Lemma If $(f_n) \subset L^p(X)$ is Cauchy seq

$\Rightarrow \exists (f_{n_k}) \subset (f_n)$: $(f_{n_k}(x))$ is conv. a.e. $x \in X$.

Proof: C. prop: $\forall \varepsilon > 0 \; \exists N \; \|f_n - f_m\|_1 \leq \varepsilon \; \forall n,m \geq N$

$(p=1)$

Fix $\varepsilon = 1$: $\exists N_1$: $\|f_n - f_m\|_1 \leq 1 \quad \forall n,m \geq N_1$

$n_1 := N_1 \qquad f_{N_1}$

Fix $\varepsilon = 1/2 \quad \exists N_2 > N_1$: $\|f_n - f_m\|_1 \leq 1/2 \quad \forall n,m \geq N_2$

$n_2 = N_2 \qquad f_{N_2}$

Notice that $\|f_{N_1} - f_{N_2}\|_1 \leq 1$ ($N_2 \geq N_1$)

Fix $\varepsilon = 1/4 \quad \exists N_3 > N_2 > N_1$: $\|f_n - f_m\|_1 \leq 1/4 \quad \forall n,m \geq N_3$

$n_3 = N_3 \qquad f_{N_3}$

Notice that $\|f_{N_2} - f_{N_3}\|_1 \leq \frac{1}{2}$

Notice that $\| \underbrace{f_{N_2} - f_{N_3}}_{\geq N_2} \|_1 \leq \frac{1}{2}$

$n_4 = N_4$: $\| f_{N_3} - f_{N_4} \|_1 \leq 1/4$

$\| f_{N_4} - f_{N_5} \|_1 \leq \frac{1}{8}$

$\vdots$

$$\int_X |f_{N_k} - f_{N_{k+1}}| = \boxed{\| f_{N_k} - f_{N_{k+1}} \|_1 \leq \frac{1}{2^{k-1}}}$$

Claim: $(f_{N_k}(x))$ converges a.e. $x \in X$

?

$$f_{N_k}(x) = f_{N_k}(x) - f_{N_{k-1}}(x) + f_{N_{k-1}}(x)$$
$$+ f_{N_{k-1}}(x) - f_{N_{k-2}}(x)$$
$$+ f_{N_{k-2}}(x)$$

$$\boxed{f_{N_k}(x) = \sum_{j=2}^{k} \left(f_{N_j}(x) - f_{N_{j-1}}(x) \right) + f_{N_1}(x)}$$

$\Big\downarrow$ a.e. $x \in X$, $k \to +\infty$

$$f(x) = \sum^{+\infty} \left(f_{N_j}(x) - f_{N_{j-1}}(x) \right) + f_{N_1}(x)$$

$$f(x) = \underbrace{\sum_{j=2}^{+\infty} \left(f_{N_j}(x) - f_{N_{j-1}}(x) \right)}_{} + f_{N_1}(x)$$

We prove that this series is conv a.e. $x \in X$.

$$\int_X \left(\sum_{j=2}^{+\infty} \left| f_{N_j}(x) - f_{N_{j-1}}(x) \right| \right) d\mu =$$

$\Big\lceil$ Monot conv: $\displaystyle \int_X \sum \overset{0}{\overset{\leq}{g_i}} = \sum \int_X g_i \Big\rfloor$

$$= \sum_{j=2}^{+\infty} \underbrace{\int_X \left| f_{N_j}(x) - f_{N_{j-1}}(x) \right| d\mu}_{\| f_{N_j} - f_{N_{j-1}} \|_1}$$

$$= \sum_{j=2}^{+\infty} \underbrace{\| f_{N_j} - f_{N_{j-1}} \|_1}_{\leq \frac{1}{2^{j-2}}}$$

$$\leq \sum_{j=2}^{+\infty} \frac{1}{2^{j-2}} = \sum_{j=0}^{\infty} \frac{1}{2^j} = 2$$

$$\leq \sum_{j=2} \frac{1}{2^{j-2}} = \sum_{j=0} \frac{1}{2^j} = 2$$

$$\Rightarrow \int_X \left(\sum |f_{N_j}(x) - f_{N_{j-1}}(x)|\right) d\mu < +\infty$$

$$\Downarrow$$

$\sum a_j$ conv.

