Exercise 1. Let

$$X := \left\{ f \in \mathscr{C}([0,1]) \ : \ \|f\|_* := \sup_{t\in]0,1]} \frac{|f(t)|}{t} < +\infty \right\}.$$

i) Check that $\|\cdot\|_*$ is a well defined norm on X.

ii) Let f_n be defined as

$$f_n(t) := \begin{cases} nt, & 0 \leqslant t \leqslant \frac{1}{n^2}, \\ \sqrt{t}, & \frac{1}{n^2} \leqslant t \leqslant 1. \end{cases}$$

Is $(f_n) \subset X$? If yes, is (f_n) convergent to some $f \in X$ in the $\|\cdot\|_*$ norm?

iii) On X is also defined the $\|\cdot\|_\infty$ norm. Show that $\|\cdot\|_*$ is stronger than $\|\cdot\|_\infty$. Are the two also equivalent? (prove or disprove)

iv) Discuss if X is a Banach space under $\|\cdot\|_*$.

Exercise 2. Let $H = L^2([-1,1])$ endowed with usual scalar product $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\,dx$.

i) Let U be the subspace of H generated by functions x, x^2, x^4. Determine an orthonormal base for U.

ii) Determine the best approximation of 1 in U.

Exercise 3. Let

$$H := \left\{ f : [0,+\infty[\longrightarrow \mathbb{R} \ : \ f \text{ Leb. meas.}, \int_0^{+\infty} f(x)^2 e^{-x}\,dx < +\infty \right\}.$$

On H we define

$$\langle f, g\rangle := \int_0^{+\infty} f(x)g(x)e^{-x}\,dx.$$

i) Check that $\langle \cdot, \cdots, \rangle$ is a well defined scalar product with vanishing in the sense that $\langle f, f\rangle = 0$ iff $f = 0$ a.e.

We accept H is Hilbert. Let $U := \{g \in H \ : \ \int_0^{+\infty} g(x)e^{-x}\,dx = 0\}$.

ii) Is U closed? Justify your answer.

iii) Determine the orthogonal projection on U of $f(x) = e^{-2x}$.