Analytical Methods — Exam Simulation

Exercise 1. Let $f_n(x) := \frac{1}{1+x^n}$, $x \in [0,+\infty[$, $n \in \mathbb{N}$, $n \geqslant 2$. Plot quickly the graph of f_n. Is $f_n \in L^1([0,+\infty[)$? Is (f_n) convergent (and, in the case, to what) in $L^1([0,+\infty[)$? Justify your answer.

Exercise 2. Let $H = L^2([0,\pi])$. Solve

$$\min_{a,b\in\mathbb{R}} \|x - (a\cos x + b\sin x)\|_2 .$$

Exercise 3. State precisely the differentiation under integral sign theorem. Let now

$$F(t) := \int_0^{+\infty} e^{-tx}\frac{1-\cos x}{x}\,dx.$$

i) Determine the domain of definition of F, that is the set of $t \in \mathbb{R}$ such that $F(t)$ is well defined.
ii) Is F continuous on its domain? Justify carefully your answer.
iii) Determine for which t is well defined $F'(t)$ and compute it.
iv) Determine F explicitly.

Exercise 4. The goal is to compute the FT of $f(x) = \frac{1}{1+x^4}$.

i) Does $\widehat{f}$ exists? If yes, which of the following statements are true/false and why: $\widehat{f} \in L^1(\mathbb{R})$; $\widehat{f} \in L^2(\mathbb{R})$; $\widehat{f} \in \mathscr{C}^1(\mathbb{R})$; $\widehat{f} \in \mathscr{S}(\mathbb{R})$.
ii) By reducing to suitable Cauchy distributions, compute FT of

$$\frac{1}{x^2 \pm \sqrt{2}x + 1}.$$

iii) Noticed that $1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$, express $\frac{1}{1+x^4}$ in terms of $\frac{1}{x^2\pm\sqrt{2}x+1}$. Use this to determine $\widehat{f}$.

Time: 2h.