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Few facts on Abstract Meas/Int

Let $(X, \mathcal{F}, \mu)$ be a measure space.

Thm: (monot. conv)

Let $(f_n) \subset L(X)$ (meas. functs on X)

be s.t.

$$0 \leq f_n(x) \leq f_{n+1}(x) \quad \text{a.e. } x \in X, \ \forall n \in \mathbb{N}$$

($\forall x \in X$ except for a $\mu = 0$ set)

Then

$$\exists \lim_n \int_X f_n \, d\mu = \int_X \lim_n f_n \, d\mu.$$

Corollary: If $(f_n) \subset L(X)$, $f_n \geq 0$

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n \, d\mu.$$

Thm (dominated conv)

Let $(f_n) \subset L^1(X)$ $\left(L^1(X) = \left\{ f \in L(X) : \int_X |f| \, d\mu < +\infty \right\} \right)$

s.t.

1. $\exists \lim_n f_n(x) =: f(x)$ μ-a.e. $x \in X$

2. $\exists\, g \in L^1(X) : \quad |f_n(x)| \leq g(x) \quad \mu\text{ a.e } x \in X$

$\Rightarrow \quad f \in L^1(X)$ and

$$\lim_n \int_X f_n \, d\mu = \int_X f \, d\mu \quad \left(= \int_X \lim_n f_n \, d\mu \right)$$

<u>Corollary</u>: Let $(f_n) \subset L^1(X)$ be s.t.

$$\sum \int_X |f_n| \, d\mu < +\infty$$

Then $\sum f_n \in L^1(X)$ and

$$\int_X \sum f_n \, d\mu = \sum_n \int_X f_n \, d\mu$$

Assume now

$$f : X \times \Lambda \longrightarrow \mathbb{C}$$
$$(x, \lambda)$$

and define

$$F(\lambda) := \int_X f(x, \lambda) \, d\mu(x)$$

(this $F : \Lambda \longrightarrow \mathbb{C}$ provided $\boxed{f(\#, \lambda) \in L^1(X, \mu) \quad \forall \lambda \in \Lambda}$)

$$\boxed{f(\#,\lambda) \in L^1(X,\mu) \quad \forall \lambda \in \Lambda}\Big)$$

Pb: Under which conds is it true that

$$\partial_\lambda F(\lambda) = \partial_\lambda \int_X f(x,\lambda)\, d\mu(x) \overset{?}{=} \int_X \partial_\lambda f(x,\lambda)\, d\mu(x).$$

Thm: (deriv. thm)

Assume that

1. $f(\#,\lambda) \in L^1(X,\mu) \quad \forall \lambda \in \Lambda \subset \mathbb{R}\ (\mathbb{R}^d, \mathbb{C})$
2. $\exists\, \partial_\lambda f(x,\lambda)$ μ-a.e $x \in X$, $\forall \lambda \in \Lambda$
3. $\exists\, g \in L^1(X,\mu)$: $|\partial_\lambda f(x,\lambda)| \leq g(x)$ μ.a.e. $x \in X$ $\forall \lambda \in \Lambda$

$$\Rightarrow \quad \exists\, \partial_\lambda \int_X f(x,\lambda)\, d\mu(x) = \int_X \partial_\lambda f(x,\lambda)\, d\mu(x).$$

Prop: (Čebishev Inequality)

$f \in L(X)$, $f \geq 0$.

$$\mu(f \geq \alpha) \leq \frac{1}{\alpha} \int f\, d\mu(x) \qquad \alpha > 0$$

$$\mu(f \geq \alpha) \leq \frac{1}{\alpha} \int_{f \geq \alpha} f \, d\mu(x) \qquad \alpha > 0$$

