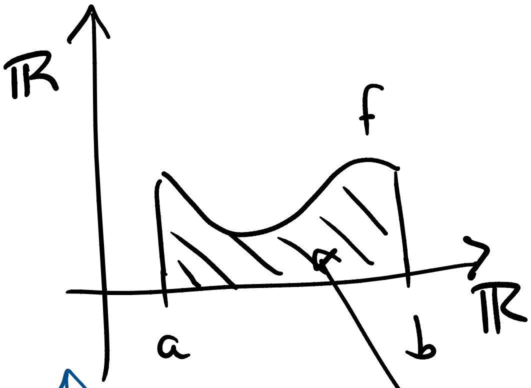
Lebesgue Measure and Integral



$$\int_a^b f(x)\,dx \quad \text{(Riemann Int)}$$

$$\parallel$$

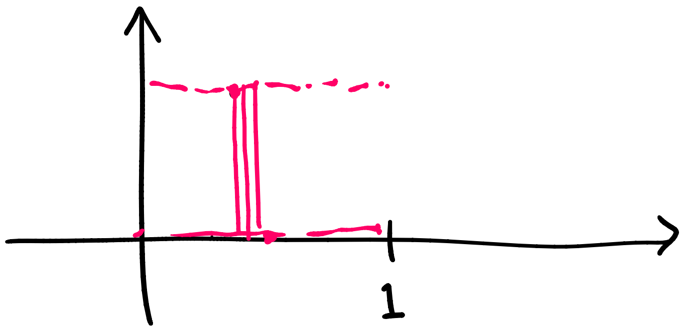
Area of

$$= \{ (x,y) \in \mathbb{R}\times\mathbb{R} : \ x \in [a,b], \ 0 \le y \le f(x) \}$$

(f must be integrable, not every fuct is integrable, cont functs are int., discont. fcts not always int.

Example (Dirichlet funct)

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \cap [0,1] \\ 1 & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$



$\int_0^1 f(x)\,dx$ is not defd according to R. def.

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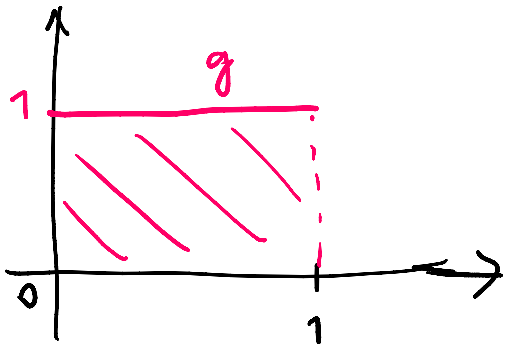
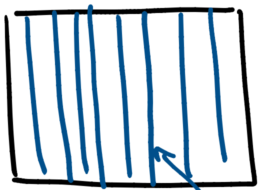
However, naturally $\int_0^1 f(x)\,dx$ should be $= 1$.



