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Def: (Abstract Measure)

Let X be a generic set, $\mathcal{F} \subset \mathcal{P}(X)$ σ-alg of sets in X. A function

$$\mu : \mathcal{F} \longrightarrow [0, +\infty]$$

is called measure on $\mathcal{F}$ if

i) $\mu(\emptyset) = 0$

ii) μ is countably additive : if $(E_n) \subset \mathcal{F}$ s.t. $E_n \cap E_m = \emptyset$, $n \neq m$, then

$$\mu\left(\bigcup_n E_n \right) = \sum_{n=0}^{\infty} \mu(E_n)$$

(notation: $\bigsqcup_n E_n$ stands for $\bigcup_n E_n$ in the case $E_n \cap E_m = \emptyset$, $n \neq m$)

The triplet $(X, \mathcal{F}, \mu)$ is called measure space

We say also that μ is finite, if

$$\mu(X) < +\infty.$$

If $\mu(X) = 1$ we say that μ is a probability measure

E.g. Probability different notations are used:

In Probability different notations are used:

$X \longleftrightarrow \Omega$

$\mathcal{F} \longleftrightarrow \mathcal{F}$

$\mu \longleftrightarrow \mathbb{P}$

$E \in \mathcal{F}$ $\longleftrightarrow$ $E \in \mathcal{F}$

measurable set $\qquad$ event

Examples

- $(\mathbb{R}^d, \mathcal{M}_d, \lambda_d)$ is a measure sp.
- Take $X = \mathbb{R}^d$, $\mathcal{F} = \mathcal{M}_d$, $f \in L(\mathbb{R}^d)$, $f \geq 0$

$$\mu : \mathcal{M}_d \longrightarrow [0, +\infty]$$

$$\mu(E) := \int_E f \, dx \quad \left(= \lambda_{d+1}(\mathrm{Trap}(f)) \right)$$

μ is a measure on $\mathcal{F} = \mathcal{M}_d$.

Indeed:

- $\mu(\phi) = \int_\phi f \, dx = 0$
- $\mu(\bigsqcup E_n) \stackrel{?}{=} \sum_n \mu(E_n)$

$\quad \|$ by def

$$\int_{\bigsqcup E_n} f \, dx = \int_{\mathbb{R}^d} f \, \mathbb{1}_{\bigsqcup_n E_n} \, dx$$

$$\mathbb{1}_{\bigsqcup_n E_n}(x) = \sum_n \mathbb{1}_{E_n}(x)$$

$$= \int_{\mathbb{R}^d} f \sum_n \mathbb{1}_{E_n}$$

$$= \int_{\mathbb{R}^d} \sum f \mathbb{1}_{E_n} \quad \geq 0$$

(Recall: $g_n \in L(\mathbb{R}^d)$, $g_n \geq 0$ a.e. $\Rightarrow$

$$\int_{\mathbb{R}^d} \sum g_n = \sum \int_{\mathbb{R}^d} g_n \quad \text{(monot conv)}$$
)

$$= \sum_n \int_{\mathbb{R}^d} f \, \mathbb{1}_{E_n}$$

$$= \sum_n \underbrace{\int_{E_n} f}_{=: \mu(E_n)} = \sum_n \mu(E_n).$$

∎

Some examples of this type of meas.

- $f(x) = e^{-\frac{(x-m)^2}{2\sigma}}$, $\mathbb{R}$

- $$f(x) = \frac{e^{-\frac{(x-m)^2}{2\sigma}}}{\sqrt{2\pi\sigma}} \qquad x \in \mathbb{R}.$$

$$\mu_f(E) := \int_E f = \int_E \frac{e^{-\frac{(x-m)^2}{2\sigma}}}{\sqrt{2\pi\sigma}}\, dx$$

μ_f is called gaussian meas. with mean m, variance σ

$$\mu_f =: \mathcal{N}(m, \sigma).$$

it's a prob. measure on $(\mathbb{R}, m_1)$.

