Analytical Methods — Homework 3

Exercise 1. Let

$$X := \left\{ f \in \mathscr{C}([0,1]) \ : \ \|f\|_* := \sup_{t\in]0,1]} \frac{|f(t)|}{t} < +\infty \right\}.$$

i) Check that $\|\cdot\|_*$ is a well defined norm on X.

ii) Let f_n be defined as

$$f_n(t) := \begin{cases} nt, & 0 \leqslant t \leqslant \frac{1}{n^2}, \\ \sqrt{t}, & \frac{1}{n^2} \leqslant t \leqslant 1. \end{cases}$$

Is $(f_n) \subset X$? If yes, is (f_n) convergent to some $f \in X$ in the $\|\cdot\|_*$ norm?

iii) On X is also defined the $\|\cdot\|_\infty$ norm. Show that $\|\cdot\|_*$ is stronger than $\|\cdot\|_\infty$. Are the two also equivalent? (prove or disprove)

iv) Discuss if X is a Banach space under $\|\cdot\|_*$.

Exercise 2. Let $H = L^2([-1,1])$ endowed with usual scalar product $\langle f,g\rangle = \int_{-1}^1 f(x)g(x)\,dx$.

i) Let U be the subspace of H generated by functions x, x^2, x^4. Determine an orthonormal base for U.

ii) Determine the best approximation of 1 in U.

Exercise 3. Let

$$H := \left\{ f : [0,+\infty[\longrightarrow \mathbb{R} \ : \ f \text{ Leb. meas.}, \int_0^{+\infty} f(x)^2 e^{-x}\,dx < +\infty \right\}.$$

On H we define

$$\langle f,g\rangle := \int_0^{+\infty} f(x)g(x)e^{-x}\,dx.$$

i) Check that $\langle \cdot, \cdots, \rangle$ is a well defined scalar product with vanishing in the sense that $\langle f,f\rangle = 0$ iff $f = 0$ a.e.

We accept H is Hilbert. Let $U := \{g \in H \ : \ \int_0^{+\infty} g(x)e^{-x}\,dx = 0\}$.

ii) Is U closed? Justify your answer.

iii) Determine the orthogonal projection on U of $f(x) = e^{-2x}$.

Solution

Exercise 1. i) Clearly, $\|f\|_*$ is well defined for every $f \in X$ and $\|f\|_* \geqslant 0$. Let's check the characteristic property of a norm:

- vanishing: $\|f\|_* = 0$ iff $\sup_{t\in]0,1]} \frac{|f(t)|}{t} = 0$, that is $|\frac{f(t)|}{t} \equiv 0$ on $]0,1]$ thus, in particular, $f \equiv 0$ on $]0,1]$. Being f continuous, this implies also $f(0) = 0$, thus $f \equiv 0$ on $[0,1]$.
- homogeneity:
$$\|\lambda f\|_* = \sup_{]0,1]} \frac{|\lambda f(t)|}{t} = \sup_{]0,1]} |\lambda| \frac{|f(t)|}{t} = |\lambda| \sup_{]0,1]} \frac{|f(t)|}{t} = |\lambda| \|f\|_*.$$
- triangular inequality: first notice that
$$\frac{\|f(t) + g(t)|}{t} \leqslant \frac{|f(t)|}{t} + \frac{|g(t)|}{t} \leqslant \|f\|_* + \|g\|_*,\ \forall t \in]0,1],$$
therefore
$$\|f+g\|_* = \sup_{]0,1]} \frac{|f(t)+g(t)|}{t} \leqslant \|f\|_* + \|g\|_*.$$

ii) Clarly, each $f_n \in \mathscr{C}([0,1])$. Furthermore,
$$\|f_n\|_* = \sup]0,1] \frac{|f_n(t)|}{t} = \sup_{]0,1]} g_n(t),$$
where
$$g_n(t) = \begin{cases} n, & 0 \leqslant t \leqslant \frac{1}{n^2}, \\ \\ \frac{1}{\sqrt{t}}, & \frac{1}{n^2} \leqslant t \leqslant 1. \end{cases}$$
Clearly, $\sup_{]0,1]} g_n(t) = n$, thus $\|f_n\|_* = n < +\infty$, that is $f_n \in X$ for every $n \in \mathbb{N}$. Because (f_n) is unbounded in $\|\cdot\|_*$ norm, it is not convergent in X.
iii) We've to show that
$$\exists C > 0,\ :\ \|f\|_\infty \leqslant C\|f\|_*,\ \forall f \in X.$$
The bound is quite easy: because $t \in]0,1]$ we just notice that
$$|f(t)| \leqslant \frac{|f(t)|}{t} \leqslant \|f\|_*,\ \forall t \in]0,1],$$
and because f is continuous this bound holds true also at $t = 0$. Thus
$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| \leqslant \|f\|_*.$$
We guess the two norms are not equivalent because it seems impossible to bound uniformly
$$\frac{\|f(t)\|}{t} \leqslant C\|f\|_\infty,$$

