Analytical Methods — Homework 2

Exercise 1. Let $(f_n) \subset L([0,1])$. Which of the following statements hold true?

i) If $f_n \xrightarrow{L^2} 0$ then $f_n \xrightarrow{L^1} 0$.
ii) If $f_n \xrightarrow{L^\infty} 0$ then $f_n \xrightarrow{L^2} 0$.
iii) If $f_n \xrightarrow{L^1} 0$ then $f_n \xrightarrow{L^2} 0$.

For each true statement provide a proof, otherwise exhibit a counterexample.

Exercise 2. Let $X := \mathscr{C}([-1,1])$ and $Y := \mathscr{C}^1([-1,1])$ (that is functions f continuous on $[-1,1]$ with f' continuous on $[-1,1]$), both endowed with uniform norm $\|f\|_\infty := \max_{[-1,1]} |f|$. Consider the sequence $f_n(x) := \sqrt{x^2 + \frac{1}{n}}$, $x \in [-1,1]$, $n \in \mathbb{N}$, $n \geqslant 1$.

i) Is $f_n \xrightarrow{X} f$ for some $f \in X$?
ii) Is $f_n \xrightarrow{Y} g$ for some $g \in Y$?

Exercise 3. Let $X := \left\{ f \in \mathscr{C}^1([0,1]) \ : \ f(0) = 0 \right\}$. On X we define

$$\|f\|_1 = \int_0^1 |f(x)|\, dx, \quad \|f\|_* := \int_0^1 |f'(x)|\, dx.$$

i) It is well known that $\|\cdot\|_1$ is a norm on $\mathscr{C}([0,1])$. Is this true also on X? Justify your answer.
ii) Show that $\|\cdot\|_*$ is a norm on X.
iii) Prove that $\|\cdot\|_*$ is stronger than $\|\cdot\|_1$, that is

$$\exists C > 0, \ : \ \|f\|_1 \leqslant C\|f\|_*, \ \forall f \in X.$$

iv) Discuss if $\|\cdot\|_*$ and $\|\cdot\|_1$ are equivalent. Hint: consider $f_n(x) = x^n$, $n \in \mathbb{N}$.

Solution due by Monday 4th of November.

SOLUTION.

Exercise 1. i) True: because of the Cauchy–Schwarz inequality,

$$\|f\|_{L^1} = \int_0^1 |f| \leqslant \left(\int_0^1 1^2\, dx\right)^{1/2} \left(\int_0^1 |f|^2\, dx\right)^{1/2} = \|f\|_{L^2},$$

by which we deduce that $\|\cdot\|_{L^2}$ is stronger than $\|\cdot\|_{L^1}$. It's a standard fact that by this it follows that if (f_n) converges in the L^2 norm it converges in L^1 norm as well.
ii) True: just notice that $|f(x)| \leqslant \|f\|_{L^\infty}$ a.e., then

$$\|f\|_{L^1} = \int_0^1 |f|\, dx \leqslant \int_0^1 \|f\|_{L^\infty}\, dx = \|f\|_{L^\infty},$$

by which, again, the conclusion follows being $\|\cdot\|_{L^\infty}$ stronger than $\|\cdot\|_{L^1}$.
iii) False: take $f_n = n^{2/3}\chi_{[0,\frac{1}{n}]}$. Then

$$\|f_n\|_{L^1} = \int_0^1 |f_n|\, dx = n^{2/3}\int_0^{1/n} dx = \frac{1}{n^{1/3}} \longrightarrow 0, \text{ but } \|f_n\|_{L^2}^2 = n^{4/3}\int_0^{1/n} dx = n^{1/3} \longrightarrow +\infty.$$

In particular, (f_n) cannot be convergent in L^2 otherwise it should be bounded. ■

Exercise 2. i) We claim $f_n \xrightarrow{X} f$ where $f(x) = |x|$. Indeed:

$$f_n(x) - f(x) = \sqrt{x^2 + \frac{1}{n}} - |x| = \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} = \frac{1/n}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}},$$

hence

$$\|f_n - f\|_\infty = \max_{[-1,1]} \frac{1/n}{\sqrt{x^2 + \frac{1}{n}} + \sqrt{x^2}} \leqslant \sqrt{\frac{1}{n}} \longrightarrow 0.$$

ii) However, (f_n) cannot have any limit in Y. Indeed: if $f_n \xrightarrow{Y} g \in Y$, in particular, $f_n \xrightarrow{X} g$. But $f_n \xrightarrow{X} f = |x|$ and because the limit is unique, $g \equiv |x|$. But such $g \notin Y = \mathscr{C}^1([-1,1])$, leading to a contradiction. ■

Exercise 3. i) Yes, X is just a subspace of $\mathscr{C}([0,1])$, properties of $\|\cdot\|_1$ remain unchanged.
ii) To check that $\|\cdot\|_*$ is a norm, we first notice that it is well defined: indeed, because $f \in X$, $f \in \mathscr{C}^1$, thus $f' \in \mathscr{C}([0,1])$ it is integrable. To complete the check, we have to prove the fundamental properties of a norm:

- vanishing: $\|f\|_* = 0$ iff $\int_0^1 |f'| = 0$. Because $f' \in \mathscr{C}$, by a well known result this happens iff $|f'| \equiv 0$, that is $f' \equiv 0$, again iff f is constant. Because $f \in X$ in particular $f(0) = 0$, thus $f \equiv 0$.
- homogeneity: trivial.
- triangular inequality: straightforward,

$$\|f+g\|_* = \int_0^1 |f'+g'| \leqslant \int_0^1 (|f'| + |g'|) = \|f\|_* + \|g\|_*.$$

iii) We have to bound $\int_0^1 |f|$ by $C \int_0^1 |f'|$. Notice that, according to the fundamental thm of integral calculus,

$$f(x) = f(0) + \int_0^x f'(y)\,dy = \int_0^x f'(y)\,dy,$$

thus

$$|f(x)| \leqslant \left| \int_0^x f'(y)\,dy \right| \leqslant \int_0^x |f'(y)|\,dy \leqslant \int_0^1 |f'| = \|f\|_*.$$

Therefore

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \leqslant \int_0^1 \|f\|_*\,dx = \|f\|_*.$$

iv) Let $f_n(x) = x^n$. Clearly $f_n \in X$ for every $n \in \mathbb{N}$. Notice that

$$\|f_n\|_1 = \int_0^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_{x=0}^{x=1} = \frac{1}{n+1},$$

while

$$\|f_n\|_* = \int_0^1 nx^{n-1}\,dx = [x^n]_{x=0}^{x=1} = 1.$$

Therefore, if $\|\cdot\|_*$ and $\|\cdot\|_1$ were equivalent, being already checked that $\|\cdot\|_*$ is stronger than $\|\cdot\|_1$, we would have

$$\|f\|_* \leqslant C\|f\|_1,\ \forall f \in X,$$

and a suitable universal constant C. However, letting $f = f_n$ we have

$$1 \leqslant C\frac{1}{n+1},\ \forall n \in \mathbb{N},$$

which is impossible. ∎