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Ex 3.5.1 On $\mathbb{R}^2$ def

$$\|(x,y)\|_* := \left(\sqrt{|x|} + \sqrt{|y|}\right)^2 = \left(\sum |x_j|^{1/2}\right)^2$$

Is $\|\cdot\|_*$ a norm on $\mathbb{R}^2$? $\quad \left(\sum |x_j|^p\right)^{1/p}$

Sol: First, $\|\cdot\|_*$ is well defd on $\mathbb{R}^2$ ($\|(x,y)\|_*$ makes sense $\forall (x,y) \in \mathbb{R}^2$)

Let's the three fund props of every norm:

1. (vanishing) $\|(x,y)\|_* = 0 \iff \left(\sqrt{|x|} + \sqrt{|y|}\right)^2 = 0$

$$\iff \underbrace{\sqrt{|x|}}_{\geq 0} + \underbrace{\sqrt{|y|}}_{\geq 0} = 0$$

$$\iff \sqrt{|x|} = \sqrt{|y|} = 0 \iff x = y = 0$$

$$\iff (x,y) = (0,0) = 0_{\mathbb{R}^2}.$$

2. (homog) : $\|\lambda(x,y)\|_* \overset{?}{=} |\lambda| \, \|(x,y)\|_*$, where $\lambda(x,y) = (\lambda x, \lambda y)$

$\|\lambda(x,y)\|_*$ = by def

$$\left(\sqrt{|\lambda x|} + \sqrt{|\lambda y|}\right)^2$$

$$= \left(|\lambda|^{1/2}\left(\sqrt{|x|} + \sqrt{|y|}\right)\right)^2 = |\lambda| \, \|(x,y)\|_*.$$

$\forall \lambda \in \mathbb{R} \quad \forall (x,y) \in \mathbb{R}^2$

3. ($\Delta$ ineq)

$$\|(x_1,y_1) + (x_2,y_2)\|_* \leq \|(x_1,y_1)\|_* + \|(x_2,y_2)\|_*$$

$$\|$$

$$\|(x_1+x_2, y_1+y_2)\|_* \overset{?}{\leq}$$

$$\|$$

$$\left(\sqrt{|x_1+x_2|} + \sqrt{|y_1+y_2|}\right)^2 \underset{?}{\leq} \left(\sqrt{|x_1|} + \sqrt{|y_1|}\right)^2 + \left(\sqrt{|x_2|} + \sqrt{|y_2|}\right)^2$$

$$\Updownarrow$$

$$|x_1+x_2| + |y_1+y_2| + 2\sqrt{|x_1+x_2||y_1+y_2|} \overset{?}{\leq} |x_1|+|x_2| + |y_1|+|y_2| + 2\left(\sqrt{|x_1||y_1|} + \sqrt{|x_2||y_2|}\right)$$

Because

$$|x_1+x_2| \leq |x_1| + |x_2|$$
$$|y_1+y_2| \leq |y_1| + |y_2|$$

Let's see if

$$\sqrt{|x_1+x_2||y_1+y_2|} \leq \sqrt{|x_1||y_1|} + \sqrt{|x_2||y_2|}$$

$$\Leftrightarrow \quad |x_1+x_2||y_1+y_2| \overset{?}{\leq} |x_1||y_1| + |x_2||y_2| + 2\sqrt{|x_1||x_2||y_1||y_2|} \ ?$$

$$\wedge\!\!/$$

$$(|x_1|+|x_2|)(|y_1|+|y_2|)$$

$$\|$$

$$|x_1||y_1| + |x_1||y_2| + |x_2||y_1| + \ldots$$

$$\underbrace{|x_1||y_1|} + \underbrace{|x_2||y_2|} \neq \underbrace{|x_1||y_2| + |y_1||x_2|}$$

$\Delta$ would be true if

$$|x_1||y_2| + |y_1||x_2| \leq 2\sqrt{|x_1||x_2||y_1||y_2|}$$

$$\Updownarrow {}^2$$

$$|x_1|^2|y_2|^2 + |y_1|^2|x_2|^2 \overset{-}{\cancel{+}} 2|x_1||x_2||y_1||y_2| \leq \cancel{4|x_1||x_2||y_1||y_2|}$$

$$\Leftrightarrow \left( |x_1||y_2| - |y_1||x_2| \right)^2 \leq 0$$

May be $\Delta$ ineq is false...