where C is an universal constant. Take the example at ii): we already checked (f_n) is not convergent under $\|\cdot\|_*$. However, as it is easy to check,

$$\|f_n\|_\infty = \sup_{t\in[0,1]} |f_n(t)| = f_n\left(\frac{1}{n^2}\right) = \frac{1}{n} \longrightarrow 0,$$

thus $f_n \overset{\|\cdot\|_\infty}{\longrightarrow} 0$. Were $\|\cdot\|_\infty$ stronger than $\|\cdot\|_*$, (f_n) should converge to 0 also according to $\|\cdot\|_*$, which is false.
iv) Let $(f_n) \subset X$ be a Cauchy sequence under $\|\cdot\|_*$, this meaning

$$\forall \varepsilon > 0,\ \exists N\ :\ \|f_n - f_m\|_* \leqslant \varepsilon,\ \forall n, m \geqslant N.$$

In particular, (f_n) is Cauchy also under $\|\cdot\|_\infty$ norm and because $\mathscr{C}([0,1])$ is a Banach space respect to this norm, (f_n) converges to some f in $\|\cdot\|_\infty$. In particular, $f_n(t) \longrightarrow f(t)$ for every $t \in [0,1]$. Returning to the Cauchy property in the $\|\cdot\|_*$ norm, because

$$\frac{|f_n(t) - f_m(t)|}{t} \leqslant \varepsilon,\ \forall t \in]0,1],\ \forall n, m \geqslant N,$$

letting $m \longrightarrow +\infty$,

$$\frac{|f_n(t) - f(t)|}{t} \leqslant \varepsilon,\ \forall t \in]0,1],\ \forall n \geqslant N,$$

that is

$$\|f_n - f\|_* \leqslant \varepsilon,\ \forall n \geqslant N,\ \Longleftrightarrow\ f_n \overset{\|\cdot\|_*}{\longrightarrow} f.$$

This shows that X is Banach. ■

Exercise 2. i) To compute an orthonormal base for $\mathrm{Span}\langle x, x^2, x^4\rangle$ we use the Gram–Schmidt algorithm. Set

$$e_0 = \frac{x}{\|x\|},$$

where

$$\|x\|^2 = \int_{-1}^{1} x^2\, dx = 2\left[\frac{x^2}{2}\right]_{x=0}^{x=1} = 1,$$

then $e_0 = x$. Next,

$$e_1 = \frac{x^2 - \langle x^2, e_0\rangle e_0}{\|x^2 - \langle x^2, e_0\rangle e_0\|}.$$

Because $\langle x^2, e_0\rangle = \int_{-1}^{1} x^2 x\, dx = 0$, hence

$$\|x^2 - \langle x^2, e_0\rangle e_0\|^2 = \|x^2\|^2 = \int_{-1}^{1} x^4\, dx = 2\left[\frac{x^5}{5}\right]_{x=0}^{x=1} = \frac{2}{5},$$

we have

$$e_1 = \sqrt{\frac{5}{2}} x^2.$$

Finally, let

$$e_2 = \frac{x^4 - \left(\langle x^4, e_0\rangle e_0 + \langle x^4, e_1\rangle e_1\right)}{\|x^4 - \left(\langle x^4, e_0\rangle e_0 + \langle x^4, e_1\rangle e_1\right)\|}.$$

We have

$$\langle x^4, e_0\rangle = \int_{-1}^{1} x^4 x \, dx = 0, \;\; \langle x^4, e_1\rangle = \sqrt{\frac{5}{2}} \int_{-1}^{1} x^4 x^2 \, dx = \sqrt{\frac{5}{2}} 2 \left[\frac{x^7}{7}\right]_{x=0}^{x=1} = \frac{\sqrt{10}}{7}.$$