$\Uparrow$

$\sum |a_j| < +\infty$

$$\sum |f_{N_j}(x) - f_{N_{j-1}}(x)| < +\infty \quad \text{a.e. } x \in X$$

$\ulcorner \int_X g \, d\mu < +\infty \Rightarrow g < +\infty$ a.e. $x \in X \lrcorner$

$g \geq 0$

$$\sum (f_{N_j}(x) - f_{N_{j-1}}(x)) \quad \text{conv a.e. } x \in X$$

$\blacksquare$

Thm: $L^p(X)$ is a Banach sp.

Proof: ($p=1$) Take $(f_n) \subset L^1$ a C. seq

$\exists \, (f_{n_k}) \subset (f_n)$ s.t.

$$f_{n_k}(x) \xrightarrow[\text{a.e. } x \in X]{} f(x) = \underbrace{\sum_{j=2}^{\infty} (f_{n_j}(x) - f_{n_{j-1}}(x)) + f_{n_1}(x)}$$

$\cancel{\ }$

$f \in L^1$? $\quad \left[\sum^{k-1} (\ldots) + f \ldots \right] = f_{N_k}$

$$f_{n_k} \xrightarrow[\;]{L^1} f \; ?$$

$$\| f - f_{N_k} \|_1 = \qquad = \boxed{\sum_{j=2}^{k-1} (f_{n_j} - f_{n_{j-1}}) + f_{n_1}} = f_{N_k}$$

$$+ \sum_{j=k}^{\infty} (f_{n_j} - f_{j-1})$$

$$= f_{N_k} + \sum_{j=k}^{\infty} (f_{n_j} - f_{n_{j-1}})$$

$$\Rightarrow \quad f - f_{N_k} = \sum_{j=k}^{\infty} (f_{n_j} - f_{n_{j-1}})$$

$$\| f - f_{N_k} \|_1 = \int_X | f - f_{N_k} | \, d\mu$$

$$= \int_X \left| \sum_{j=k}^{\infty} (f_{n_j} - f_{n_{j-1}}) \right| d\mu$$

$$\leq \sum_{j=k}^{\infty} | f_{n_j} - f_{n_{j-1}} |$$

$$\leq \int_X \sum_{j=k}^{\infty} \underbrace{| f_{n_j} - f_{n_{j-1}} |}_{\geq 0} \, d\mu$$

(monot conv)

$$\leq \sum_{j=k}^{+\infty} \int_X | f_{n_j} - f_{n_{j-1}} | \, d\mu$$

$$\| f_{n_j} - f_{n_{j-1}} \|_1 \leq \frac{1}{2^{j-2}}$$

$$\leq \sum_{j=k}^{+\infty} \frac{1}{2^{j-2}} = \frac{4}{2^k} \sum_{j=k}^{+\infty} \frac{1}{2^{j-k\cancel{+k}}}$$

$$= \frac{4}{2^k} \underbrace{\left(\sum_{i=0}^{\infty} \frac{1}{2^i} \right)}_{=2} = \frac{8}{2^k}$$

$$\| f - f_{N_k} \|_1 \leq \frac{8}{2^k} \xrightarrow[k \to +\infty]{} 0$$

$$\Rightarrow f_{N_k} \xrightarrow{L^1} f$$

<u>Lemma</u> (general property of C. seqs)

If (f_n) is C and $\exists (f_{n_k}) \subset (f_n)$ s.t.

$$f_{n_k} \xrightarrow{\|\cdot\|} f$$

$$\Rightarrow f_n \xrightarrow{\|\cdot\|} f.$$

(thus, in part, in the case of prev proof

$$f_n \xrightarrow{\|\cdot\|_1} f \quad)$$

Proof of Lemma: Hp: $\|f_{n_k} - f\| \to 0 \quad k \to +\infty$.

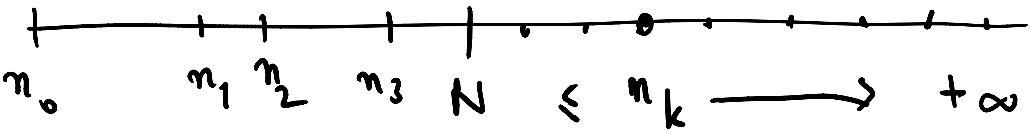
Th: $\|f_n - f\| \longrightarrow 0 \quad n \to +\infty$.

