tfrac{5}{8}}\textstyle\int_{-1}^{1} \cos x(3x^2-1)\, dx = \sqrt{\tfrac{5}{8}}\left([3x^2\sin x]_{-1}^{1} - \int_{-1}^{1} 6x\sin x\, dx - 2\sin 1\right)\\
&= \sqrt{\tfrac{5}{8}}\left([6x\cos x]_{-1}^{1} - \int_{-1}^{1} 6\cos x\, dx + 4\sin 1\right)\\
&= \sqrt{\tfrac{5}{8}}\,(12\cos 1 - 8\sin 1)\,.
\end{aligned}$$

thus, in conclusion

$$\Pi \sin x = (\sin 1)x + \frac{5}{8}(12\cos 1 - 8\sin 1)(x^3 - x). \quad \blacksquare$$

Exercise 14. i) Clearly $\|f\|$ is well defined. Let's check the characteristic properties of a norm:

- positivity: $\|f\| \geqslant 0$, trivial.
- vanishing: $\|f\| = 0$ means $t^{1/2}|f'(t)| \equiv 0$ on $[0,1]$, thus in particular $f' \equiv 0$ on $]0,1]$ and because $f' \in \mathscr{C}$, $f' \equiv 0$ on $[0,1]$. In particular, f is constant and because $f(0) = 0$ ($f \in X$), we conclude $f \equiv 0$.
- homogeneity: $\|\lambda f\| = \max t^{1/2}|(\lambda f)'(t)| = \max t^{1/2}|\lambda||f'(t)| = |\lambda| \max t^{1/2}|f'(t)| = |\lambda|\|f\|$.
- triangular inequality: notice first that if $f, g \in X$ we have

  $$|(f+g)'(t)| = |f'(t) + g'(t)| \leqslant |f'(t)| + |g'(t)|,$$

  thus

  $$t^{1/2}|(f+g)'(t)| \leqslant t^{1/2}|f'(t)| + t^{1/2}|g'(t)| \leqslant \|f\| + \|g\|, \ \forall t \in [0,1],$$

  hence, taxing maximum, $\|f+g\| \leqslant \|f\| + \|g\|$.

ii) We have to prove that there exists a universal constant C such that $\|f\|_\infty \leqslant C\|f\|$ for every $f \in X$. We start recalling that

$$f(t) = f(0) + \int_0^t f'(s)\, ds = \int_0^t f'(s)\, ds,$$

therefore

$$|f(t)| = \left|\int_0^t f'(s)\,ds\right| \leqslant \int_0^t |f'(s)|\,ds = \int_0^t \frac{1}{s^{1/2}} s^{1/2}|f'(s)|\,ds \leqslant \int_0^t \frac{1}{s^{1/2}}\|f\|\,ds = 2t^{1/2}\|f\|,$$

thus, finally

$$\|f\|_\infty = \max_{t\in[0,1]} |f(t)| \leqslant 2\|f\|.$$

iii) Trivially $\|f_n\|_\infty = 1$. Since

$$f_n'(t) := \begin{cases} \frac{1}{4}t^{-3/4}, & t \in [\frac{1}{n}, 1], \\ \\ \frac{n^{3/4}}{4}, & t \in [0, \frac{1}{n}[. \end{cases} \implies t^{1/2}|f_n'(t)| := \begin{cases} \frac{1}{4}t^{-1/4}, & t \in [\frac{1}{n}, 1], \\ \\ \frac{n^{3/4}}{4}t^{1/2}, & t \in [0, \frac{1}{n}[. \end{cases}$$

thus

$$\|f_n\| = \max_{t\in[0,1]} t^{1/2}|f_n'(t)| = \frac{1}{4}n^{1/4} \longrightarrow +\infty,$$

hence it is impossible that there exists a constant c such that $\|f\| \leqslant c\|f\|_\infty$. In conclusion, $\|\cdot\|$ and $\|\cdot\|_\infty$ are not equivalent. ■

Exercise 15. If $f \in L^1(\mathbb{R})$ the FT of f is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}\,dx,\ \xi \in \mathbb{R}.$$

i) Informally

$$\partial_\xi \widehat{f}(\xi) = \partial_\xi \int_{\mathbb{R}} f(x)e^{-i2\pi\xi x}\,dx = \int_{\mathbb{R}} f(x)(-i2\pi x)e^{-i2\pi\xi x}\,dx = \widehat{-i2\pi\sharp f}(\xi).$$

Precise assumptions under which this derivation is correct are $f, \sharp f \in L^1$. Indeed, under these assumptions we may apply differentiation under integral sign. Calling

$$F(x,\xi) = f(x)e^{-i2\pi\xi x},\ \ \partial_\xi F = f(x)(-i2\pi x)e^{-i2\pi\xi x},$$

and because

$$|\partial_\xi F(x,\xi)| \leqslant 2\pi|xf(x)| \in L^1,\ \forall \xi \in \mathbb{R},$$

we deduce the conclusion.

ii) We know $\widehat{f}(\xi) = \frac{\xi}{1+\xi^4}$. First we should start with some "technicalities", namely: are we sure such f exists? Hopefully, $f(x) = \widehat{\widehat{f}}(-x)$ according to inversion thm. This holds true provided $f, \widehat{f} \in L^1$. The latter is obvious ($\widehat{f} \in \mathscr{C}(\mathbb{R})$ and $\widehat{f}(\xi) \sim_{\pm}\infty \frac{1}{\xi^3}$ which is integrable at $\pm\infty$); about the former, we recall that if $g, g', g'' \in L^1$ then $\widehat{g} \in L^1$. We apply this to $g = \widehat{f}$: clearly g, g', g'' are well defined and continuous, and

$$g'(\xi) = \partial_\xi \widehat{f}(\xi) = \frac{(1+\xi^4) - 4\xi^4}{(1+\xi^4)^2} = \frac{1-3\xi^4}{(1+\xi^4)^2} \sim_{\pm\infty} \frac{-3}{\xi^4},\ \text{integrable at } \pm\infty,$$

and, similarly,

$$g''(\xi) = \frac{-12\xi^2(1+\xi^4)^2 - (1-3\xi^4)2(1+\xi^4)4\xi^3}{(1+\xi^4)^4} \sim_{\pm\infty} \frac{24\xi^{11}}{\xi^{16}} = \frac{24}{\xi^5}, \text{ integrable at } \pm\infty$$

This ensures f exists and $f \in L^1$. Now, since

$$\int_{-\infty}^{\infty} x f(x)\, dx = \frac{1}{-i2\pi} \int_{\mathbb{R}} (-i2\pi x) f(x) e^{-i2\pi 0 \cdot x}\, dx = \frac{i}{2\pi} \widehat{-i2\pi \sharp f}(0) = \frac{i}{2\pi} \partial_\xi \widehat{f}(0) = \frac{i}{2\pi}.$$

About $f(0)$ we may use inversion formula:

$$f(0) = \widehat{\widehat{f}}(-0) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{i2\pi\xi\cdot 0}\, d\xi = \int_{\mathbb{R}} \frac{\xi}{1+\xi^4}\, d\xi = 0,$$

being $\widehat{f}$ odd. ∎