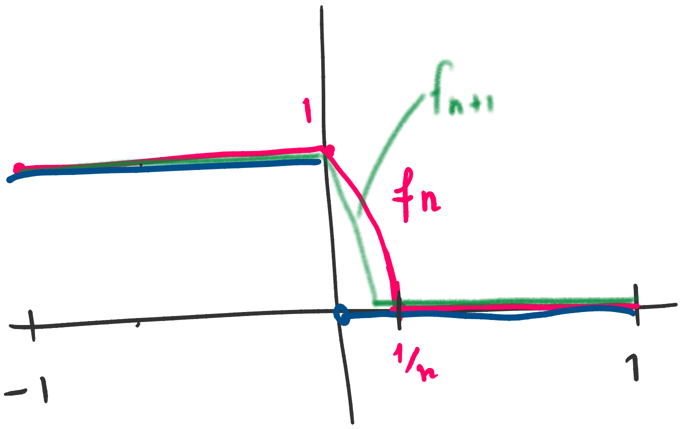
Ex. 3.5.7. $f_n(x) = 1 \cdot \mathbb{1}_{[-1,0]}(x) + \sqrt{1-nx}\, \mathbb{1}_{[0,1/n]}(x)$

$x \in [-1,1]$

Discuss conv of (f_n) in $L^2([-1,1])$.

Sol:



$f(x) = \mathbb{1}_{[-1,0]}$. **Guess:** $f_n \xrightarrow{L^2} f$.

To check this we have to compute

$$\| f_n - f \|_2^2 = \int_{-1}^{1} |f_n - f|^2 \, dx$$

$$= \int_0^{1/n} \left|\sqrt{1-nx}\right|^2 dx$$

$$= \int_0^{1/n} (1 - nx) \, dx$$

$$= \frac{1}{n} - n \int_0^{1/n} x \, dx$$

$\qquad\qquad\qquad\qquad \| \quad 1/n$

$$n \qquad \dot{0}$$

$$\| \quad \left[\frac{x^2}{2}\right]_0^{1/n}$$

$$= \frac{1}{n} - \cancel{n} \cdot \frac{1}{2} \frac{1}{n^2}$$

$$= \frac{1}{2n}.$$

$$\Rightarrow \|f_n - f\|_2 = \frac{1}{\sqrt{2n}} \to 0 \Rightarrow f_n \xrightarrow{L^2} f \quad \boxtimes$$

Rmk: Can we have 2 or more limits?

Prop: The limit, if it $\exists$, is unique.

Proof: If $f_n \xrightarrow{\|\cdot\|} f \Rightarrow \|f_n - f\| \to 0$

$f_n \xrightarrow{\|\cdot\|} g \Rightarrow \|f_n - g\| \to 0$

$$0 \leq \|f - g\| = \|(f - f_n) + (f_n - g)\|$$

$$\leq \|f - f_n\| + \|f_n - g\| \longrightarrow 0$$

$$\Rightarrow \|f - g\| = 0 \Rightarrow f = g. \quad \boxtimes$$

Rmk: If we have the $L^p(X)$ space $1 \leq p < +\infty$.

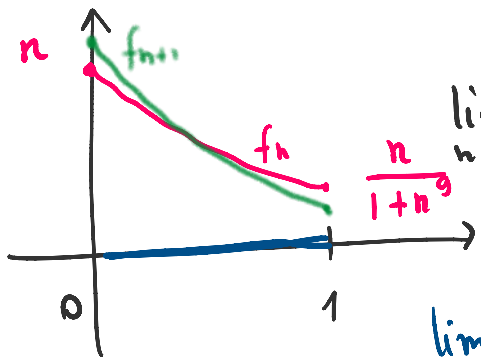
this proves that

$$\left.\begin{array}{l} f_n \to f \\ f_n \to g \end{array}\right\} \Rightarrow f = g \text{ a.e.} \qquad \boxtimes$$

<u>Ex 3.5.8</u> . $f_n(x) = \dfrac{n}{1+n^9x^3}$ $\quad x \in [0,1]$.

Discuss conv of (f_n) in $L^p([0,1])$ $\quad p \in [1,+\infty[$.