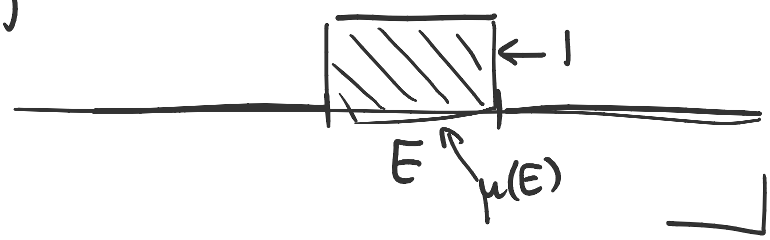
$$\mu(f \geq \alpha) = \{x \in X : f(x) \geq \alpha\}$$

$$\left(\leq \frac{1}{\alpha} \int_X f \, d\mu(x) \right)$$

<u>Proof:</u>

$$\mu(f \geq \alpha) = \int_X 1 \cdot \mathbb{1}_{\{f \geq \alpha\}} \, d\mu$$

$$\ulcorner \quad \mu(E) \overset{\leftarrow}{=} \int_X \mathbb{1}_E \, d\mu(x)$$



$$x \in \{f \geq \alpha\} \qquad f(x) \geq \alpha \underset{\alpha > 0}{\iff} 1 \leq \frac{f(x)}{\alpha}$$

$$1 \cdot \mathbb{1}_{\{f \geq \alpha\}}(x) \leq \frac{f(x)}{\alpha} \cdot \mathbb{1}_{\{f \geq \alpha\}} \qquad \forall x \in X$$

$$\Rightarrow \mu(\{f \geq \alpha\}) \leq \int_X \frac{f(x)}{\alpha} \mathbb{1}_{\{f \geq \alpha\}} \, d\mu(x)$$

$$\frac{1}{\alpha} \cdot \int \underset{\overset{\wedge\!\shortparallel}{0}}{f(x)} \, d\mu(x)$$

$$= \frac{1}{\alpha}\cdot\int_{\{f\geq\alpha\}} \overset{\geq}{f}(x)\,d\mu(x)$$

$$\underset{\{f\geq\alpha\}\subset X}{\leq} \frac{1}{\alpha}\int_X f\,d\mu \,. \qquad \boxtimes$$

In part, according to Čeb., if $f\in L^1(X)$, $f\geq 0$

$$\left(\mu(\{f\geq\alpha\}) \leq \frac{1}{\alpha}\underbrace{\int_X f\,d\mu}\right.$$

$$(*) \quad \mu(\{f\geq\alpha\}) \leq \frac{C}{\alpha}. \qquad \forall\, 0<\alpha<+\infty.$$

By this it follows that

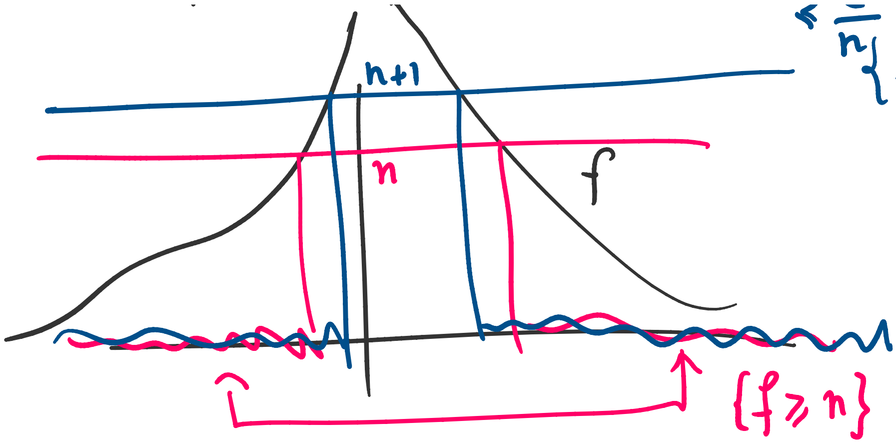
$$\boxed{\mu(\{f=+\infty\})=0}$$

This follows by (*) once we realize that

$$\{f=+\infty\} = \bigcap_{n\in\mathbb{N}}^{+\infty} \overset{E_n}{\{f\geq n\}} = E$$

$$\Rightarrow \quad \mu(\overset{E}{\{f=+\infty\}}) = \lim_n \overset{\mu(E_n)}{\mu(\{f\geq n\})} = 0 \quad \text{(by cont from above)}$$

$$\leq \frac{C}{n} \qquad \{f\geq n+1\}\subset\{f\geq n\}$$



$\{f \geq n+1\} \subseteq \{f \geq n\}$

$\{f \geq n\} \searrow$

$$\Rightarrow \quad \mu(\{f = +\infty\}) = 0.$$

Vocabulary

<u>Meas Th</u>	<u>Prob.</u>
X	Ω (sample space)
$\mathcal{F}$ (meas. sets)	$\mathcal{F}$ (family of events)
μ	$\mathbb{P}$ (finite meas with $\mathbb{P}(\Omega) = 1$, called <u>probability</u>)
$f : X \to \mathbb{R}$ <u>meas funct</u>	$X : \Omega \to \mathbb{R}$ random var.
$\int_X f \, d\mu$ <u>integral</u>	$\int_\Omega X \, d\mathbb{P} =: \mathbb{E}[X]$ (expected value of X)
μ a.e. $x \in X$	$\mathbb{P}$-almost surely (a.s.)