By C. prop

$\forall \varepsilon > 0 \quad \exists N : \quad \|f_n - f_m\| \leq \varepsilon \quad \forall n, m \geq N$

Now if $\quad n \geq N$

$$\|f_n - f\| \leqslant \|f_n - f_{n_k}\| + \underbrace{\|f_{n_k} - f\|}_{\downarrow \atop 0}$$



$$\leqslant \varepsilon + \varepsilon \qquad\qquad k \to +\infty$$

$$= 2\varepsilon \qquad\qquad k \to +\infty$$

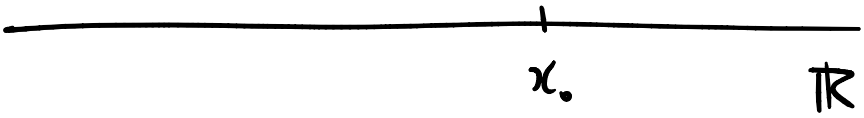
$\Rightarrow \quad \|f_n - f\| \leqslant 2\varepsilon \qquad \forall n \geqslant N.$

$\Rightarrow \quad f_n \xrightarrow{\|\cdot\|} f.$ $\blacksquare$

$L^\infty(X)$ $\quad$ $\cdot$ $+\infty$ $\quad$ ("bded meas functs")

$$f(x) = \begin{cases} 0 & x \neq x_0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \neq x_0 \\ +\infty & x = x_0 \end{cases}$$



is f bded or not?

Def: $(X, \mathcal{F}, \mu)$ be a meas. space.
$f: X \to [-\infty, +\infty]$ meas. is said to be <u>essentially bded</u> if

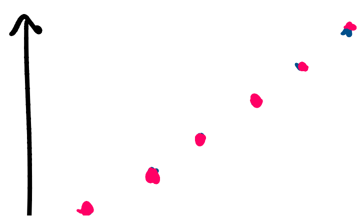
$$\exists M : \quad |f(x)| \leq M \quad \text{a.e. } x \in X.$$

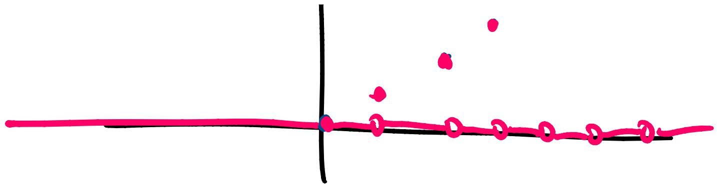
Rmk: If $f: X \to [-\infty, +\infty]$ is <u>bded</u>

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \quad \text{is a norm}$$

If $f: X \to [-\infty, +\infty]$ is ess. bded,
$\|f\|_\infty$ could be $= +\infty$.

$$\left(\quad f(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus \mathbb{N} \\ n & x = n \end{cases} \right.$$

