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Banach Spaces

We have seen that, if $(f_n) \subset V$ is a conv. seq in $(V, \|\cdot\|)$ normed space then (f_n) fulfills the Cauchy prop

$$\boxed{\forall \varepsilon > 0 \quad \exists N : \quad \|f_n - f_m\| \leq \varepsilon \quad \forall n, m \geq N}$$

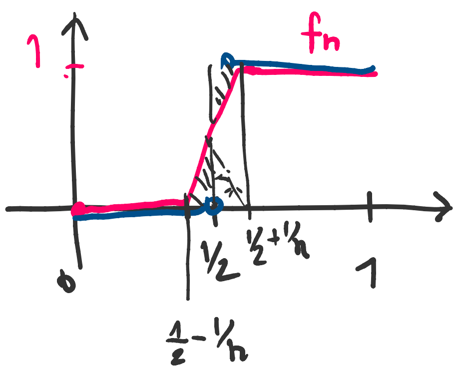
Pb: Is this cond also sufficient to ensure (f_n) is conv ?

In gen. the answer is NO.

Example $\quad V = \mathcal{C}([0,1]) \; [\subset L^1([0,1])]$

$$\|f\|_1 := \int_0^1 |f|$$

Let's see an example of a C. seq which is not conv. in V.



$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq 1/2 - 1/n \\ 2n\left(x - \left(\frac{1}{2} - 1/n\right)\right) & \frac{1}{2} - \frac{1}{n} \leq x \leq \frac{1}{2} + 1/n \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

$$f_n \overset{L^1}{\longrightarrow} 1_{[1/2, 1]}$$

$(f_n) \subset \mathcal{C}([0,1])$.

Let's check that

$$\| f_n - \mathbb{1}_{[1/2,1]} \|_1 \longrightarrow 0$$

$$\|$$

$$\int_0^1 | f_n - \mathbb{1}_{[1/2,1]} | \, dx = \int_{1/2-1/n}^{1/2+1/n} | \qquad |$$

$$= \frac{1}{2}\frac{2}{n} \cdot \frac{1}{2} = \frac{1}{2n} \longrightarrow 0$$

$$\Rightarrow \quad f_n \xrightarrow{L^1([0,1])} \mathbb{1}_{[1/2,1]}$$

in part: (f_n) is a C. seq resp. to $\|\cdot\|_1$

$$\forall \varepsilon > 0 \quad \exists N : \quad \|f_n - f_m\|_1 \leq \varepsilon \quad \forall n, m \geq N.$$

Claim: (f_n) cannot be conv in $\mathcal{C}([0,1])$.
(resp to $\|\cdot\|_1$)

$$\nexists f \in \mathcal{C}([0,1]) : \quad f_n \xrightarrow{\|\cdot\|_1} f.$$

We proved that

$$f_n \xrightarrow{\|\cdot\|_1} \mathbb{1}_{[1/2,1]} \notin \mathcal{C}([0,1])$$

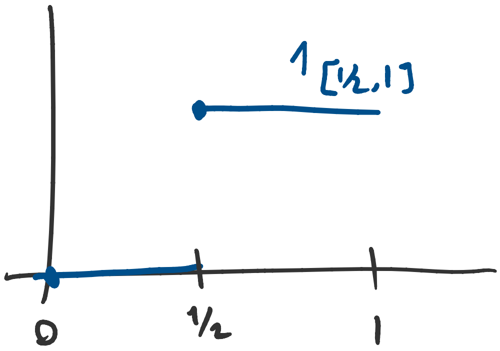
Is it possible that $f_n \xrightarrow{\|\cdot\|_1} g \in \mathcal{C}([0,1])$?

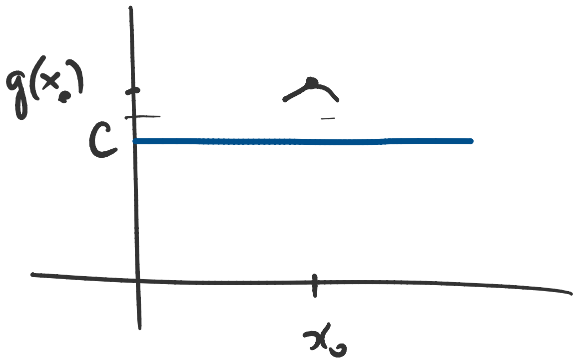
If $\exists g \in \mathcal{C}([0,1]) : \quad f_n \xrightarrow{\|\cdot\|_1} g$

