Ex $L^p(X) := \{ f : X \to \mathbb{R} : f \text{ meas and } \int_X |f|^p \, d\mu < +\infty \}$

$1 \leq p < +\infty$.

$L^p(X)$ is a vect space.

Try to prove this: the diff part is to prove that if

$$\int_X |f|^p \, d\mu, \int_X |g|^p \, d\mu < +\infty \Rightarrow \int_X |f+g|^p \, d\mu < +\infty$$

$\alpha, \beta \in \mathbb{R}$ $\quad \alpha, \beta > 0$ $\quad \exists C > 0 :$

$$(*) \quad (\alpha + \beta)^p \leq C(\alpha^p + \beta^p) \quad \forall \alpha, \beta > 0$$

If $(*)$ is true

$$\int_X |f+g|^p \, d\mu$$

$$|f+g|^p \leq (\underbrace{|f|}_{\alpha} + \underbrace{|g|}_{\beta})^p \leq C(|f|^p + |g|^p)$$

$$\Downarrow$$

$$\int_X |f+g|^p \, d\mu \leq C \int_X (|f|^p + |g|^p) \, d\mu < +\infty$$

because $f, g \in L^p$.

$f,g\in L^p$.

Proof of $\boxed{(\alpha+\beta)^p \le C(\alpha^p+\beta^p)}$ $\quad \alpha,\beta>0$

$$\Updownarrow$$

$$\cancel{\beta^p}\left(\frac{\alpha}{\beta}+1\right)^p \le C\cancel{\beta^p}\left(\left(\frac{\alpha}{\beta}\right)^p+1\right)$$

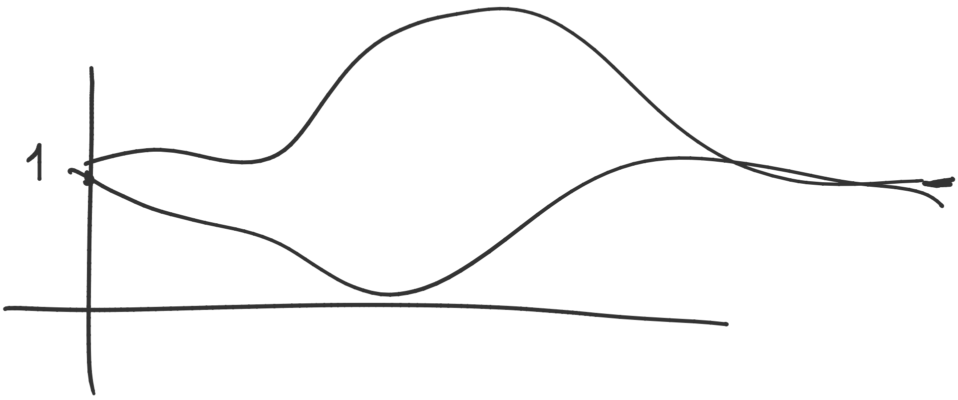
$t=\frac{\alpha}{\beta}$ $\quad (t+1)^p\le C(t^p+1) \quad \forall t\ge 0$

$$\Updownarrow$$

$$\varphi(t):=\frac{(t+1)^p}{t^p+1}\le C \quad \forall t>0$$

Conclusion follows once we prove this φ bdd.

$\varphi\in\mathcal{C}([0,+\infty[)$, $\varphi\ge 0$, $\varphi(t)\xrightarrow[t\to+\infty]{}1$



φ bdd. ∎

Def: Let V be a vect. sp. on $\mathbb{R}$ ($\mathbb{C}$)
A function $\|\cdot\|: V\longrightarrow[0,+\infty[$ is called **norm** on V if

1. (vanishing) $\|f\| = 0 \iff f = 0_V$

2. (homogeneity) $\|\lambda f\| = |\lambda| \, \|f\| \quad \forall \lambda \in \mathbb{R} \ (\mathbb{C})$
$\forall f \in V$