Then

$$\begin{aligned} \|x^4 - \left(\langle x^4, e_0\rangle e_0 + \langle x^4, e_1\rangle e_1\right)\|^2 &= \left\|x^4 - \tfrac{5}{7}x^2\right\|^2 = \int_{-1}^{1} \left(x^4 - \tfrac{5}{7}x^2\right)^2 \, dx \\ &= 2\left[\tfrac{x^9}{9}\right]_{x=0}^{x=1} - \tfrac{20}{7}\left[\tfrac{x^7}{7}\right]_{x=0}^{x=1} + \tfrac{50}{49}\left[\tfrac{x^5}{5}\right]_{x=0}^{x=1} \\ &= \tfrac{2}{9} - \tfrac{20}{49} + \tfrac{25}{49} = \tfrac{143}{441} \end{aligned}$$

and

$$e_2 = \sqrt{\frac{441}{143}} \left(x^4 - \frac{5}{7}x^2\right).$$

ii) The best approximation of 1 in U is its orthogonal projection, namely,

$$\Pi_U 1 = \langle 1, e_0\rangle e_0 + \langle 1, e_1\rangle e_1 + \langle 1, e_2\rangle e_2,$$

and because

$$\begin{aligned} \langle 1, e_0\rangle &= \textstyle\int_{-1}^{1} 1 \cdot x \, dx = 0, \\ \langle 1, e_1\rangle &= \textstyle\int_{-1}^{1} 1 \cdot \sqrt{\tfrac{5}{2}} x^2 \, dx = \sqrt{\tfrac{5}{2}} \tfrac{2}{3} = \tfrac{\sqrt{10}}{3}, \\ \langle 1, e_2\rangle &= \sqrt{\tfrac{441}{143}} \textstyle\int_{-1}^{1} 1 \cdot \left(x^4 - \tfrac{5}{7}x^2\right) \, dx = \sqrt{\tfrac{441}{143}} 2 \left(\tfrac{1}{5} - \tfrac{5}{7}\tfrac{1}{3}\right) = \tfrac{32}{105}\sqrt{\tfrac{441}{143}}. \end{aligned}$$

we have

$$\Pi_U 1 = \frac{5}{3}x^2 + \frac{1568}{2145}\left(x^4 - \frac{5}{7}x^2\right). \quad \blacksquare$$

Exercise 3. i) Check is straightforward. Just the vanishing: $\langle f, f\rangle = 0$ iff

$$\int_0^{+\infty} f(x)^2 e^{-x} \, dx = 0, \iff f(x)^2 e^{-x} = 0, \text{ a.e. } x \in [0, +\infty[,$$

that is $f = 0$ a.e. on $[0, +\infty[$.
ii) We may see U as

$$U = \{g \in H \ : \ \langle g, 1\rangle = 0\}\,.$$

It is easy to check that U is closed: if $(g_n) \subset U$ is such that $g_n \longrightarrow g$ (in H), then

$$0 = \langle g_n, 1\rangle \longrightarrow \langle g, 1\rangle, \implies \langle g, 1\rangle = 0, \implies g \in U.$$

iii) We may notice that U is the space of vectors orthogonal to 1. Take $V = \mathrm{Span}\langle 1\rangle$. Because

$$f = \Pi_V f + (f - \Pi_V f)$$

and $f - \Pi_V f$ is orthogonal to V, we claim that $\Pi_U f = f - \Pi_V f$. Indeed: first notice that

$$\Pi_V f = \langle f, 1\rangle \frac{1}{\|1\|},$$

where

$$\|1\|^2 = \int_0^{+\infty} 1^2 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_{x=0}^{x=+\infty} = 1.$$

Thus

$$\Pi_V f = \langle f, 1\rangle 1.$$

To show that $\Pi_U f = f - \Pi_V f \equiv f - \langle f, 1\rangle 1$ notice that

- $f - \Pi_V f \in U$ (trivial: $f - \Pi_V f$ is, by construction, orthogonal to V, that is to 1, thus it belongs to U, the space of vectors orthogonal to 1;
- $f - (f - \Pi_V f) = \Pi_V f$ is orthogonal to U because if $g \in U$, namely $\langle g, 1\rangle = 0$, then

$$\langle g, \Pi_V f\rangle = \langle g, \langle f, 1\rangle 1\rangle = \langle f, 1\rangle\langle g, 1\rangle = \langle f, 1\rangle \cdot 0 = 0.$$

Being U closed, there exists a unique element $\Pi_U f \in U$ such that $f - \Pi_U f \perp U$, thus it must be $\Pi_U f = f - \Pi_V f$. In conclusion

$$\Pi_U f = f - \langle f, 1\rangle 1. \quad \blacksquare$$