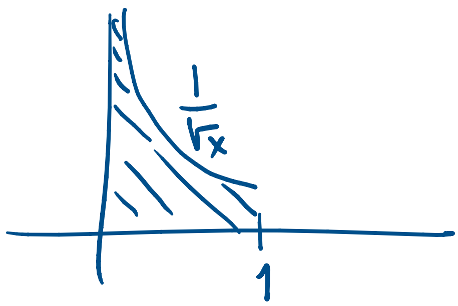
$g(x) \equiv 1 \quad x \in [0,1]$

$\int_0^1 g(x)\,dx = 1$

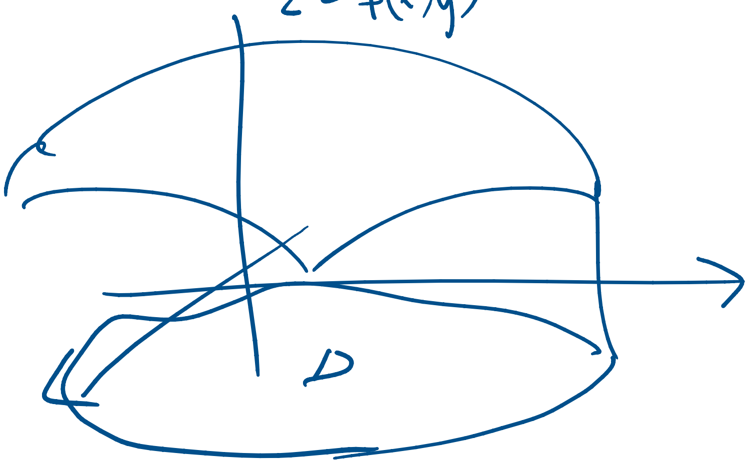


$\{(x,y) : x \in \mathbb{Q} \cap [0,1],\ y \in [0,1]\}$

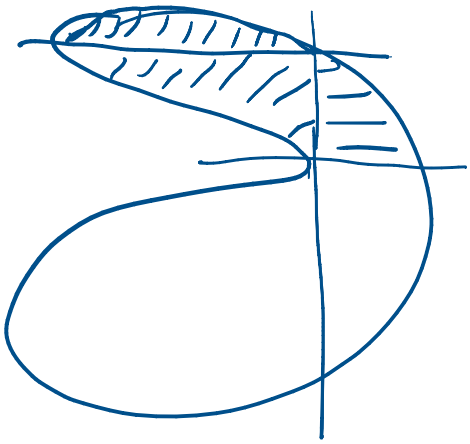
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$



$z = f(x,y)$



Pb: Def. a general method



Pb: Def. a general method of computing

areas (of plane fig)

volumes (of solid fig)

[$\underbrace{\text{hyper volumes}}_{\text{measure}}$ of a set in $\mathbb{R}^d$]

$$E \subset \mathbb{R}^d \rightsquigarrow \text{measure of } E$$

We look for a function

$$\lambda : \text{sets} \subset \mathbb{R}^d \longrightarrow \text{numbers}$$

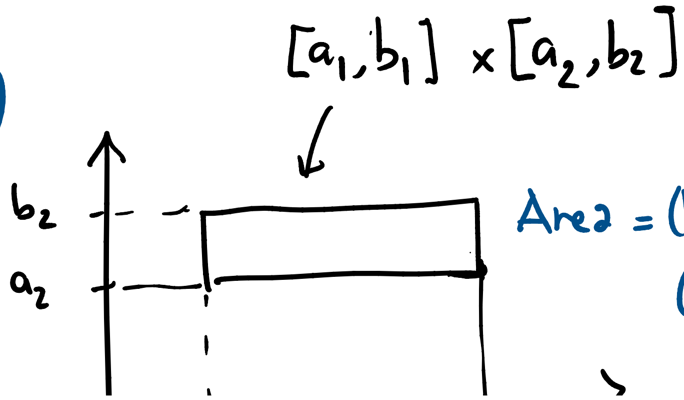
$$\lambda_d : \mathcal{P}(\mathbb{R}^d) \longrightarrow [0, +\infty[$$

(parts of $\mathbb{R}^d$: $\mathcal{P}(\mathbb{R}^d) = \{E \subset \mathbb{R}^d\}$

We want to define such a λ_d fulfilling certain nat conds :

Def: (Interval in $\mathbb{R}^d$)

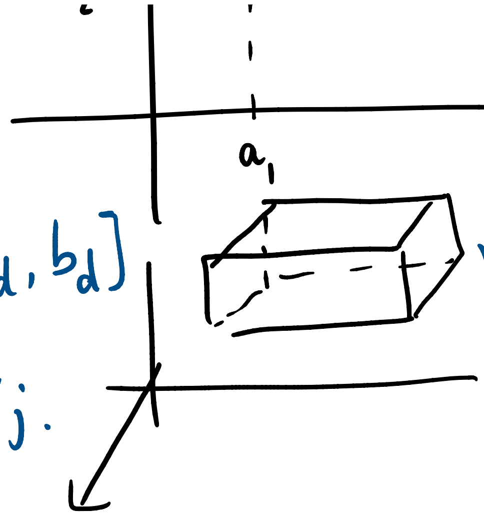
We call interval set of type



Area $= (b_1-a_1)\cdot(b_2-a_2)$

We call [cut off]

a set of type



$[a_1,b_1]\times[a_2,b_2]\times\dots\times[a_d,b_d]$

$\mathrm{Vol} = (b_1-a_1)\cdot(b_2-a_2)\cdot(b_3-a_3)$

where $a_j \leq b_j \quad \forall j$.