- $\mathcal{N}(m, C)$ $\quad m \in \mathbb{R}^d$, C is $d \times d$, pos. def. symmetric matrix

$$f(x) := \frac{e^{-\frac{1}{2}(C^{-1}(x-m))\cdot(x-m)}}{\sqrt{(2\pi)^d \det C}} \qquad x \in \mathbb{R}^d$$

$$\equiv \frac{e^{-\frac{1}{2} C^{-1}(x-m)\cdot(x-m)}}{\sqrt{(2\pi)^d \det C}}$$

$$\mu_f(E) = \int \frac{e^{-\frac{1}{2} C^{-1}(x-m)\cdot(x-m)}}{}\, dx$$

$$\mu_f(E) = \int_E \frac{e^{\cdot}}{\sqrt{(2\pi)^d \det C}} \, dx$$

$E \subset \mathfrak{M}_d$

it is a well def'd prob. meas. denoted by

$\mathcal{N}(m, C)$.

mean covariance matrix

- $X = \mathbb{N}$, $\mathcal{F} = \mathcal{P}(\mathbb{N})$

$$\mu(E) = \sum_{x \in E} 1 .$$ (counting measure)

$$= \begin{cases} \# E & \text{if } E \text{ is finite} \\ +\infty & \text{if } E \text{ is infinite.} \end{cases}$$

$\mu(\phi) = 0$.

μ is a measure on E.

- $\Omega = \{\omega_1, \omega_2, \cdots, \omega_n, \cdots\}_{n \in \mathbb{N}}$

$\omega_1 \to p_1$, $\omega_2 \to p_2$, $\omega_n \to p_n$

$0 \leq p_n \leq 1$. $\sum^{\infty} p_n = 1$

$$0 \leq p_n \leq 1, \quad \sum_{n=0}^{\infty} p_n = 1$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P}(E) = \sum_{\omega_n \in E} p_n .$$

$(\Omega, \mathcal{F}, \mathbb{P})$ is a prob. space.

Some general properties of a measure:

Prop: Let $(X, \mathcal{F}, \mu)$ be a meas. sp.

Then:

i) (additivity) $\mu(E \sqcup F) = \mu(E) + \mu(F)$

(agreement $c + (+\infty) = +\infty$)

ii) (sub-add) $\mu(\bigcup E_n) \leq \sum_n \mu(E_n)$

iii) (monotonicity) $E \subset F \Rightarrow \mu(E) \leq \mu(F)$

iv) (cont from below) $(E_n) \subset \mathcal{F}$, $E_n \subset E_{n+1} \Rightarrow$

$$\lim_n \mu(E_n) = \mu(E) \qquad E = \bigcup E_n$$

v) (cont from above) $(E_n)\subset\mathcal{F}$, $E_n \supset E_{n+1}$, $\mu(E_1)<+\infty$