3. (triang ineq) $\|f+g\| \leq \|f\| + \|g\| \quad \forall f, g \in V.$

__Examples__

$V = \mathbb{R}^d$, $\quad \|(x_1, \dots x_d)\|_2 = \sqrt{\sum_{j=1}^{d} x_j^2}$

(euclidean norm)

$$\|(x_1, \dots x_d)\|_1 := \sum_{j=1}^{d} |x_j|$$

$$\|(x_1, \dots, x_d)\|_\infty := \max_{j=1 \dots d} |x_j|.$$

Let's check that $\|\cdot\|_1$ is a norm on $\mathbb{R}^d$

1. (vanish.) $\|(x_1 \dots x_d)\|_1 = \sum_{j=1}^{d} |x_j| = 0$

$\iff |x_j| = 0 \quad \forall j = 1 \dots d \implies x_j = 0 \quad \forall j = 1 \dots d$

$\iff (x_1 \text{---} x_d) = 0_{\mathbb{R}^d}$

2. (homog) $\|\lambda (x_1 \text{---} x_d)\|_1 = \|(\lambda x_1, \dots \lambda x_d)\|_1$

$= \sum^{d} |\lambda x_j| = \sum^{d} (|\lambda|) |x_j|$

$$= \sum_{j=1}^{d} |\lambda x_j| = \sum_{j=1}^{d} |\lambda| |x_j|$$

$$= |\lambda| \sum_{j=1}^{d} |x_j| = |\lambda| \, \|(x_1 \text{—}, x_d)\|_1 .$$

3. (Δ ineq) : $\|x + y\|_1 = \|(x_1+y_1, \ldots x_d+y_d)\|_1$

$x = (x_1 - x_d)$, $y = (y_1 - \cdot y_d)$

$$= \sum_{j=1}^{d} |x_j + y_j| \le \sum_{j=1}^{d} (|x_j| + |y_j|) = \|x\|_1 + \|y\|_1 .$$

$$|x_j + y_j| \underset{\Delta\, |\cdot|}{\le} |x_j| + |y_j|$$

Ex: Check that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^d$

Let's take now

$$\|(x_1 - - , x_d)\|_2 = \sqrt{\sum_{j=1}^{d} x_j^2}$$

Vanishing and homog. are evident.

About Δ ineq :

$$\|x + y\|_2 \overset{?}{\le} \|x\|_2 + \|y\|_2$$

$x = (x_1 \text{—} x_d)$, $y = (y_1, \text{—} y_d)$

$(x_1 \text{—} x_d) \quad (y_1, \text{—} y_d)$

$$\| \quad \|_2^2 \leq \left(\| \quad \|_2 + \| \quad \|_2\right)^2$$

$$\sum_j (x_j + y_j)^2 \leq \cancel{\sum_j x_j^2} + \cancel{\sum_j y_j^2} + \cancel{2}\left(\sum x_j^2\right)^{1/2}\left(\sum y_j^2\right)^{1/2}$$

$$= \cancel{x_j^2 + y_j^2} + \cancel{2} x_j y_j$$

$$\iff \quad \sum_j x_j y_j \leq \underbrace{\left(\sum_j x_j^2\right)^{1/2}}_{\|x\|} \underbrace{\left(\sum_j y_j^2\right)^{1/2}}_{\|y\|} \quad \text{(CS)}$$

$\forall \ (x_1 \text{—} x_d)$, $(y_1 \text{—} y_d)$

(Cauchy - Schwarz inequality)

Proof: We notice that if $\sum x_j^2 = 0$ or $\sum y_j^2 = 0$ the CS ineq is trivial.

