Homework 1

Exercise 1. Compute
$$\lim_{n\to+\infty} n\int_0^{+\infty} \log\left(1+\frac{e^{-x}}{n}\right)\,dx.$$

Exercise 2. Let
$$F(x) = \int_0^{+\infty} \frac{1-e^{-xt^2}}{t^2}\,dt.$$

i) Determine the set of $x \in \mathbb{R}$ such that $F(x)$ is well defined (domain of F).
ii) Discuss carefully the derivability of F on its domain and compute F'. State the general theorems you need.
iii) Use the previous results to determine explicitly F.

Exercise 3. Let
$$f(x,y) := \frac{1}{1-xy}, \text{ a.e. } (x,y) \in [0,1]^2$$
Recall that $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$ for every $|q| < 1$. Use this to discuss if $f \in L^1([0,1]^2)$.

Solution

Exercise 1. Let

$$f_n(x) := \log\left(1+\frac{e^{-x}}{n}\right)^n.$$

We have to compute

$$\lim_n \int_0^{+\infty} f_n(x)\,dx.$$

To compute this we may use, in this case, either the monotone or the dominated convergence theorems. Let's see how:

- **monotone convergence.** Recall that $\left(1+\frac{t}{n}\right)^n \nearrow$ in n for every $t \geqslant 0$. Therefore, being $\log \nearrow$,

$$f_n(x) = \log\left(1+\frac{e^{-x}}{n}\right)^n \leqslant \log\left(1+\frac{e^{-x}}{n+1}\right)^{n+1} = f_{n+1}(x),\ \forall n \in \mathbb{N},\ n \geqslant 1,\ \forall x \in \mathbb{R}.$$

  MOreover, being $\left(1+\frac{e^{-x}}{n}\right)^n \geqslant 1$, $f_n(x) \geqslant 0$ for every n and $x \in [0,+\infty[$. We're then in conditions to apply monotone convergence to conclude that

$$\begin{aligned}\lim_n \int_0^{+\infty} f_n(x)\,dx &= \int_0^{+\infty} \lim_n f_n(x)\,dx = \int_0^{+\infty} \lim_n \log\left(1+\tfrac{e^{-x}}{n}\right)^n\,dx \\ &= \int_0^{+\infty} \log\left(e^{e^{-x}}\right)\,dx \\ &= \int_0^{+\infty} e^{-x}\,dx = 1.\end{aligned}$$

- **dominated convergence.** As above, we have

$$\lim_n f_n(x) = \log\left(e^{e^{-x}}\right) = e^{-x},\ \forall x \geqslant 0.$$

  We need then to find an integrable dominant $g \in L^1([0,+\infty[)$ such that

$$|f_n(x)| \leqslant g(x),\ \text{a.e. } x \in [0,+\infty[,\ \forall n \geqslant n_0,$$

  for some $n_0 \geqslant 1$. We recall the remarkable inequality

$$\log(1+t) \leqslant t,\ \forall t > -1.$$

  If you don't know this bound, let's see how how to deduce it. You may first notice that $y = \log(1+t)$ (as for $y = \log t$) is concave. Therefore, by a remarkable property of concavity, $y = \log(1+t)$ is below every of its tangents (sometimes this is considered as definition of concavity). Easily, the tangent at $t = 0$ is $y = t$, therefore $\log(1+t) \leqslant t$ for every $t > -1$. Applying this inequality,

$$n \log\left(1 + \frac{e^{-x}}{n}\right) \leqslant n \cdot \frac{e^{-x}}{n} = e^{-x} =: g(x),\ \forall x \geqslant 0,\ \forall n \geqslant 1.$$

Because $g \in L^1([0,+\infty[)$ we're done. The conclusion follows as above. ■

Exercise 2. i) Let $f(x,t) := \frac{1-e^{-xt^2}}{t^2}$. In order F be well posed we need to check for which $x \in \mathbb{R}$ we have $f(x,\sharp) \in L^1([0,+\infty[)$. Notice that $f(0,t) \equiv 0 \in L^1([0,+\infty[)$. For $x \neq 0$, $f(x,t)$ is well defined for $t \neq 0$, $f(x,\sharp) \in \mathscr{C}(]0,+\infty[)$ hence $f(x,\sharp) \in L([0,+\infty[)$. Because $f(x,\sharp)$ is continuous we invoke the following fact:

$$\int_0^{+\infty} |f(x,t)|\, dt < +\infty \text{ in generalized sense} \implies f(x,\sharp) \in L^1([0,+\infty[).$$

To check this we need to discuss the behavior of f as $t \longrightarrow 0, +\infty$ (because $f \in \mathscr{C}(]0,+\infty[)$). Recalling that $e^s = 1 + s + o(s)$ we have that, as $t \longrightarrow 0$,

$$f(x,t) = \frac{1-(1-xt^2+o(t^2))}{t^2} = x + \frac{o(t^2)}{t^2} \longrightarrow x \in \mathbb{R},$$

so $f(x,\sharp)$ is absolutely integrable in $t = 0$, for every $x \in \mathbb{R}$. As $t \longrightarrow +\infty$ we may notice that if $x < 0$,

$$f(x,t) \sim \frac{-e^{-xt^2}}{t^2} \longrightarrow +\infty,$$

whence $f(x,\sharp) \notin L^1([0,+\infty[)$ for $x < 0$. If $x > 0$, being $0 \leqslant e^{-xt^2} < 1$, easily

$$|f(x,t)| = f(x,t) \leqslant \frac{1}{t^2},$$

which is integrable at $+\infty$. Conclusion: $f(x,\sharp) \in L^1([0,+\infty[)$ iff $x \geqslant 0$, thus the domain of F is $[0,+\infty[$.