$(1,0)$ $\quad \|(1,0)\|_* = 1$

$(0,1)$ $\quad \|(0,1)\|_* = 1$

$$4 = \|(1,0) + (0,1)\|_* = \|(1,1)\|_*$$

$$= (2)^2 = 4$$

$$\overset{?}{\leq} \|(1,0)\|_* + \|(0,1)\|_*$$

$$1 + 1 = 2$$

No!

$\Rightarrow$ $\Delta$ ineq false. $\blacksquare$

$\Rightarrow$ $\Delta$ ineq false. $\blacksquare$

FYI: We defined

$$L^p(X) = \left\{ f : X \to \underset{(\mathbb{C})}{\mathbb{R}} \ : \ f \text{ meas}, \ \int_X |f|^p d\mu < +\infty \right\}$$

$0 < p \le +\infty$.

The quantity

$$\|f\|_p := \left( \int_X |f|^p d\mu \right)^{1/p}$$

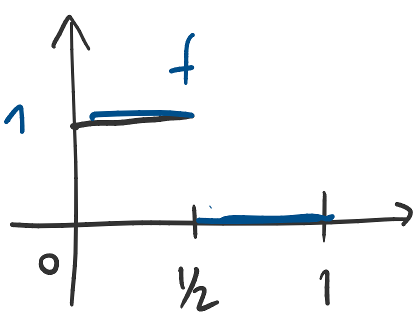
is a norm only for $p \geq 1$

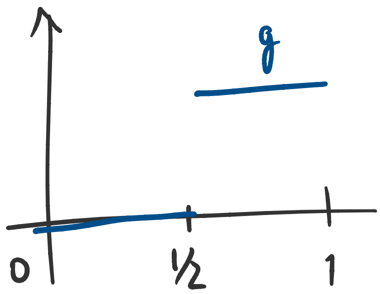
For inst, $p = 1/2$

$$\|f\|_{1/2} = \left( \int_X |f|^{1/2} d\mu \right)^2 \quad \left( \sim \left( \sum |x_j|^{1/2} \right)^2 \right)$$

Pb: $X = [0,1]$, $\mu =$ Leb meas.

Try to find $f, g \in L^{1/2}$ : $\|f+g\|_{1/2} > \|f\|_{1/2} + \|g\|_{1/2}$

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3.5.3 $X = \mathcal{C}^1([0,1]) = \{ f : [0,1] \to \mathbb{R}, \ f, f' \in \mathcal{C}([0,1]) \}$

On $X$ we consider

$\|\cdot\|_\infty$ $\quad \|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \max_{x\in[0,1]} |f(x)|$ $\quad f\in\mathcal{C}$

$$\|f\|_v := \|f\|_\infty + \|f'\|_1 = \|f\|_\infty + \int_0^1 |f'(x)|\,dx$$

$$\|f\|_* := \|f\|_\infty + \|f'\|_\infty .$$

i) Check $\|\ \|_*$ is a norm

ii) $\exists\ c, C > 0$ const. such that

$$\|f\|_\infty \leq c\|f\|_v \leq C\|f\|_* \quad \forall f\in X$$

iii) By using $f_k(x) = c_k \sin(k\pi x)$

$g_k(x) = c_k x^k$

show that $\nexists\ m, M > 0$ const s.t.

$$\|f\|_v \leq m\|f\|_\infty, \quad \|f\|_* \leq M\|f\|_v . \ \forall f\in X.$$

Sol: i) $\|f\|_* = \|f\|_\infty + \|f'\|_\infty$ it is well defd $\forall f\in X = \mathcal{C}^1$. To check that $\|\cdot\|_*$ is a norm we have to check vanish., homog, $\Delta$ ineq.