$$\Rightarrow \quad \lim_n \mu(E_n) = \mu(E) \qquad E = \bigcap_n E_n.$$

vi) if $F\subset E$ and $\mu(E)<+\infty$

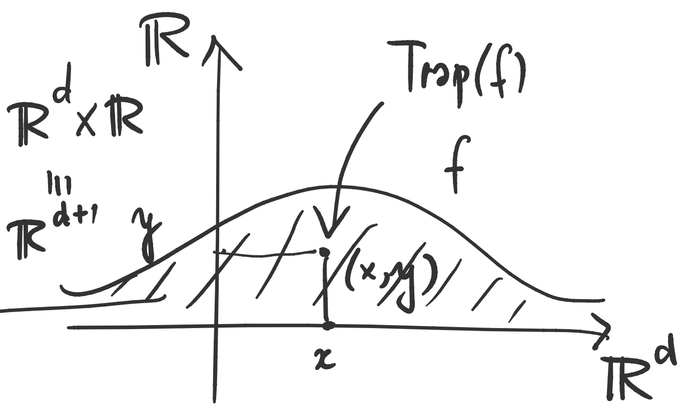
$$\mu(E\setminus F) = \mu(E)-\mu(F).$$

Once we have a measure, we wish to introduce a concept of int

$$\int_X f\, d\mu$$

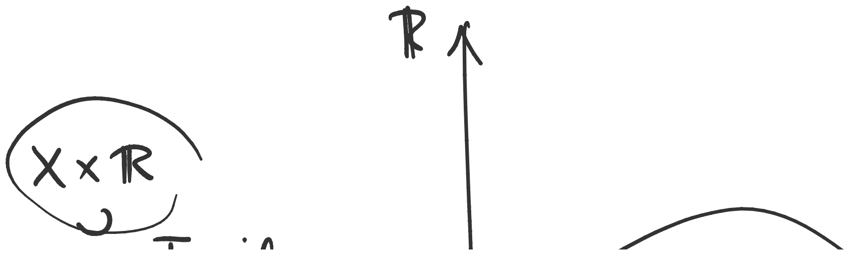
$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f(x)\, dx$$

$$f:\mathbb{R}^d \longrightarrow [0,+\infty]$$



$$\text{Trap}\, f = \{(x,y)\in\mathbb{R}^d\times\mathbb{R} : 0\le y \le f(x)\}$$

$$\int_{\mathbb{R}^d} f = \lambda_{d+1}\left(\text{Trap}(f)\right)$$



$$f : X \longrightarrow [0,+\infty]$$

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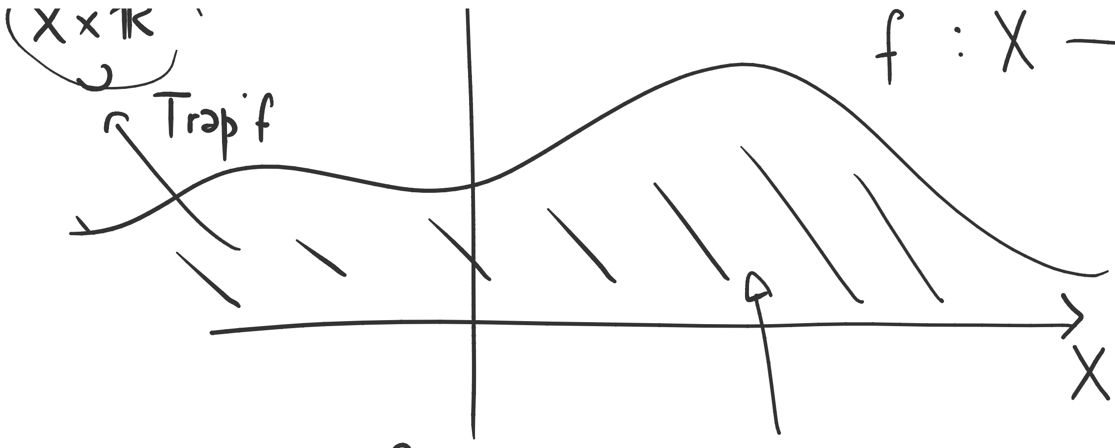
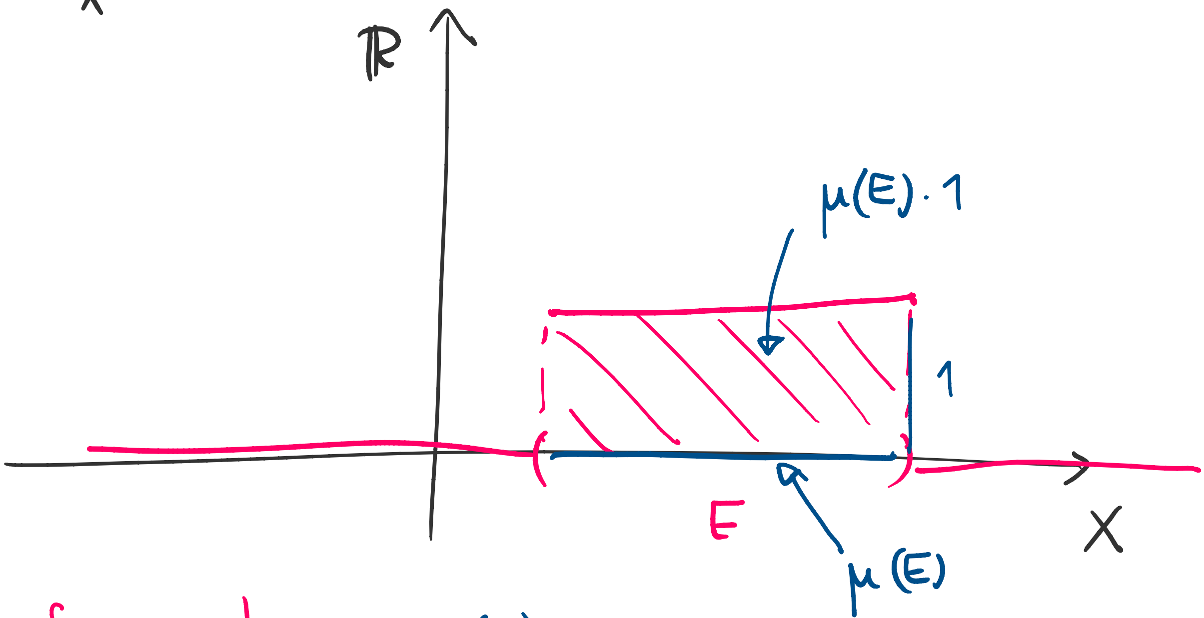