ii) If derivation under integral sign applies at some $x \geqslant 0$,

$$F'(x) = \int_0^{+\infty} \partial_x \frac{1-e^{-xt^2}}{t^2}\, dt = \int_0^{+\infty} \frac{1}{t^2}(t^2 e^{-xt^2})\, dt = \int_0^{+\infty} e^{-xt^2}\, dt \overset{y=\sqrt{2x}t}{=} \frac{1}{\sqrt{2x}} \int_0^{+\infty} e^{-\frac{y^2}{2}}\, dy$$

$$= \frac{1}{\sqrt{2x}} \frac{1}{2} \sqrt{2\pi} = \sqrt{\frac{\pi}{2}} x^{-1/2}.$$

This formula would make sense for $x > 0$. Now, to check that derivation under integral applies for $x > 0$, we need to check the conditions:

- $f(x,\sharp) \in L^1([0,+\infty[)\ \forall x > 0$ (already done);
- $\partial_x f(x,t)$ exists for every $x > 0$ and a.e. $t \in]0,+\infty[$: yes, $\partial_x f(x,t) = e^{-xt^2}$;

- $\exists g \in L^1([0,+\infty[)$ such that $|\partial_x f(x,t)| \leqslant g(t)$, for every $x > 0$, a.e. $t \in [0,+\infty[$. About this, we understand that if $x > 0$ the unique possible bound valid for every $x > 0$ is $g(t) \equiv 1$, which is not in L^1. However, if we restrict to $x \geqslant r$ for $r > 0$ fixed, we have immediately

$$|\partial_x f(x,t)| = e^{-xt^2} \leqslant e^{-rt^2} =: g_r(t),\ \forall x \geqslant r,\ a.e.\ t \in [0,+\infty[.$$

  Therefore, assumptions are fulfilled on $[r,+\infty[$ and the previous derivation holds true for every $x \geqslant r$. Now, because $r > 0$ is arbitrary we can easily conclude that actually $F'(x)$ exists for every $x > 0$.

iii) By ii) it follows that

$$F(x) = -\sqrt{2\pi}x^{1/2} + C,\ \forall x > 0,$$

where C is an arbitrary constant. Now, how can we determine C? Formally, we cannot set $x = 0$ because this is not allowed by the previous formula. However, by the same

$$\lim_{x\to 0+} F(x) = C.$$

We can compute the limit directly on F by mean of some limit theorem. The question is: can we say that

$$\lim_{x\to 0+} F(x) = \lim_{x\to 0+} \int_0^{+\infty} f(x,t)\,dt = \int_0^{+\infty} \lim_{x\to 0+} f(x,t)\,dt.$$

It this is true, because $\lim_{x\to 0+} f(x,t) = f(0,t) = 0$ we would conclude $\lim_{x\to 0+} F(x) = 0$, that is $C = 0$. Now, in the present case we can justify the passage of the limit into the integral in two ways. Let's see both for convenience (just one is enough for formal justification).

- **By monotone convergence:** we profit of the fact that $f(x,t)$ is *increasing in x*. In fact, $\partial_x f(x,t) = G^{-xt^2} \geqslant 0$, hence when $x \searrow 0$, $f(x,t) \searrow 0$. So, if (x_n) is a sequence decreasing to 0, setting $f_n(t) := f(x_n,t)$, we have a decreasing sequence of functions. Because $f_0 = f(x_0,\sharp) \in L^1([0,+\infty[)$ we can apply monotone convergence with decreasing sequences to obtain that

$$\lim_n F(x_n) = \lim_n \int_0^{+\infty} f(x_n,t)\,dt = \lim_n \int_0^{+\infty} f_n(t)\,dt = \int_0^{+\infty} \lim_n f_n(t)\,dt = 0.$$

  Being the sequence (x_n) arbitrary, it follows that $\lim_{x\to 0+} F(x) = 0$.
- **By dominated convergence:** here we need a bound like $|f(x,t)| \leqslant g(t)$ with $g \in L^1([0,+\infty[)$ for all $x \in [0,1]$ for example. Also in this case we may notice that because e^{-xt^2} decreases with x, $0 \leqslant e^{-xt^2} \leqslant e^{-t^2}$ for all $x \in [0,1]$, therefore

$$|f(x,t)| \leqslant \frac{1-e^{-t^2}}{t^2},\ \forall x \in [0,1],\ a.e. t \geqslant 0.$$

We can now apply the continuity theorem to deduce that F is continuous on $[0,1]$, whence in particular

$$\lim_{x\to 0+} F(x) = F(0) = 0. \quad \blacksquare$$

Exercise 3. Notice first that f is well defined and continuous on $[0,1[^2$, hence it is $L([0,1]^2)$. Furthermore, being $\geqslant 0$, the integral of f is well defined (eventually $= +\infty$). By the hint,

$$f(x,y) = \sum_{n=0}^{\infty} (xy)^n,$$

thus

$$\int_{[0,1]^2} f(x,y)\, dxdy = \int_{[0,1]^2} \sum_{n=0}^{\infty} (xy)^n\, dxdy.$$

Applying monotone convergence for series (we recall: if $(f_n) \subset L(D)$, $f_n \geqslant 0$ a.e. on D, then $\int_D \sum_n f_n = \sum_n \int_D f_n$) we have

$$\begin{aligned} \int_{[0,1]^2} f &= \sum_{n=0}^{\infty} \int_{[0,1]^2} (xy)^n\, dxdy = \sum_{n=0}^{\infty} \int_0^1 x^n \left(\int_0^1 y^n\, dy \right) dx = \sum_{n=0}^{\infty} \left(\int_0^1 x^n\, dx \right)^2 \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < +\infty. \end{aligned}$$

We conclude $f \in L^1([0,1]^2)$. $\blacksquare$