$\cap \qquad \|\cdot\|_1 |$

If $\exists\, g \in C([0,1])$: Th , g

$$\underset{L^1([0,1])}{\cap} \qquad \|\cdot\|_1 \downarrow \quad 1_{[1/2,1]} \qquad \Rightarrow \qquad g = 1_{[1/2,1]} \ \text{a.e. } [0,1]$$





$$\left.\begin{array}{l} g = C \ \text{a.e.} \\ g \in C([0,1]) \end{array}\right\} \Rightarrow g \equiv C$$

Indeed if

$g(x_0) \neq C$

$$\Rightarrow \exists\, I_{x_0} : \ g(x) \neq C \ \ \forall x \in I_{x_0}$$

$$\Rightarrow \{g \neq C\} \supset I_{x_0} \Rightarrow \lambda(\{g \neq C\}) \geq \underset{0}{\underset{\vee}{\lambda(I_{x_0})}}$$

$\Rightarrow g = C$ a.e. is false !

So if $g \overset{\text{a.e.}}{=} 1_{[1/2,1]} \Rightarrow g \overset{\text{a.e.}}{=} 0$ on $[0,1/2]$ $\Rightarrow g \equiv 0$ on $[0,1/2]$

$g \overset{\text{a.e.}}{=} 1$ on $[1/2,1]$

$$g \overset{a.e.}{=} 1 \quad \text{on } [1/2, 1]$$
$$\Downarrow$$
$$g \equiv 1 \quad \text{on } [1/2, 1]$$

$\Rightarrow g$ is not cont at $x = 1/2$

$\blacksquare$

<u>Def</u>: A normed space is called <u>Banach sp</u> if every C. seq is conv.

A fund fact to know is if principal normed sp. are B. sp. or not.

<u>Thm</u>: $\mathbb{R}^d$ with any norm is a Banach sp.

(more in gen, V finite dim normed sp)

<u>Thm</u>: $B(X)$ = space of bounded functs on set X

$$\|f\|_\infty := \sup_{x \in X} |f(x)| .$$

$(B(X), \|\cdot\|_\infty)$ is a Banach sp.

<u>Proof</u>: Let $(f_n) \subset B(X)$ be a C. seq

$$\boxed{\forall \varepsilon > 0 \ \exists N : \quad \|f_n - f_m\|_\infty \leq \varepsilon \quad \forall n, m \geq N}$$

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<u>Goal</u>: $\exists f \in B(X) : \quad f_n \xrightarrow{\|\cdot\|_\infty} f.$

$\Leftrightarrow \exists \boxed{f \in B(X)} : \forall \varepsilon>0 \ \exists N : \quad \|f_n - f\|_\infty \leq \varepsilon \quad \forall n \geq N$

$\forall \varepsilon>0 \ \exists N : \quad \sup_{x\in X} |f_n(x) - f_m(x)| \leq \varepsilon \quad \forall n,m \geq N$

$$\Downarrow$$

$$\boxed{\forall \varepsilon>0 \quad \exists N \quad |f_n(x) - f_m(x)| \leq \varepsilon \quad \forall n,m \geq N} \ \forall x \in X$$

$$\Downarrow$$

$(f_n(x)) \subset \mathbb{R} \quad (\mathbb{C}, \mathbb{R}^d, \mathbb{C}^d)$

is a C. seq in $\mathbb{R}$ $\quad \forall x \in X$ fixed

$\Downarrow$ $\mathbb{R}$ is a B. sp. (prev thm)

$(f_n(x))$ is conv in $\mathbb{R}$, $\forall x \in X$

Define : $\quad f(x) := \lim_n f_n(x)$

$f : X \longrightarrow \mathbb{R}.$

Claim: 1. $f \in B(X)$ $\quad$ 2. $f_n \xrightarrow{\|\cdot\|_\infty} f$

Claim: 1. $f \in B(X)$ 2. $f_n \xrightarrow{\|\cdot\|_\infty} f$.

1. By C prop

$$|f_n(x) - f_m(x)| \leq \varepsilon \qquad \forall m \geq N \quad \forall n \geq N$$

$$f_m(x) \xrightarrow{m \to +\infty} f(x)$$

$$(*) \quad \boxed{|f_n(x) - f(x)| \leq \varepsilon \qquad \forall n \geq N \quad \forall x \in X}$$

$$|f(x)| = |f(x) - f_{\hat{n}}(x) + f_{\hat{n}}(x)|$$

$$\leq \underbrace{|f(x) - f_{\hat{n}}(x)|}_{\leq \varepsilon} + \underbrace{|f_{\hat{n}}(x)|}_{\leq \|f_{\hat{n}}\|_\infty}$$