• vanish: $\|f\|_* = 0 \iff \underbrace{\|f\|_\infty}_{\geq 0} + \underbrace{\|f'\|_\infty}_{\geq 0} = 0$

$$\iff \|f\|_\infty = \|f'\|_\infty = 0$$

$$\iff f \equiv 0 \quad (f \in \mathcal{C})$$

. homog: $\|\lambda f\|_* = \|\lambda f\|_\infty + \|\underbrace{(\lambda f)'}_{\lambda f'}\|_\infty$

$$= |\lambda|\,\|f\|_\infty + |\lambda|\,\|f'\|_\infty = |\lambda|\,\|f\|_*.$$

• $\Delta$-ineq: $\|f+g\|_* = \|f+g\|_\infty + \|\underbrace{(f+g)'}_{= f'+g'}\|_\infty$

$$\overset{\Delta \text{ ineq } \|\ \|_\infty}{\leq} \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty$$

$$= \|f\|_* + \|g\|_*.$$

ii) $\exists c, C > 0$ : $\|f\|_\infty \leq c\,\|f\|_V \leq C\,\|f\|_*$.

$\|f\|_\infty \leq c\,\|f\|_V$

Trivial: $\|f\|_v = \|f\|_\infty + \|f'\|_1 \geq \|f\|_\infty$

$\Rightarrow \quad c = 1$

$\|f\|_v \overset{?}{\leq} C \|f\|_* \qquad \forall f \in X$

$\|f\|_v = \|f\|_\infty + \|f'\|_1$

$\|f\|_* = \|f\|_\infty + \|f'\|_\infty .$

The point is: how do we control

$$\|f'\|_1 \leq C \left( \|f\|_\infty + \|f'\|_\infty \right)$$

$$\|f\|_\infty = \max|f| \qquad \|f'\|_\infty = \max|f'|$$

$$\|f'\|_1 = \int_0^1 |f'(x)|\, dx$$

$$|f'(x)| \leq \|f'\|_\infty \qquad \forall x$$

$$\Rightarrow \int_0^1 |f'(x)|\, dx \leq \int_0^1 \|f'\|_\infty\, dx = \|f'\|_\infty$$

$$\|f'\|_1 \leq \|f'\|_\infty$$

$$\Rightarrow \|f\|_v = \|f\|_\infty + \|f'\|_1 \leq \|f\|_\infty + \|f'\|_\infty = \|f\|_*$$

$$\Rightarrow C = 1.$$

$f_k = c_k \sin(k\pi x) \qquad g_k = c_k x^k$

iii) $f_k = c_k \sin(k\pi x)$ $\qquad g_k = c_k x^k$

Pb: $\nexists \; m, M$ :

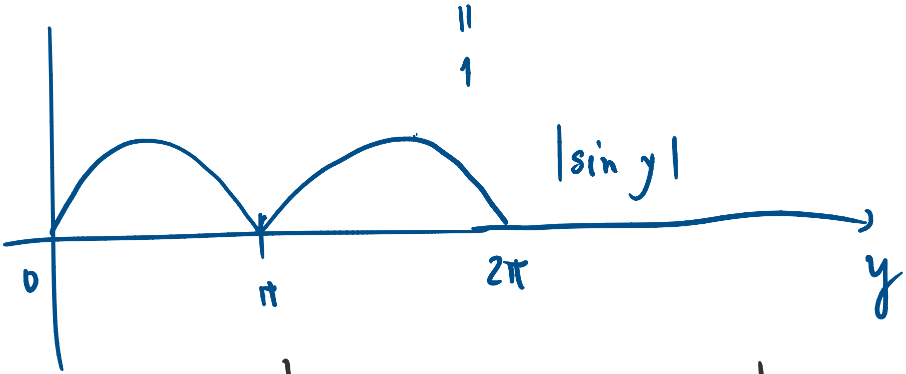
$$\|f\|_V \leq m \|f\|_\infty \qquad \forall f \in X$$

$$\|f_k\|_V = \|f_k\|_\infty + \|f_k'\|_1 .$$

$|c_k| |\sin(k\pi x)|$

$f_k = c_k \sin(k\pi x) \qquad \|f_k\|_\infty = \max\limits_{x \in [0,1]} |c_k \sin(k\pi x)|$

$$= |c_k| \max_{x\in[0,1]} \underbrace{|\sin(k\pi x)|}_{=1} = |c_k|$$




$$\|f_k'\|_1 = \int_0^1 |f_k'(x)|\, dx = \int_0^1 \underbrace{|c_k| k\pi} |\cos(k\pi x)| dx$$