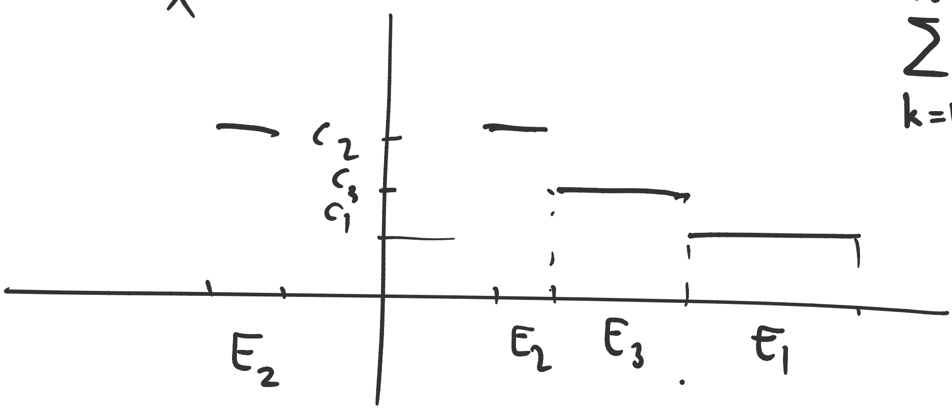
$$\int_X f \, d\mu = \; ?$$



$$\int_X \mathbb{1}_E \, d\mu = \mu(E)$$

$$\int_X c \, \mathbb{1}_E \, d\mu = c \, \mu(E) \qquad (0 \cdot \infty = 0)$$

$$\sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$$



$$\int \sum^{n} c_k \mathbb{1}_{E_k} \, d\mu = \sum^{n} c_k \, \mu(E_k) \qquad c_k \geq 0$$

$$\int_X \sum_{k=1}^{n} c_k \mathbb{1}_{E_k} \, d\mu = \sum_{k=1}^{n} c_k \, \mu(E_k) \qquad c_k \geq 0, \quad E_k \in \mathcal{F}$$

Def: A function $s : X \longrightarrow [-\infty, +\infty]$ s.t.

$s(X)$ is finite

$\| \; \{c_1, c_2, \dots, c_n\}$

is called simple funct.