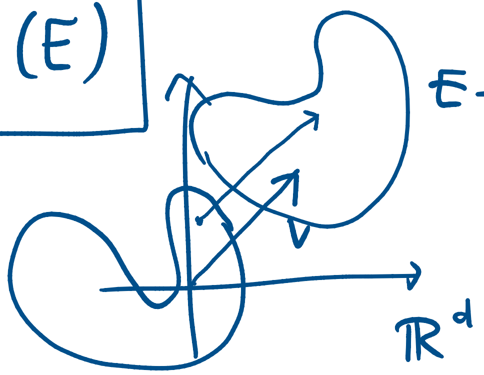
(1). $$\boxed{\lambda_d\left([a_1,b_1]\times\dots\times[a_d,b_d]\right) = \prod_{j=1}^{d}(b_j-a_j)}$$

Furthermore we want λ_d to respect some natural invariances:

Translations invariance $\forall E\subset\mathbb{R}^d$ $\left(\forall E\in\mathcal{P}(\mathbb{R}^d)\right)$

(2) $$\boxed{\lambda_d(E+v) = \lambda_d(E)}$$

$\forall v\in\mathbb{R}^d$.



Rotation Invariance $\forall E\in\mathcal{P}(\mathbb{R}^d)$

$$\lambda_d(TE) = \lambda_d(E)$$

$\lambda_d (\cdots) \cdots {}_d$

$\forall T$ orthogonal matrix, that is such that

$$T^t T = \mathbb{I}_d.$$

<u>Other simple invariances</u>

T flips two coords $\Longrightarrow$ $\lambda_d(TE) = \lambda_d(E)$ $\forall E \subset \mathcal{P}(\mathbb{R}^d)$

$$T \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} c_1 x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \qquad \lambda_d(TE) = |c_1| \lambda_d(E)$$

$$= \begin{bmatrix} c_1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{bmatrix}$$

$$T = \mathrm{diag}(c_1 \cdots c_d) = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & c_d \end{bmatrix}$$

$$\lambda_d(TE) = |c_1 \cdots c_d| \, \lambda_d(E).$$

In general, if T is <u>any invertible matrix</u> it can be proved that

$$\lambda_d(TE) = |\det T| \, \lambda_d(E) \quad \forall E \subset \mathcal{P}(\mathbb{R}^d)$$

This contains the case of rotations.

(Rmk: this contains the case of rotations.
in that case, because ${}^tT\,T = \mathbb{I}$

$$\det T \det {}^tT = \det({}^tT \cdot T) = \det \mathbb{I} = 1$$

$\det {}^tT = \det T$

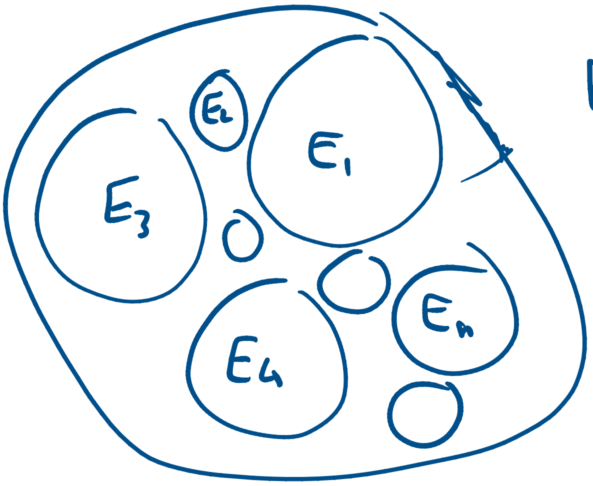
$$\Rightarrow \quad (\det T)^2 = 1 \quad \Rightarrow \quad |\det T| = 1.$$

Summarizing:

$$\boxed{(3) \quad \lambda_d(TE + v) = |\det T|\,\lambda_d(E) \quad \forall v \in \mathbb{R}^d, \ \forall T \text{ invertible}}$$

<u>What else?</u> we expect also that <u>countable additivity</u> holds for λ_d:

$$\lambda_d\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=0}^{\infty} \lambda_d(E_n)$$



$E_n \cap E_m = \phi$