<u>Rmk:</u> In the case of a prob space we have a short version of the dom conv.

Rmk: In ... ... , | | have a part version of the dom conv.

Corollary (bded conv thm)

Assume $(\Omega, \mathcal{F}, \mathbb{P})$ be a prob. space (X_n) a seq of random vars. such that

1. $\lim_{n \to \infty} X_n(\omega) =: X(\omega)$ $\mathbb{P}$-a.s.
2. $\exists\, C > 0$: $|X_n(\omega)| \leq C$ $\mathbb{P}$-a.s. $\forall n \in \mathbb{N}$

$$\Rightarrow \quad \lim_n \int_\Omega X_n \, d\mathbb{P} = \int_\Omega X \, d\mathbb{P}$$

$$\lim_n \mathbb{E}[X_n] = \mathbb{E}[X]. \qquad \blacksquare$$

Do Ex 2.5.1/2/3/5 (*)/6

Basic Banach Spaces

Most of Advanced applications of Math to Modeling pbs are studied in spaces of functions. In general a space of functs has some remarkable algebraic structure as that one of vector space.

A vector space V is a set whose elements

A vector space V is a set whose elements are called vectors on which are defined two main operations:

sum: $v, w \in V \Rightarrow v + w \in V$

product by scalars: $v \in V$, $\lambda \in \mathbb{C}\ (\mathbb{R}) \Rightarrow \lambda v \in V$

Ex: $V = \mathbb{R}^d = \{ \underbrace{(x_1, \dots, x_d)}_{v} : x_j \in \mathbb{R},\ j = 1 \dots d \}$

$$v = (x_1 \,—\, x_d) \qquad w = (y_1 \,—\, y_d) \qquad v + w = (x_1 + y_1, \dots, x_d + y_d)$$

$$\lambda \in \mathbb{R} \qquad \lambda v = (\lambda x_1, \dots, \lambda x_d)$$

$V = \mathbb{C}^d = \{ (z_1, \dots, z_d) : z_j \in \mathbb{C}\ \ j = 1 \dots d \}$

with same defs of $+$ and $\cdot$. In this case we may use scalars $\lambda \in \mathbb{R}, \mathbb{C}$

Ex. $V = \mathcal{C}([0,1]; \mathbb{R}) = \{ f : [0,1] \to \mathbb{R} :$ $\underline{f \text{ cont on } [0,1]}$ $\}$

In this case we have a natural structure

In this case we have a natural structure of vect sp on $\mathbb{R}$ (as scalars)

$f, g \in V \quad (f, g \in \mathscr{C}([0,1]))$

$$f + g \in V \quad \text{where}$$

$$(f+g)(x) := f(x) + g(x) \qquad \forall x \in [0,1]$$

$$f+g : [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto f(x) + g(x)$$

Notice that the sum is well defined because if $f, g \in V \ (f, g \in \mathscr{C}([0,1]))$ $\Rightarrow$ $f+g \in V$ being $f+g \in \mathscr{C}([0,1])$. This follows by a non trivial fact "the sum of cont funcs is a cont funct"

Similarly the prod by scalars is defd

$\lambda \in \mathbb{R}, \quad f \in V \qquad (\lambda f)(x) := \lambda f(x) \quad \forall x \in [0,1]$

$\lambda f \in V \iff \lambda f \in \mathscr{C}([0,1])$

$\Rightarrow V$ is a vector sp.

Ex $V = L^1(X,\mu) = \left\{ f : X \longrightarrow \mathbb{R} \;:\; f \text{ meas and } \int_X |f|\, d\mu < +\infty \right\}$

V is a vector sp. with + and · defined as above.