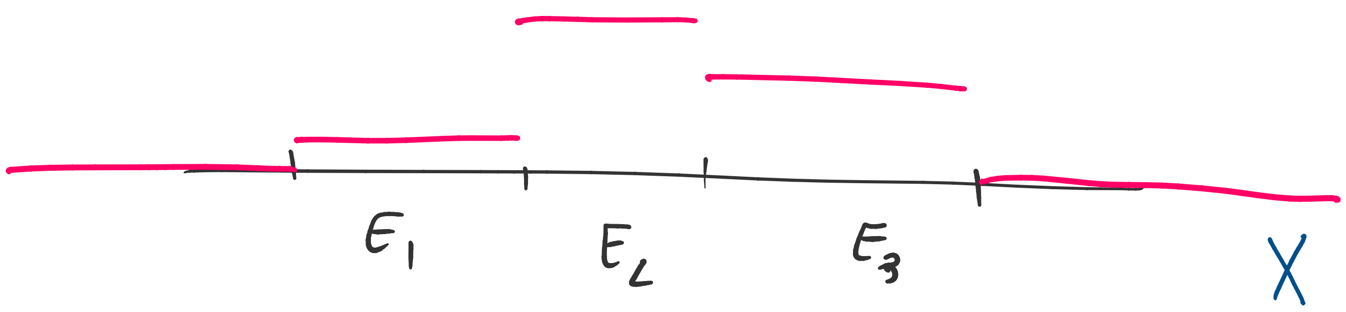
$$s(x) = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k} \quad \text{where } E_k = \{s = c_k\}$$

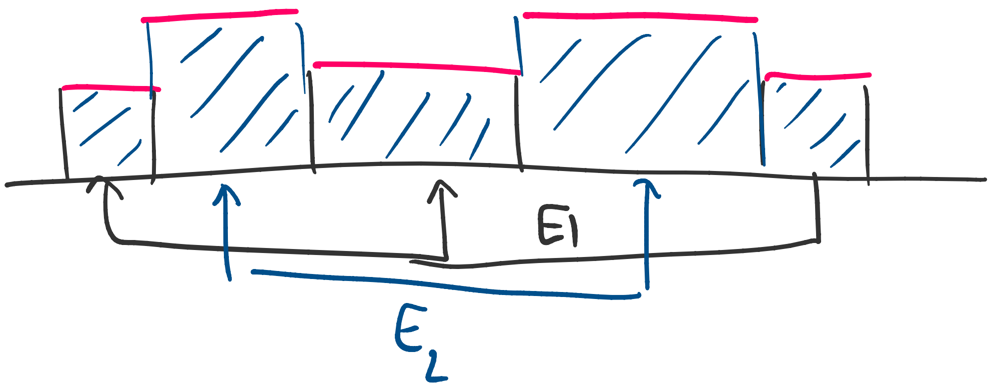
(for inst $\mathbb{1}_E$ is a simple funct)

Def: (Int. for a simple funct)

Let $s = \sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$ $\quad E_i \cap E_j \neq \emptyset \;\; i \neq j$

be a simple funct, $s \geq 0$ $(c_k \geq 0 \;\; \forall k)$





$$\int_X s \, d\mu = \sum_{k=1}^{n} c_k \, \mu(E_k) \qquad \left(\begin{array}{c}\text{provided}\\ E_k \in \mathcal{F}\end{array}\right)$$