<u>Sol:</u>



$$\lim_{n\to+\infty} f_n(x) = \begin{cases} x=0 & \lim n = +\infty \\ x\in]0,1] \text{ fixed} & = 0 \end{cases}$$

$x \in [0,1]$

$\lim |f_n(x) - f(x)| = 0$ (point wise lim)

Idea : $f_n \xrightarrow{L^p} 0$?

$$\Updownarrow$$

$$\|f_n - 0\|_p^p = \int_0^1 |f_n(x) - \cancel{0}|^p \, dx$$

$$= \int_0^1 \left(\frac{n}{1+n^9x^3}\right)^p dx$$

$$= \int_0^1 \frac{n^p}{(1+n^9x^3)^p} dx \qquad f_n^p$$

<u>Pb:</u> $\lim_{n\to+\infty} \|f_n - 0\|_p^p = \lim_{n\to+\infty} \int_0^1 \frac{n^p}{(1+n^9x^3)^p} dx$

$$\stackrel{?}{=} 0$$

$$= \lim_n n^p \int_0^1 \frac{1}{(1+\underbrace{n^9 x^3}_{(n^3x)^3})^p}\,dx$$

$y = n^3 x$

$dx = \frac{1}{n^3}\,dy$

$$= \lim_n n^p \int_0^{n^3} \frac{1}{(1+y^3)^p}\,\frac{1}{n^3}\,dy$$

$$= \lim_n n^{p-3} \underbrace{\int_0^{n^3} \frac{1}{(1+y^3)^p}\,dy}$$

$$\downarrow$$

$$\int_0^{+\infty} \frac{1}{(1+y^3)^p}\,dy < +\infty$$

$$\sim \frac{1}{y^{3p}} \qquad p \geq 1$$

$$n^{p-3} \longrightarrow \begin{cases} p-3>0 & +\infty \\ p-3=0 & 1 \\ p-3<0 & 0 \end{cases}$$

$$\Rightarrow \lim_n \|f_n - 0\|_p^p = \begin{cases} +\infty & p>3 \leftarrow \\ \int_0^{+\infty} \frac{1}{(1+y^3)^3} =: C & p=3 \leftarrow \\ 0 & 1 \leq p < 3 \end{cases}$$

$$\|f_n\|_p^p$$

$\Rightarrow$ $f_n \xrightarrow{L^p} 0 \qquad 1 \leq p < 3$

$\|f_n\|_p \longrightarrow +\infty \qquad p>3$

$\Downarrow$

(f_n) is unbded in L^p $\forall p>3$ $\Rightarrow$ (f_n) is not conv in

$\Downarrow$

(f_n) is unbded in L^p $\forall p>3$ $\Rightarrow$ (f_n) is not conv in L^p $p>3$

$\boxed{p=3}$ In this case we proved

$$\|f_n\|_p = \|f_n - 0\|_p \longrightarrow C^{1/p} > 0.$$

$$\boxed{f_n \overset{L^3}{\not\to} 0}$$

<u>Rmk:</u> It is possible to prove that

if $f_n \xrightarrow{L^p(X)} f$ $\cancel{\overset{\Rightarrow}{\Leftarrow}}$ $f_n \to f$ a.e.

$$\Rightarrow \exists (f_{n_k}) \subset (f_n) : f_{n_k}(x) \to f(x) \text{ a.e. } x \in X$$

With this we can conclude that (f_n) is not conv in L^3: indeed if $f_n \xrightarrow{L^3} f$

$\Downarrow$

$\exists (f_{n_k}) \subset (f_n)$: $f_{n_k}(x) \to f(x)$ a.e.

$\downarrow$

0 a.e. because of the initial arg.

$f = 0$ a.e. $\Leftarrow$

$$\text{if } f_n \overset{\Downarrow}{\xrightarrow{L^3}} f \Rightarrow f_n \xrightarrow{L^3} 0 \quad \text{but this is false}$$

$$\|f_n - 0\|_3 \longrightarrow C^{1/3} > 0$$

$\boxtimes$

<u>Def:</u> Given two norms on V, $\|\cdot\|, \|\cdot\|_*$, we say that $\|\cdot\|_*$ is <u>stronger</u> than $\|\cdot\|$, if

$$\exists\, C > 0 : \quad \|f\| \leq C \|f\|_* \quad \forall f \in V.$$

We say that $\|\cdot\|, \|\cdot\|_*$ are equivalent if

$$\exists\, c, C > 0 : \quad c\|f\|_* \leq \|f\| \leq C \|f\|_* \quad \forall f \in V.$$

<u>Prop:</u> If $\|\cdot\|_*$ is stronger than $\|\cdot\|$ $\Rightarrow$

$$(*) \quad f_n \xrightarrow{\|\cdot\|_*} f \quad \Rightarrow \quad f_n \xrightarrow{\|\cdot\|} f$$