Assume then $\sum x_j^2, \ \sum y_j^2 \neq 0$

$$\iff \quad \boxed{\sum_j \frac{x_j}{\|x\|_2} \cdot \frac{y_j}{\|y\|_2} \overset{?}{\leq} 1}$$

$$j \qquad \|x\|_2 \qquad \|y\|_2$$

Recall that $ab \leq \frac{1}{2}(a^2+b^2)$

$$\frac{x_j}{\|x\|_2} \cdot \frac{y_j}{\|y\|_2} \leq \frac{1}{2}\left(\frac{x_j^2}{\|x\|_2^2} + \frac{y_j^2}{\|y\|_2^2} \right)$$

$$\Rightarrow \sum_j \frac{x_j}{\|x\|_2} \cdot \frac{y_j}{\|y\|_2} \leq \frac{1}{2}\left(\underbrace{\sum_j \frac{x_j^2}{\|x\|_2^2}}_{= 1} + \underbrace{\sum_j \frac{y_j^2}{\|y\|_2^2}}_{= 1} \right)$$

$$\sum_j \frac{x_j^2}{\|x\|_2^2} = \cancel{\frac{1}{\|x\|_2^2}} \cdot \sum x_j^2, \qquad \sum x_j^2 = \cancel{\|x\|_2^2}$$

$$= 1. \qquad \blacksquare$$

Rmk: Because $ab = \frac{1}{2}(a^2+b^2) \iff (a-b)^2 = 0$

$$\iff a = b$$

In part, in the CS, $=$ holds $\iff$

$$\frac{x_j}{\|x\|_2} \cdot \frac{y_j}{\|y\|_2} = \frac{1}{2}\left(\frac{x_j^2}{\|x\|_2^2} + \frac{y_j^2}{\|y\|_2^2} \right)$$

$$\Updownarrow$$

$$x_j \qquad \ldots \qquad \|y\|_2 =: \lambda$$

$$\frac{x_j}{\| \ \|} = \frac{y_j}{\| \ \|} \iff y_j = \frac{\|y\|_2}{\|x\|_2} x_j \quad \forall j = 1 \ldots d$$

$$\iff \boxed{y = \lambda x}$$

So $=$ holds in CS $\iff y = \lambda x$.
$(y /\!/ x)$

$\blacksquare$

Ex. $V = \mathscr{C}([0,1]; \mathbb{R})$

On V we def.

$$\|f\|_\infty := \max_{x \in [0,1]} |f(x)|.$$

(Rmk: $\|\cdot\|_\infty$ is well def'd $\forall f \in \mathscr{C}([0,1])$ because of Weierstrass thm.)

$\|\cdot\|_\infty$ is a norm on V.

Sol:

1. vanishing: $\|f\|_\infty = 0 \iff \max\limits_{x \in [0,1]} |f(x)| = 0$

$$\max\{|f(x)| : x\in[0,1]\}$$

Because $|f|\in\mathcal{C} \Rightarrow \exists\, \hat{x}\in[0,1]$:

$$0\leq |f(x)| \leq \max|f(x)| = |f(\hat{x})| = 0$$
$\forall x\in[0,1]$

$$\Rightarrow f(x) = 0 \quad \forall x\in[0,1] \Rightarrow f = 0_V.$$

2. Homog (evid)

$$\|\lambda f\|_\infty = \max|\lambda f(x)| = |\lambda| \underbrace{\max|f(x)|}_{\|f\|_\infty} = |\lambda|\,\|f\|_\infty.$$

$$|\lambda f(x)| = |\lambda|\,|f(x)|$$

3. Triang ineq:

$$\|f+g\|_\infty = \max|f(x)+g(x)| \overset{?}{\leq} \|f\|_\infty + \|g\|_\infty.$$

$$|f(x)+g(x)| \underset{\Delta|\cdot|}{\leq} |f(x)| + |g(x)|$$

$$\|f\|_\infty = \max_{x\in[0,1]}|f(x)| \geq |f(x)| \quad \forall x\in[0,1]$$

$$\leq \|f\|_\infty + \|g\|_\infty$$

$$\Rightarrow \quad |f(x)+g(x)| \leq \|f\|_\infty + \|g\|_\infty \quad \forall x \in [0,1]$$

$$\Rightarrow \quad \underbrace{\max_{x\in[0,1]} |f(x)+g(x)|}_{\|f+g\|_\infty} \leq \|f\|_\infty + \|g\|_\infty$$

<u>Ex.</u> (Space of bded functions)

X be a generic set,

$$\underset{\substack{\uparrow \\ \text{bded}}}{B(X)} := \left\{ f : X \to \mathbb{R} \ : \ \sup_{x\in X} |f(x)| < +\infty \right\}$$

$B(X)$ is a vector sp. (check this)

and it can be endowed with a norm

$$\|f\|_\infty := \sup_{x\in X} |f(x)|.$$

$\|\cdot\|_\infty$ is a norm on $B(X)$. (check.)