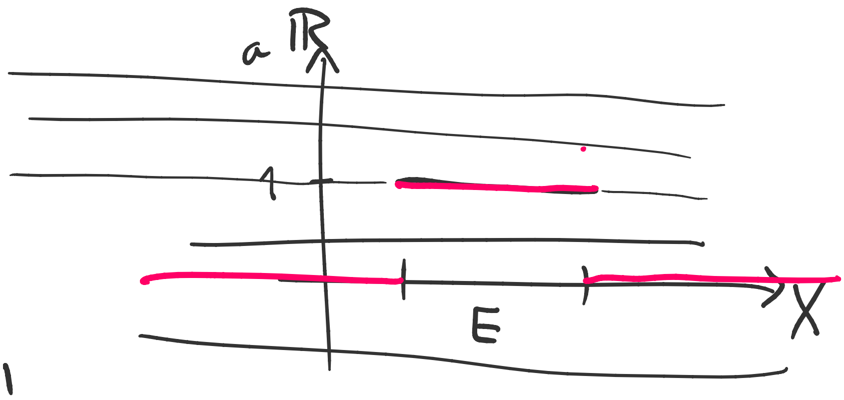
<u>Def:</u> Let $\mathcal{F}$ be a σ-alg. $f : X \to [-\infty, +\infty]$

We say that f is $\mathcal{F}$-meas if

$$\{f > a\} \equiv \{x \in X : f(x) > a\} \in \mathcal{F} \qquad \forall a \in \mathbb{R}$$

<u>Examples:</u>

- $\mathbb{1}_E$ is $\mathcal{F}$-meas $\iff$ $E \in \mathcal{F}$



$$\{f > a\} = \begin{cases} \emptyset \in \mathcal{F} & a \geq 1 \\ E & 0 \leq a < 1 \\ X \in \mathcal{F} & a < 0 \end{cases}$$

$$\{f > a\} \in \mathcal{F} \quad \forall a \iff E \in \mathcal{F} \qquad \blacksquare$$

$\{f>a\} \in \mathcal{F} \quad \forall a$ ...

• Every simple funct $s=\sum_{k=1}^{n} c_k \mathbb{1}_{E_k}$ is $\mathcal{F}$-meas
$\Longleftrightarrow \quad E_k \in \mathcal{F} \quad \forall\, k=1,\dots,n.$

• X = any set

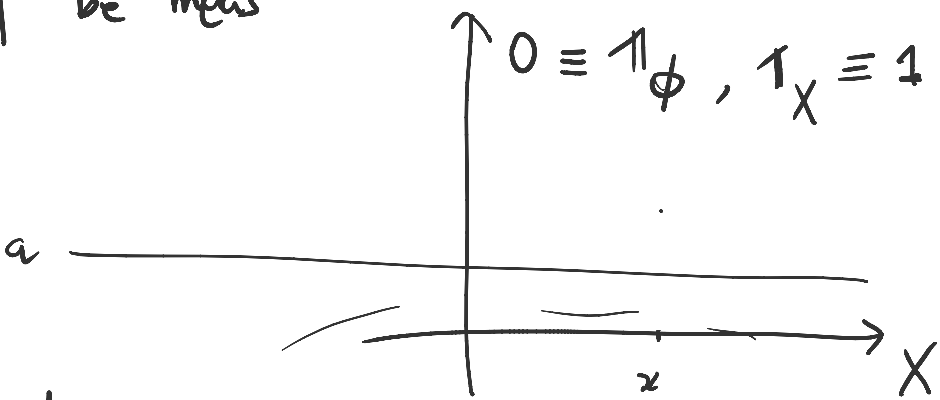
$\mathcal{F} = \{\emptyset, X\}$

$f : X \longrightarrow [-\infty, +\infty]$ is meas $\overset{?}{\Longleftrightarrow}$

$\{f>a\} = \emptyset, X \quad \forall a$ $\Big(\mathbb{1}_E$ is meas $\Leftrightarrow$ $E \in \mathcal{F} = \{\emptyset, X\}$

in order f be meas

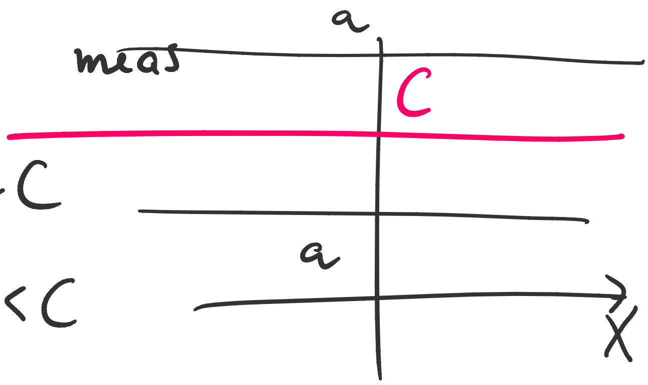
$0 \equiv \mathbb{1}_\emptyset \,,\ \mathbb{1}_X \equiv 1 \Big)$