$$|f(x)| \leq \varepsilon + \|f_{\hat{n}}\|_\infty \equiv C \qquad \forall x \in X$$

$$\Rightarrow f \in B(X)$$

~~Moreover~~

$$\boxed{(*) \Rightarrow \underbrace{\sup_{x \in X} |f_n(x) - f(x)|}_{\|f_n - f\|_\infty} \leq \varepsilon \qquad \forall n \geq N}$$

$$\|f_n - f\|_\infty \leq \varepsilon \qquad \forall n \geq N$$

$$\Rightarrow \quad f_n \xrightarrow{\|\cdot\|_\infty} f . \quad \blacksquare$$

Thm: $\mathscr{C}(K)$ $\quad K \subset \mathbb{R}^d$ closed and bounded (compact)

$$\mathscr{C}(K) = \{ f : K \to \mathbb{R}/\mathbb{C}/\mathbb{R}^d/\mathbb{C}^d : f \text{ continuous on } K \}$$

$$\|f\|_\infty = \sup_{x \in K} |f(x)| \overset{\text{Weier.}}{=} \max_{x \in K} |f(x)|$$

$(\mathscr{C}(K), \|\cdot\|_\infty)$ is a Banach sp.

Proof: $\mathscr{C}(K)$ in the case $K = [a,b] \subset \mathbb{R}$.

$\mathscr{C}([a,b]) \subset B([a,b])$

vect space $\subset$ vect sp. $\quad$ (we say $\mathscr{C}([a,b])$ is a subspace of $B([a,b])$)

So if $(f_n) \subset \mathscr{C}([a,b])$ is a C. seq in the $\|\cdot\|_\infty$ $\Rightarrow (f_n) \subset B([a,b])$ " " " " " "

" $\Rightarrow$ (prev. thm) $\exists f \in B([a,b]) : f_n \xrightarrow{\|\cdot\|_\infty} f$.

Is $\underline{f \in \mathscr{C}([a,b])}$? If yes the proof is finished.

Let's check this: because $f_n \xrightarrow{\|\cdot\|_\infty} f \Rightarrow$

$\underline{\forall \varepsilon>0}\ \exists N : \sup\limits_{x\in[a,b]} |f_n(x) - f(x)| \leqslant \varepsilon \qquad \forall n \geqslant N$

in part: $n = N$

$$\underbrace{|f_N(x)}_{f_N\in\mathcal{C} \dashrightarrow f\in\mathcal{C}} - f(x)| \leqslant \varepsilon \qquad \forall x \in X$$

Fix $x_0 \in [a,b]$ and consider $f(x) - f(x_0)$

<u>Goal:</u> prove $f(x) - f(x_0)$ small when $x \sim x_0$.

$$|f(x) - f(x_0)| = \Big|\big(f(x) - f_N(x)\big) + \big(f_N(x) - f_N(x_0)\big) + \big(f_N(x_0) - f(x_0)\big)\Big|$$

$$\overset{\Delta \text{ ineq}}{\leqslant} \varepsilon + |f_N(x) - f_N(x_0)| + \varepsilon$$

$$= 2\varepsilon + |f_N(x) - f_N(x_0)|$$

$$|f(x) - f(x_0)| \leqslant 2\varepsilon + |f_N(x) - f_N(x_0)|$$

$x \to x_0$ $\qquad\qquad\qquad\qquad\qquad\qquad \downarrow f_N \in \mathcal{C}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$

$$\Downarrow$$

$$\forall \varepsilon>0 \ \cancel{\exists N:}\ \lim_{x\to x_0} |f(x) - f(x_0)| \leqslant 2\varepsilon \qquad \cancel{\forall n \geqslant N}$$

$$\forall \varepsilon>0 \quad \lim_{x\to x_0} |f(x)-f(x_0)| \leq 2\varepsilon$$

$$\Downarrow$$

$$\lim_{x\to x_0} |f(x)-f(x_0)| = 0 \Rightarrow f(x) \xrightarrow[x\to x_0]{} f(x_0) \quad \forall x_0 \in [a,b]$$

$$\Rightarrow f \in \mathcal{C}([a,b]) \quad \blacksquare$$

Thm: $L^p(X) = \{ f : X \to \mathbb{R}/\mathbb{C} : \int_X |f|^p \, d\mu < +\infty \}$

$1 \leq p < +\infty$.

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

$(L^p(X), \|\cdot\|_p)$ is a B. sp $\forall\ 1 \leq p < +\infty$.