<u>$L^p(X)$ spaces</u>

$$L^p(X) = \left\{ f : X \to \underset{(\mathbb{C})}{\mathbb{R}} \ : \ f \text{ meas}, \ \int_X |f|^p \, d\mu < +\infty \right\}$$

$(X, \mathcal{F}, \mu)$ be a meas sp. $\quad 1 \leq p < +\infty$.

On $L^p(X)$ is defined

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p}$$

<u>Rmk:</u> $X = \{1, \dots, d\}$ $\quad \mathcal{F} = \mathcal{P}(X)$, $\mu : \mathcal{F} \longrightarrow [0, +\infty]$

$f : X \to \mathbb{R}$

$\{1 \dots d\} \ni j \longmapsto f(j) = f_j$

$\mu(S) = \#$ elements of S

$\mu(X) = d$

$f \longleftrightarrow (f_1, \dots, f_d)$

$$\int_X g \, d\mu = \sum_{j=1}^{d} \underbrace{\int_{\{j\}} g \, d\mu}_{g(j)\,\mu(\{j\})} = \sum_{j=1}^{d} g(j) = \sum_{j=1} g_j$$

$$\Downarrow$$

$$\left(\int_X |f|^p \, d\mu \right)^{1/p} = \left(\sum_{j=1}^{d} |f_j|^p \right)^{1/p}$$

So for $p = 1, 2$ we get the $\|\cdot\|_1$, $\|\cdot\|_2$ of $\mathbb{R}^d$. $\boxtimes$

Let's check that $\|\cdot\|_p$ is a norm (more or less)

on $L^p(X)$ $\quad 1 \leq p < +\infty$.

$\boxed{p = 1}$ $\quad \|f\|_1 = \int_X |f| \, d\mu$.

1. Vanishing:

$$\|f\|_1 = 0 \implies \int_X |f| \, d\mu = 0 \overset{?}{\implies} f \equiv 0$$

$f(x) = 0 \quad \forall x \in X$

$$\|f\|_1 = 0 \Rightarrow \int_X |f|\,d\mu = 0 \Rightarrow f \text{ --- }$$

$$f(x) = 0 \quad \forall x \in X.$$

__Lemma 1__ $g \in L(X)$ meas, $g \geq 0$ s.t.

$$\int_X g\,d\mu = 0$$

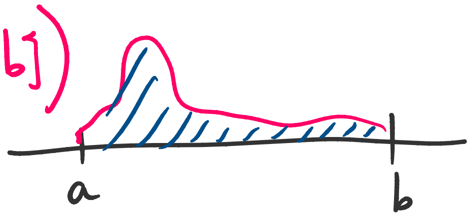
$$\Rightarrow g = 0 \quad \mu\text{. a.e. } x \in X.$$

A similar version of this with cont functs:

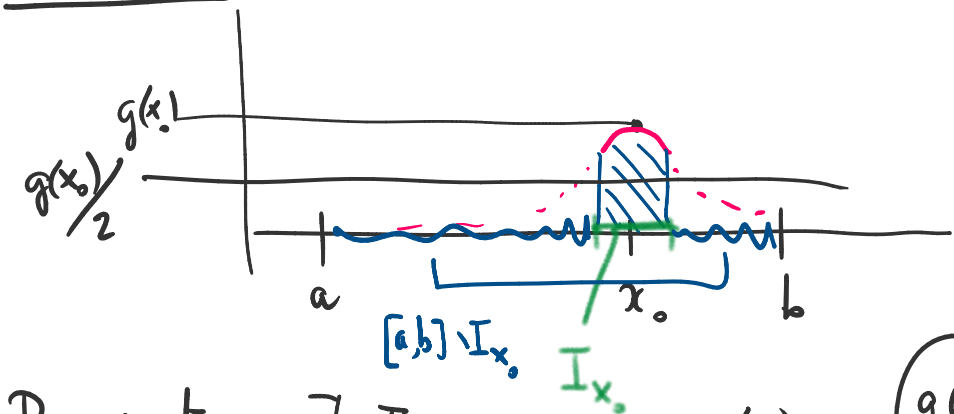
__Lemma 2:__ $g \in \mathscr{C}([a,b])$, $g \geq 0$ s.t.