? $\Updownarrow$

f constant

A $f \equiv C = C\mathbb{1}_X.$ is always meas

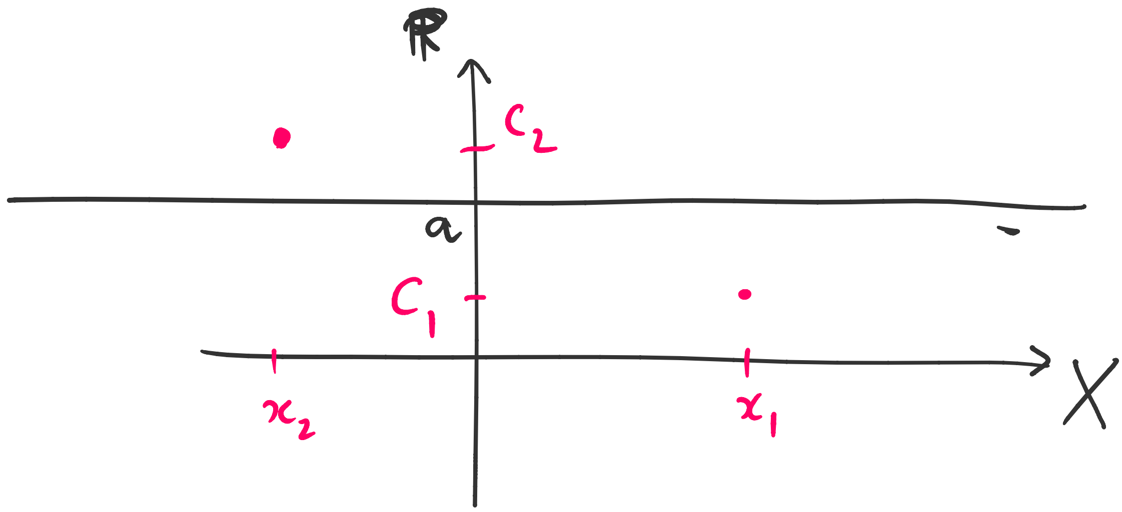


$$\{f>a\} = \begin{cases} \emptyset \in \mathcal{F} & a \geq C \\ X \in \mathcal{F} & a < C \end{cases}$$

Assume now f be meas: guess $f \equiv C.$

Indeed suppose

$$\begin{array}{lcl} \exists\, x_1 & : & f(x_1) = c_1 \\ & & \qquad\quad \wedge \\ \exists\, x_2 & : & f(x_2) = c_2 \end{array}$$



$\mathcal{F} \not\ni \{f > a\} \neq \phi$ because $x_2 \in \{f > a\}$

$\{f > a\} \neq X$ because $x_1 \notin \{f > a\}$

Conclusion: f meas $\Longleftrightarrow$ f is const. $\square$

- X any set

  $\mathcal{F} = \mathcal{P}(X)$

  $\Rightarrow \forall f : X \longrightarrow [-\infty, +\infty]$ is meas.