$$f_k' = c_k \cos(k\pi x) \cdot k\pi$$

$$= k\pi |c_k| \int_0^1 |\cos(\underbrace{k\pi x}_{y})| \; dx$$

$y = k\pi x$

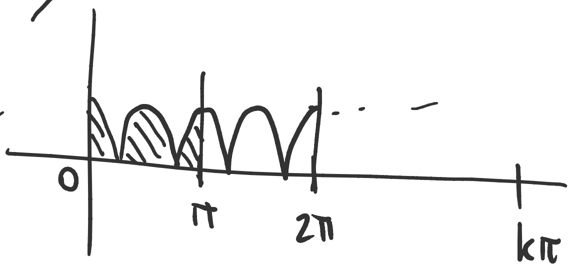
$0 \leq x \leq 1 \qquad x = \frac{1}{k\pi} y \qquad dx = \frac{dy}{k\pi}$

$0 \leq x \leq 1 \qquad x = \frac{1}{k\pi} y \qquad dx = \frac{dy}{k\pi}$

$0 \leq y = k\pi x \leq k\pi$

$$= \cancel{k\pi} |c_k| \int_0^{k\pi} |\cos y| \frac{dy}{\cancel{k\pi}}$$

$$= |c_k|\, k \int_0^{\pi} |\cos y|\, dy$$




$$= \mathbb{K} |c_k| k.$$

$$\Rightarrow \|f_k\|_v = |c_k| + \mathbb{K} |c_k| k$$

$$= |c_k| (1 + \mathbb{K} k)$$

and because $\|f_k\|_\infty = |c_k|$

if $\exists m : \quad \|f\|_v \leq m \|f\|_\infty \quad \forall f$

$$\Downarrow$$

$$\cancel{|c_k|}(1 + \mathbb{K} k) = \|f_k\|_v \leq m \|f_k\|_\infty = m \cdot \cancel{|c_k|}$$

(take $c_k = 1$)

$$\Rightarrow (1 + \mathbb{K} k) \leq m \qquad \forall k \in \mathbb{N}$$

impossible because l.h.s $\to +\infty$ as $k \to +\infty$.

<u>Pb</u>: $\exists M$ : $\|f\|_* \leq M \|f\|_v$

(use $f = g_k = c_k x^k$)

$$\|c_k \#^k\|_* = \cancel{|c_k|} \; \|\#^k\|_*$$

$$\|c_k \#^k\|_v = \cancel{|c_k|} \; \|\#^k\|_v$$

$c_k \equiv 1$

$g_k = x^k$

$$\|\#^k\|_* = \|\#^k\|_\infty + \|(\#^k)'\|_\infty$$

$$= \max_{x\in[0,1]} |x^k| + \max_{x\in[0,1]} |k x^{k-1}|$$

$$\parallel$$

$$1 \qquad + |k| \cdot 1 = 1 + k.$$

$$\|\#^k\|_v = \|\#^k\|_\infty + \|\#^k\|_1$$

$$\parallel$$

$$1 \qquad + \int_0^1 |k x^{k-1}| \, dx$$

$$\parallel$$

$$\int_0^1 k x^{k-1} \, dx$$

$$\parallel$$

$$\left( \begin{array}{c} \| \\ (x^k)' \\ = \left[x^k\right]_\partial' = 1 \end{array} \right.$$

$$= 1 + 1 = 2$$

Therefore if $\exists M : \|f\|_* \leq M \|f\|_V \quad \forall f \in X$

$$\Downarrow$$

$$1 + k = \|g_k\|_* \leq M \underbrace{\|g_k\|_V}_{= 2}$$

$$\Rightarrow \quad 1 + k \leq 2M \quad \forall k \in \mathbb{N} \quad \text{imposs.}$$

$\blacksquare$

What do we do once we have a normed space?

<u>Def:</u> Let $(V, \|\cdot\|)$ be a normed space. Given $(f_n) \subset V$ we say that $f_n \xrightarrow{\|\cdot\|} f$ if

$$\|f_n - f\| \longrightarrow 0 \qquad n \to +\infty.$$

<u>Rmk:</u> Convergence depends by the $\|\cdot\|$ you use:

[illegible] sea could change character.

