Analytical Methods — Homework 2

Exercise 1. Let $(f_n) \subset L([0,1])$. Which of the following statements hold true?

i) If $f_n \xrightarrow{L^2} 0$ then $f_n \xrightarrow{L^1} 0$.

ii) If $f_n \xrightarrow{L^\infty} 0$ then $f_n \xrightarrow{L^2} 0$.

iii) If $f_n \xrightarrow{L^1} 0$ then $f_n \xrightarrow{L^2} 0$.

For each true statement provide a proof, otherwise exhibit a counterexample.

Exercise 2. Let $X := \mathscr{C}([-1,1])$ and $Y := \mathscr{C}^1([-1,1])$ (that is functions f continuous on $[-1,1]$ with f' continuous on $[-1,1]$), both endowed with uniform norm $\|f\|_\infty := \max_{[-1,1]} |f|$. Consider the sequence $f_n(x) := \sqrt{x^2 + \frac{1}{n}}$, $x \in [-1,1]$, $n \in \mathbb{N}$, $n \geqslant 1$.

i) Is $f_n \xrightarrow{X} f$ for some $f \in X$?

ii) Is $f_n \xrightarrow{Y} g$ for some $g \in Y$?

Exercise 3. Let $X := \left\{ f \in \mathscr{C}^1([0,1]) \ : \ f(0) = 0 \right\}$. On X we define

$$\|f\|_1 = \int_0^1 |f(x)|\, dx, \quad \|f\|_* := \int_0^1 |f'(x)|\, dx.$$

i) It is well known that $\|\cdot\|_1$ is a norm on $\mathscr{C}([0,1])$. Is this true also on X? Justify your answer.

ii) Show that $\|\cdot\|_*$ is a norm on X.

iii) Prove that $\|\cdot\|_*$ is stronger than $\|\cdot\|_1$, that is

$$\exists C > 0, \ : \ \|f\|_1 \leqslant C\|f\|_*, \ \forall f \in X.$$

iv) Discuss if $\|\cdot\|_*$ and $\|\cdot\|_1$ are equivalent. Hint: consider $f_n(x) = x^n$, $n \in \mathbb{N}$.

Solution due by Monday 4th of November.