$$\int_a^b g(x)\,dx = 0$$

$$\Rightarrow g \equiv 0 \quad (g(x) = 0 \ \forall x \in [a,b])$$



__Proof L2:__ If $\exists\, x_0 \in [a,b] : g(x_0) > 0$



By cont $\exists\, I_{x_0}$: $g(x) \geq \frac{g(x_0)}{2}$ $\forall x \in I_{x_0}$

$$\Rightarrow 0 = \int_a^b g \underset{g \geq 0}{=} \underbrace{\int_{[a,b]\setminus I_{x_0}} g}_{\substack{\text{IV} \\ 0}} + \int_{I_{x_0}} g(x)$$

$$\geq \int_{I_{x_0}} g(x)\, dx$$

$$\geq \int_{I_{x_0}} \frac{g(x_0)}{2}\, dx = \frac{\overset{\substack{0 \\ \wedge}}{g(x_0)}}{2} \underbrace{\lambda_1(I_{x_0})}_{\text{length of } I_{x_0}} > 0$$

Imposs! $\Rightarrow$ $g \equiv 0$. ∎

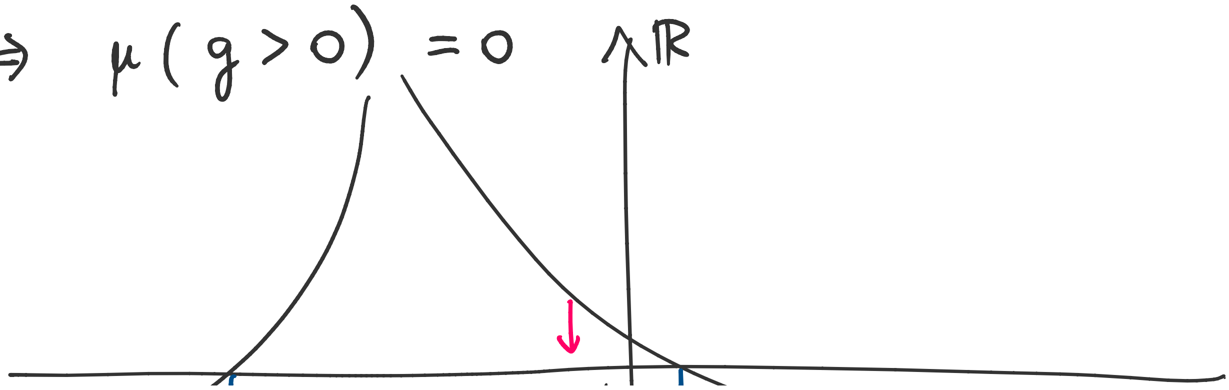
<u>Proof L1:</u> by Čeb ineq

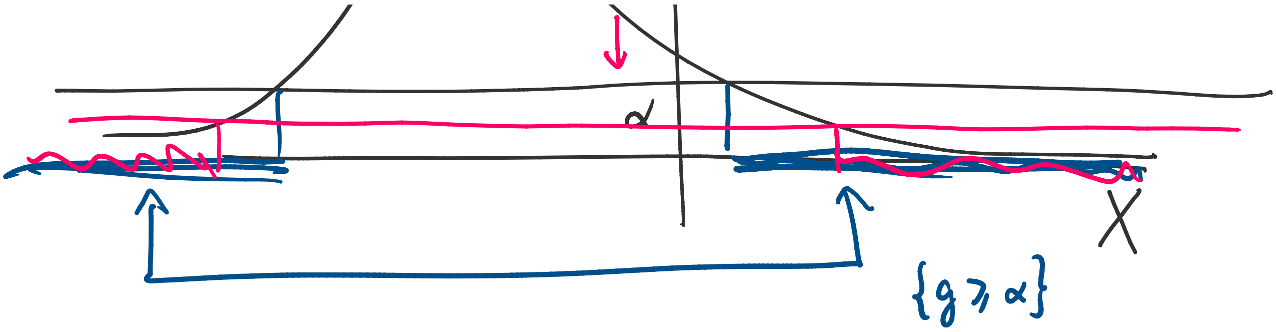
$$\mu(g \geq \alpha) \leq \frac{1}{\alpha} \int_X g\, d\mu$$

If $\int_X g = 0 \Rightarrow 0 \leq \mu(g \geq \alpha) \leq 0 \Rightarrow$

$$\Rightarrow \boxed{\mu(g \geq \alpha) = 0 \quad \forall \alpha > 0}$$

$$\Rightarrow \mu(g > 0) = 0$$





$$\{g>0\} = \bigcup_{\alpha>0} \{g \geqslant \alpha\}$$

$$\{g>0\} = \bigcup_{n\in\mathbb{N}} \underbrace{\{g \geqslant \tfrac{1}{n}\}}_{\mu=0}$$

$$\Rightarrow 0 \leqslant \mu(\{g>0\}) \leqslant \sum_n \underbrace{\mu(\{g \geqslant 1/n\})}_{=0} = 0$$

$$\Rightarrow \mu(\{g>0\}) = 0. \qquad \boxtimes$$

Ret. to vanishing:

$$\|f\|_1 = 0 \Leftrightarrow \int_X |f|\,d\mu = 0 \Leftrightarrow f = 0 \text{ a.e.}$$

Homog: $\|\lambda f\|_1 = \int_X \overset{|\lambda|\,|f(x)|}{\overset{\|}{|\lambda f(x)|}}\,d\mu$

$$= |\lambda| \int_X |f|\,d\mu = |\lambda|\,\|f\|_1$$

Δ-ineq: $\|f+g\|_1 = \int_X |f+g|\,d\mu$

$$|f(x)+g(x)| \leq |f(x)| + |g(x)|$$

$$\int_X |f+g| \, d\mu \leq \int_X |f| \, d\mu + \int_X |g| \, d\mu$$

$$\underset{\|f+g\|_1}{\parallel} \qquad \leq \qquad \underset{\|f\|_1}{\parallel} + \|g\|_1 . \qquad \blacksquare$$

$p = 2$

<u>Vanishing:</u> $\|f\|_2 = 0 \iff \int_X |f|^2 \, d\mu = 0$

$$\|f\|_2 = \left(\int_X |f|^2 d\mu \right)^{1/2} \quad \underset{L1}{\iff} |f|^2 = 0 \text{ a.e.}$$

$$\iff f = 0 \quad \text{a.e.}$$

<u>Homog.</u> $\|\lambda f\|_2 = \left(\int_X \underbrace{|\lambda f|^2}_{|\lambda|^2 |f|^2} d\mu \right)^{1/2}$

$$= |\lambda| \left(\int_X |f|^2 d\mu \right)^{1/2} = |\lambda| \, \|f\|_2$$

<u>Δ - ineq:</u> $f, g \in L^2$ Goal: $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

$$\iff \quad \|f+g\|_2^2 \leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2$$

$$\|f+g\|_2^2 = \int_X |f+g|^2 \, d\mu = \int_X (f+g)^2 \, d\mu = \int_X f^2 + g^2 + 2fg \, d\mu$$

$$\iff \quad \int_X fg \, d\mu \leq \|f\|_2 \|g\|_2 = \left(\int_X f^2\right)^{1/2} \left(\int_X g^2\right)^{1/2}$$

(CS)

The (CS) holds because of the same arguments we proved above. ∎

Do 3.5.1/2/3/4