Rmk: Convergence depends [illegible]

by changing norm a seq could change character.

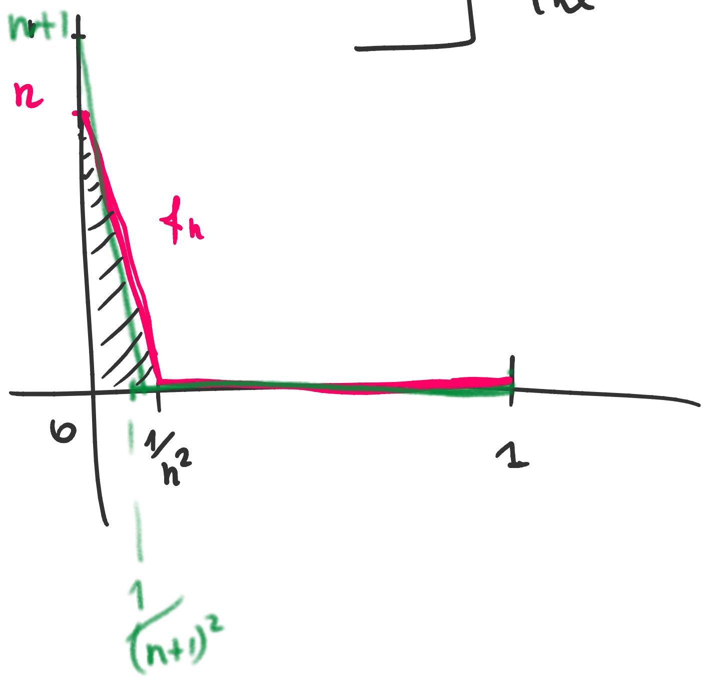
Ex 3.2.2 On $V = \mathcal{C}([0,1])$ we consider two norms: $\|\cdot\|_\infty$, $\|\cdot\|_1$

Let $(f_n) \subset V$

$$f_n(x) = \begin{cases} n - n^3 x & 0 \le x \le 1/n^2 \\ 0 & \frac{1}{n^2} \le x \le 1 \end{cases}$$

Discuss conv of $(f_n)$ in $\|\cdot\|_\infty$ and $\|\cdot\|_1$.

⌈ $a_n = \dfrac{n^2 + n - 1}{n^3 + 2n^2 + 5}$ ⌋ Let's give an idea to the $f_n = f_n(x)$.




Is $(f_n)$ convergent in $\|\cdot\|_\infty$ / $\|\cdot\|_1$ ?

Claim: $f_n \xrightarrow{\|\cdot\|_1} 0$

$$\|f_n - 0\|_1 = \|f_n\|_1 = \int_0^1 |f_n(x)|\, dx$$

$$\|f_n - 0\|_1 = \|f_n\|_1 = \int_0^1 |f_n(x)|\,dx$$

$$\overset{f_n \geq 0}{=} \int_0^1 f_n = \frac{1}{2} \cdot \frac{1}{n^2} \cdot n = \frac{1}{2n} \longrightarrow 0$$

<u>Claim</u>: $f_n \xrightarrow{\|\cdot\|_\infty} 0$ false!

To be true, we need $\|f_n - 0\|_\infty \to 0$

But $\|f_n - 0\|_\infty = \|f_n\|_\infty = \max\limits_{x \in [0,1]} |f_n(x)| = n \to +\infty.$

<u>Pb</u>: Can $f_n \xrightarrow{\|\cdot\|_\infty} f$ ? and, in this case, what is $f$ ? No!

<u>Prop</u>: If $(f_n) \subset V$, $(V, \|\cdot\|)$ normed space, converges $\Rightarrow$ $(\|f_n\|)$ must be bounded!

<u>Proof</u>: If $\|f_n - f\| \to 0 \Rightarrow$

$$\Rightarrow \exists M : \|f_n - f\| \leq M \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \|f_n\| = \|f_n - f + f\| \leq \underbrace{\|f_n - f\|}_{\leq M} + \|f\|$$

$$\leq M + \|f\| \quad \forall n \in \mathbb{N} \qquad \blacksquare$$

Do 3.5.2. / 3.5.7