Sketch of the proof: Let $(f_n) \subset L^1(X)$ be a C. seq $(p=1)$

$$\forall \varepsilon>0 \quad \exists N : \quad \|f_n - f_m\|_1 \leq \varepsilon \quad \forall n,m \geq N$$

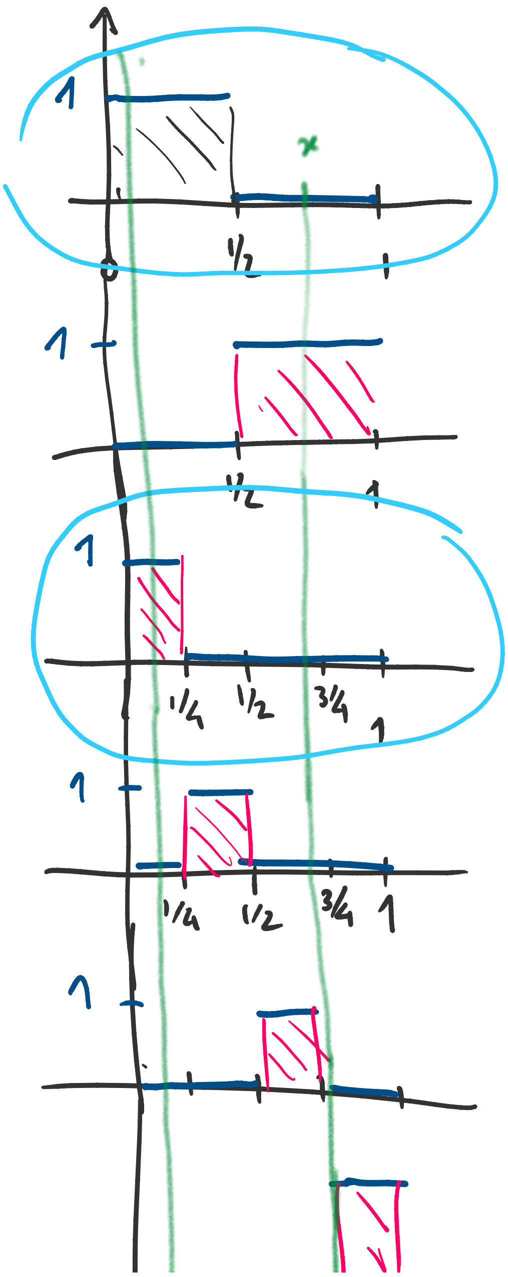
$$\| \qquad\qquad$$

$$\int_X |f_n - f_m| \leq \varepsilon$$

Pb: by $\int \leq \varepsilon$ we cannot derive any concl. on $|f_n - f_m|$

Rmk: We may have $f_n \xrightarrow{L^1} 0$ but $(f_n(x))$ is never conv $\forall x \in X$.

$L^1([0,1])$



f_0 $\quad \int_0^1 f_n$ $\quad 1/2$

f_1 $\quad 1/2$

$$f_n \xrightarrow{\|\cdot\|_1} 0$$

$$\Updownarrow$$

f_2 $\quad 1/4$

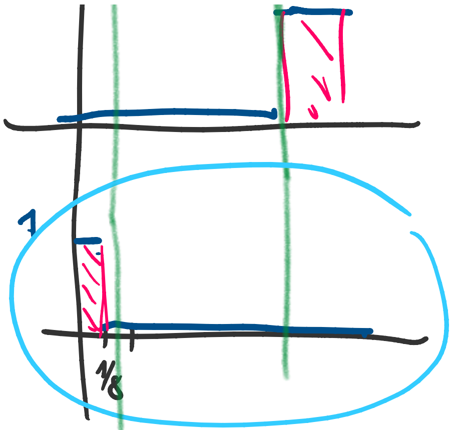
$$\|f_n - 0\|_1 \to 0$$

$$\|$$

f_3 $\quad \int_0^1 |f_n| = \int_0^1 f_n$ $\quad 1/4$

f_4 $\quad 1/4$

f_5 $\quad 1/4$



f_5 $\qquad$ $1/4$

f_6 $\qquad$ $1/8$

$\vdots$

$1/8$

$1/16$

$\vdots$

$\downarrow$

0

But: $f_n(x)$ contains ∞ many times $0, 1$ $\Rightarrow$

$$\nexists \lim_n f_n(x) \quad \forall x \in [0,1].$$

However it turns out that

Lemma: If $(f_n) \subset L^p$ is a C. seq (hence if (f_n) is conv) $\Rightarrow \exists (f_{n_k}) \subset (f_n)$ s.t.

$$(f_{n_k}(x)) \text{ conv. a.e. } x \in X.$$