In part, if $\|\cdot\|, \|\cdot\|_*$ are equiv

$$f_n \xrightarrow{\|\cdot\|_*} f \quad \Leftrightarrow \quad f_n \xrightarrow{\|\cdot\|} f.$$

<u>Proof:</u> $(*)$ If $f_n \xrightarrow{\|\cdot\|_*} f \Leftrightarrow \|f_n - f\|_* \to 0.$

$$\Rightarrow 0 \leq \|f_n - f\| \leq \underbrace{C \|f_n - f\|_*}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$
$$0 \qquad\qquad\qquad 0$$

$$\Rightarrow \quad \|f_n - f\| \longrightarrow 0 \; . \iff f_n \xrightarrow{\|.\|} f \, . \quad \blacksquare$$

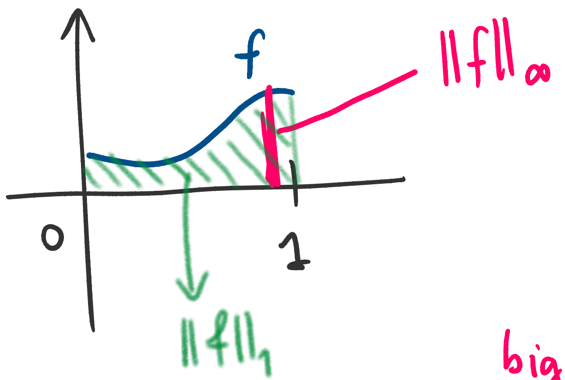
Prop: If $\dim V < +\infty$ then all norms are equivalent.

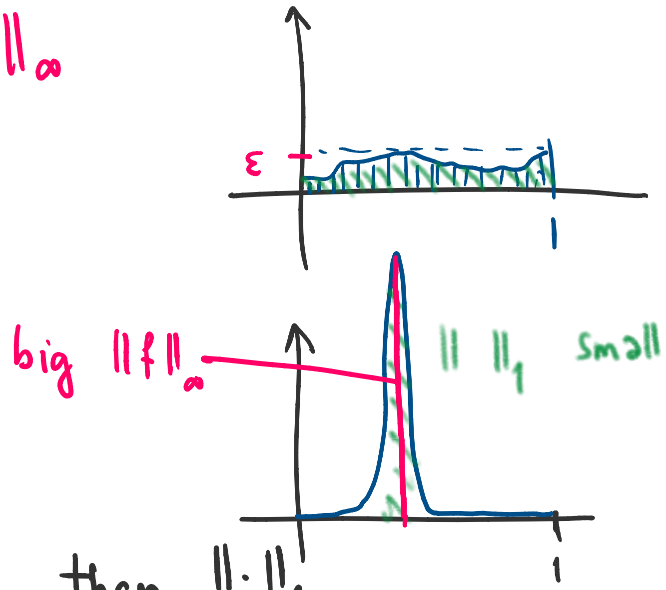
Rmk: If $\dim V = +\infty$ in general two norms are never equivalent.

Example: $V = \mathcal{C}([0,1])$

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$$

$$\|f\|_1 = \int_0^1 |f(x)|\,dx$$





Guess: $\|.\|_\infty$ is stronger than $\|.\|_1$

$$\iff \exists C : \quad \|f\|_1 \leq C \|f\|_\infty . \quad \forall f \in \mathcal{C}([0,1])$$

$$\|f\|_1 = \int_0^1 |f(x)|\,dx \le \int_0^1 \|f\|_\infty\,dx = \|f\|_\infty$$

$$\int_0^1 |f(x)|\,dx \ge \|f\|_\infty$$

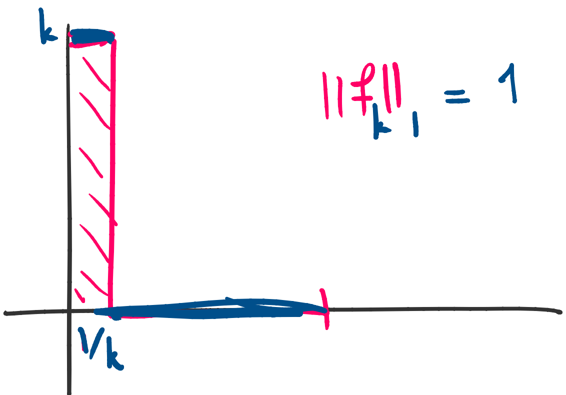
$$\|f\|_1 \le 1\cdot\|f\|_\infty \quad \forall f \in \mathcal{C}.$$

Guess: $\|\cdot\|_1$ is not str. than $\|\cdot\|_\infty$.