Here for inst take $f, g \in V = L^1$

We should check that $f+g \in L^1$

$\Leftrightarrow \underbrace{f+g \text{ is meas}}_{\text{true because}}$ and $\underbrace{\int_X |f+g|\, d\mu < +\infty.}_{\text{true because}}$

sum of meas functs is meas.

$|f+g| \leq |f| + |g|$

$|f(x)+g(x)| \leq |f(x)| + |g(x)|$

$\forall x \in X$

$\Downarrow$

$\int_X \underbrace{|f(x)+g(x)|\, d\mu}_{\wedge \; +\infty} \leq \underbrace{\int_X |f|\, d\mu}_{\wedge \; +\infty} + \underbrace{\int_X |g|\, d\mu}_{\wedge \; +\infty}$

$\Downarrow$

$f+g \in L^1$

yes because $f, g \in L^1$

E $V = L^2(X,\mu)$

Ex: $V = L^2(X;\mu)$

$= \{ f : X \to \mathbb{R} : f$ meas and $\int_X |f|^2 \, d\mu < +\infty \}$

On V we def usual $+$ and $\cdot$

Let's check that $+$ is well defd:

let $f, g \in V = L^2$ $\left(\begin{matrix} f \\ g \end{matrix} \text{ meas } \quad \begin{matrix} \int_X |f|^2 \, d\mu < +\infty \\ \int_X |g|^2 \, d\mu < +\infty \end{matrix} \right)$

Pb: Is $f + g \in L^2$?

$\Updownarrow$

1. $f+g$ meas (yes! same as in the prev ex)

2. $\int_X |f+g|^2 \, d\mu < +\infty$?

$\int_X |f+g|^2 \, d\mu \leq$

$|f+g|^2 \leq (|f| + |g|)^2$

$\leq \int (|f| + |g|)^2 \, d\mu$

$\int_X |f|^2 \, d\mu < +\infty$

$\int_X |g|^2 \, d\mu < +\infty$

$$\leq \int_X (|f|+|g|)^2 \, d\mu$$

$$= \int_X \left(|f|^2 + |g|^2 + 2|f||g|\right) d\mu$$

$$= \int_X |f|^2 \, d\mu + \int_X |g|^2 d\mu + \underbrace{2\int_X |f||g| \, d\mu}$$

$$\int_X |f||g| \, d\mu \overset{?}{\leq} \ldots \int_X |f|^2 \, d\mu \ldots \int_X |g|^2 \, d\mu.$$

$$\boxed{2ab \leq a^2 + b^2} \qquad ab \leq \frac{1}{2}(a^2+b^2)$$

$$\left(\Leftrightarrow \quad a^2 + b^2 - 2ab \geq 0 \quad \Leftrightarrow \quad (a-b)^2 \geq 0 \right)$$

$$\boxed{2|f||g| \quad \left(\equiv |f(x)||g(x)|\right) \leq |f|^2 + |g|^2}$$

$$\Rightarrow \quad 2\int_X |f||g| \, d\mu \leq \int_X \left(|f|^2 + |g|^2\right) d\mu$$

$$= \int_X |f|^2 \, d\mu + \int_X |g|^2 \, d\mu$$

$$\Rightarrow \boxed{\int_X |f+g|^2 \, d\mu \leq 2\left(\int_X |f|^2 \, d\mu + \int_X |g|^2 \, d\mu\right)}$$

$$\Rightarrow \boxed{\int_X |f+g|^2 \, d\mu \le \alpha \left(\int_X |f|^2 \, d\mu + \int_X |g|^2 \, d\mu \right)}$$

$\Rightarrow f+g \in L^2$. □

<u>Ex</u> $L^p(X) := \{ f : X \to \mathbb{R} : f$ meas and $\int_X |f|^p \, d\mu < +\infty \}$

$1 \le p < +\infty$.

$L^p(X)$ is a vect space.

Try to prove this: the diff part is to prove that if

$$\int_X |f|^p \, d\mu, \int_X |g|^p \, d\mu < +\infty \Rightarrow \int_X |f+g|^p \, d\mu < +\infty$$

$\alpha, \beta \in \mathbb{R}$ $\alpha, \beta > 0$ $\exists C > 0$:

$$(\alpha + \beta)^p \le C(\alpha^p + \beta^p) \quad \forall \alpha, \beta > 0$$

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