  $(\{f > a\} \in \mathcal{P}(X) \quad \forall a.)$

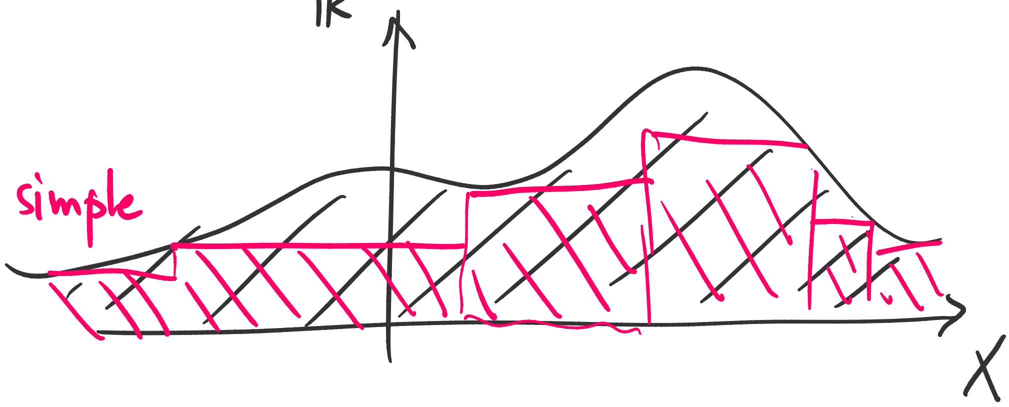
Def: (Integral for f. meas pos)

Let $f: X \longrightarrow [0, +\infty]$ be $\mathcal{F}$ meas.

Def: (Integral

Let $f: X \to [0, +\infty]$ be $\mathcal{F}$ meas.

Consider the class of all simple meas functs below f



$$\mathcal{S}(f) = \left\{ s: X \to [0,+\infty] : \begin{array}{l} s \text{ is simple and meas} \\ s \le f \\ (s(x) \le f(x) \text{ a.e.}) \end{array} \right\}$$

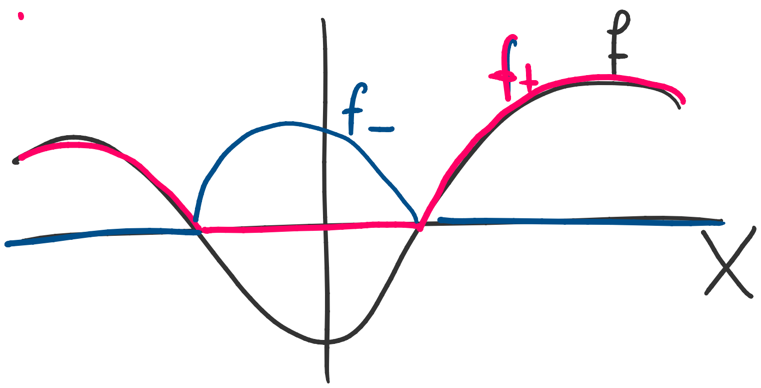
For every $s \in \mathcal{S}(f)$

we compute $\displaystyle\int_X s \, d\mu$

We def.

$$\int_X f \, d\mu := \sup \left\{ \int_X s \, d\mu : s \in \mathcal{S}(f) \right\}$$

This defines the int for a pos. and meas. f.



$$f_+ = \begin{cases} f & f > 0 \\ 0 & f < 0 \end{cases}$$

$$f_- = \begin{cases} 0 & f \ge 0 \\ -f & f < 0 \end{cases}$$

Def: Given $f : X \longrightarrow [-\infty, +\infty]$ $\mathcal{F}$-meas, we say that f is μ-int on X if

$$\int_X |f| \, d\mu < +\infty .$$

In this case we pose

$$\int_X f \, d\mu = \int_X f_+ \, d\mu - \int_X f_- \, d\mu .$$

This def can be extended to the case of $\mathbb{C}$ valued functs.

The charateristic funct. of a r.v.: $X, Y, Z, \dots$

$\updownarrow$

meas funct

$$(\Omega, \mathcal{F}, \mathbb{P}) \qquad \underset{\text{r.v.}}{X} : \Omega \longrightarrow [-\infty, +\infty]$$

$$\int_\Omega \underbrace{e^{i\xi X}}_{\in \mathbb{C}} \, d\mathbb{P} \qquad \xi \in \mathbb{R}$$