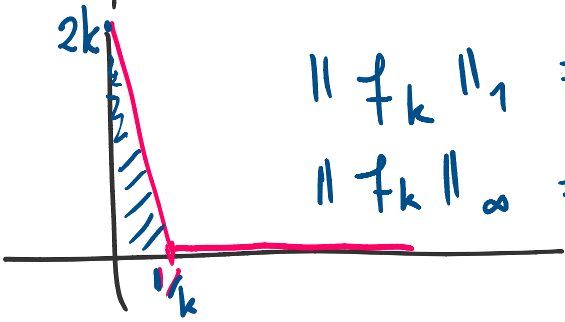
$$\left(\Leftrightarrow \nexists\, C : \underbrace{\|f\|_\infty}\le C\underbrace{\|f\|_1} \quad \forall f \in \mathcal{C}. \right)$$

Let's construct a counterexample by taking

$$f_k : \quad \|f_k\|_1 = 1, \quad \|f_k\|_\infty = k$$



$\|f_k\|_1 = 1$



$\|f_k\|_1 = 1$

$\|f_k\|_\infty = 2k$

$\Rightarrow \nexists\, C : \|f\|_\infty \le C\|f\|_1$

otherwise

$2k \le C \quad \forall k \in \mathbb{N}$

imposs! $\blacksquare$

Ex. 3.5.2. $\quad V = \mathscr{C}^1([0,1])$

i) $\|f\|_* := \|f\|_\infty + \|f'\|_\infty$

ii) $\|f\|_{**} := \|f'\|_\infty$

iii) $\|f\|_{***} := |f(0)| + \|f'\|_\infty$

iv) $\|f\|_{****} := |f(1)| + \int_0^1 |f'(x)|\,dx$

Q1: Which is really a norm?

i) yes.

ii) Let's check the key props of a norm:

- vanishing: $\|f\|_{**} = 0 \iff \|f'\|_\infty = 0$

$$\iff f' \equiv 0 \iff f \equiv C \quad \text{false!}$$

iii) • vanishing: $\|f\|_{***} = 0 \iff |f(0)| + \|f'\|_\infty = 0$

$$\iff \begin{cases} |f(0)| = 0 & \Rightarrow f(0) = 0 \\ \|f'\|_\infty = 0 & \Rightarrow f \equiv C \end{cases} \Rightarrow f \equiv 0$$

- homog $\wedge$ triang ineq are straightforward

iv) . vanish: $\|f\|_{****} = 0 \iff \underset{\geq 0}{|f(1)|} + \underbrace{\int_0^1 |f'|}_{\geq 0} = 0$

$$\iff \begin{cases} |f(1)| = 0 \\ \int_0^1 \underbrace{|f'|}_{\in \mathscr{C}} = 0 \end{cases} \Rightarrow \begin{cases} f(1) = 0 \\ |f'| \equiv 0 \end{cases} \Rightarrow \begin{cases} f' \equiv 0 \\ f(1) = 0 \end{cases}$$

$\int_0 \underbrace{|f|}_{\substack{\| \\ g \geq 0}} \in \mathcal{C}$ $\qquad$ $|f| \equiv 0$ $\qquad$ $|f(1)| = 0$

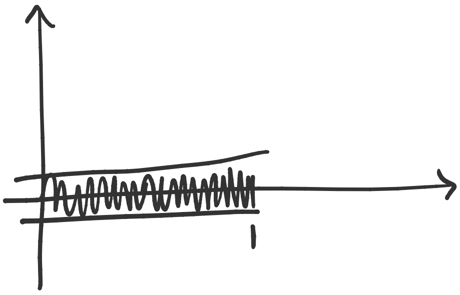
$$\Downarrow$$

$$f \equiv 0.$$

• hom $\wedge$ Δ ineq straightf.

<u>Q2:</u> $f_n(x) = \frac{1}{n}\sin(n^2 x)$. $\quad x \in [0,1]$

Discuss conv of (f_n) in $\|\cdot\|_*$, $\| \ \|_{***}$, $\| \ \|_{****}$



$$\|f_n\|_* =$$

$$= \underbrace{\|f_n\|_\infty}_{\substack{\| \\ 1/n}} + \|f_n'\|_\infty$$

$$\max_{x\in[0,1]} |f_n(x)| = \frac{1}{n} \overbrace{\max_{y\in[0,n^2]} |\sin y|}^{\substack{1 \\ \|}} = 1/n$$

$$f_n' = \frac{1}{n}\cos(n^2 x)\cdot n^2 = n\cos(n^2 x)$$

$\|f_n'\|_\infty = n \quad \Rightarrow \|f_n\|_* = \frac{1}{n} + n \longrightarrow +\infty$

$\Rightarrow (f_n)$ is unbdd in $\|\cdot\|_*$

$\Rightarrow (f_n)$ does not conv in $\|\cdot\|_*$.

$$\|f_n\|_{***} = 0 + n \longrightarrow +\infty$$

$\Rightarrow (f_n)$ not conv. in $\|\cdot\|_{***}$.

$$\|f_n\|_{****} = |f_n(1)| + \int_0^1 |f_n'(x)|\,dx$$

$$= \left|\frac{1}{n}\sin(n^2)\right| + n\int_0^1 |\cancel{n}\cos(n^2x)|\,dx$$

$$\overset{\shortparallel}{O_n}$$

$y = n^2 x$

$dx = \frac{1}{n^2}\,dy$

$$= \overset{0}{\overset{\uparrow}{O_n}} + n\int_0^{n^2} |\cos y| \frac{1}{n^2}\,dy$$

$$= \overset{0 \nearrow}{O_n} + \underbrace{\frac{1}{n}\int_0^{n^2} |\cos y|\,dy}_{\text{unbded}}$$

$$\int_0^{n^2 \to +\infty} |\cos y|\,dy \to \int_0^{+\infty} |\cos y|\,dy = +\infty$$

$$\frac{\int_0^{n^2} |\cos y|\,dy}{n} \overset{\hat{H}}{=} \frac{|\cos n^2|\,2n}{1}$$

$$\partial_t\left(\int_c^t g(y)\,dy\right) = g(t)$$

$\in \mathcal{C}$

$$\Rightarrow \|f_n\|_{****} \text{ unbded} \Rightarrow f_n \overset{\|\cdot\|_{****}}{\not\to}$$

Q3: relations among norms.

$$\|f\|_* = \|f\|_\infty + \|f'\|_\infty$$

$$\|f\|_* \geq \|f\|_{***}$$

$$\|f\|_{***} = |f(0)| + \|f'\|_\infty$$

$$\|f\|_{***} \leq \|f\|_*$$

$|f(x)|$ $\quad$ $|f(0)|$ $\quad$ $\max |f'|$

$$f(x) - f(0) = \int_0^x f'(t)\,dt$$

$$f(x) = f(0) + \int_0^x f'(t)\,dt$$

$$\Rightarrow |f(x)| \leq |f(0)| + \left|\int_0^x f'(t)\,dt\right|$$

$$\leq \int_0^x \underbrace{|f'|}_{\leq \|f'\|_\infty} dt \leq \|f'\|_\infty \cdot x$$

$$\Rightarrow |f(x)| \leq |f(0)| + \|f'\|_\infty \cdot x$$

$$\Rightarrow \|f\|_\infty = \max_{x\in[0,1]} |f(x)| \leq |f(0)| + \|f'\|_\infty .$$

$$\Rightarrow \|f\|_* \leq |f(0)| + 2\|f'\|_\infty \leq 2\left(|f(0)| + \|f'\|_\infty\right)$$

$$\leq 2\|f\|_{***} .$$

Discuss $\|\cdot\|_*$, $\|\ \|_{****}$ and $\|\cdot\|_{***}$, $\|\cdot\|_{****}$

Pb: Is there a characteristic cond for $\exists$ of $\lim_n f_n$?

2. If (f_n) is convergent $\Rightarrow$ (f_n) is a Cauchy seq

Prop: If (f_n) is convergent $\Rightarrow$ (f_n) is a Cauchy seq

that is:

$$\forall \varepsilon>0 \quad \exists N : \quad \|f_n - f_m\| \leq \varepsilon \qquad \forall n,m \geq N.$$

Proof: If $f_n \xrightarrow{\|\cdot\|} f \Rightarrow \|f_n - f\| \to 0$

$\Leftrightarrow \forall \varepsilon>0 \quad \exists N : \|f_n - f\| \leq \varepsilon \qquad \forall n \geq N.$

If $n,m \geq N$,

$$\|f_n - f_m\| = \|(f_n - f) + (f - f_m)\|$$

$$\leq \|f_n - f\| + \|f - f_m\| \leq 2\varepsilon \qquad \blacksquare$$