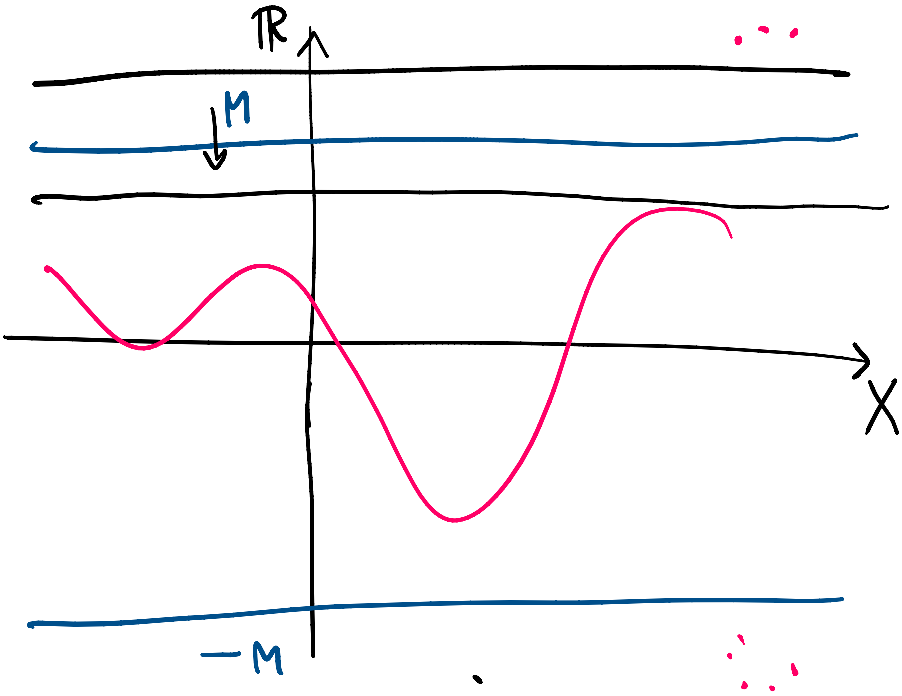




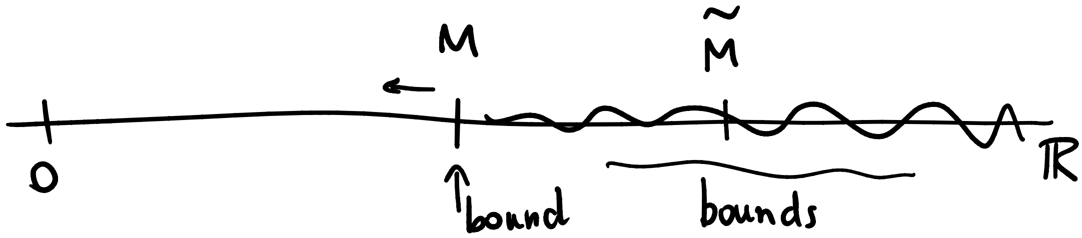
So how can be defined an analogous to $\|\cdot\|_\infty$?

Idea: If f is ess bded

$$\exists M : \quad |f(x)| \leq M \quad \text{a.e. } x \in X.$$



If such M $\exists$, $\forall \tilde{M} > M$ is still an ess bound for M



$$\|f\|_\infty := \inf \left\{ M \geq 0 : \quad |f(x)| \leq M \quad \text{a.e. } x \in X \right\}$$

Prop:

$$L^\infty(X) := \left\{ f : X \to \mathbb{R}/\mathbb{C} \;:\; f \text{ meas. and } \|f\|_\infty < +\infty \right\}$$

$(L^\infty(X), \|\cdot\|_\infty)$ is a Banach space.

($\|\cdot\|_\infty$ is a norm with vanishing

$\|f\|_\infty = 0 \iff f = 0$ a.e. on X.)

Hilbert Spaces

Basically an Hilbert sp is a Banach sp with the concept of angle between vectors.
This is induced by the concept of inner product.

Def: (scalar product)
Let V be a vector space on $\mathbb{R}$. A scalar product is a funct $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ such that

i) (positivity) $\langle f, f \rangle \geq 0 \quad \forall f \in V$

such that

i) (positivity) $\langle f, f \rangle \geq 0 \quad \forall f \in V$

ii) (vanishing) $\langle f, f \rangle = 0 \iff f = 0$

iii) (linearity) $\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$
$\forall \alpha, \beta \in \mathbb{R}, \quad \forall f, g, h \in V$

iv) (symmetry) $\langle f, g \rangle = \langle g, f \rangle \quad \forall f, g \in V$

Rmk: Combining iii) and iv), $\langle \, , \rangle$ is linear also in the second component:

$$\langle f, \alpha g + \beta h \rangle \underset{iv)}{=} \langle \alpha g + \beta h, f \rangle$$
$$\underset{iii)}{=} \alpha \langle g, f \rangle + \beta \langle h, f \rangle$$
$$\underset{iv)}{=} \alpha \langle f, g \rangle + \beta \langle f, h \rangle . \qquad \blacksquare$$

Examples

- $V = \mathbb{R} \qquad f, g \in V, \qquad \langle f, g \rangle = fg$.

- $V = \mathbb{R}^d \qquad f = (f_1 - - f_d) \qquad \langle f, g \rangle = \sum_{j=1}^{d} f_j g_j$
  $g = (g_1 - - g_d)$

• $V = \mathbb{R}^d$ $\quad f = (f_1 \cdots f_d)$, $g = (g_1 \cdots g_d)$

$$\langle f, g \rangle = \sum_{j=1} f_j g_j = f \cdot g$$

• $L^2_{\mathbb{R}}(X) = \left\{ f : X \to \mathbb{R} : f \text{ meas s.t. } \int_X |f|^2 < +\infty \right\}$

Given $f, g \in L^2$

$$\langle f, g \rangle = \int_X f(x) g(x) \, d\mu(x)$$

In this example, it is not evident that $\langle f, g \rangle \in \mathbb{R}$ $\forall f, g \in L^2$. This is true because $f \cdot g \in L^1$,

$$\int_X |f(x) g(x)| \, d\mu = \int_X |f||g| \, d\mu \underset{CS}{\leq} \underbrace{\left(\int_X |f|^2\right)^{1/2}}_{<+\infty \; f \in L^2} \underbrace{\left(\int_X |g|^2\right)^{1/2}}_{<+\infty \; g \in L^2}$$

$$\Leftrightarrow \int_X \underbrace{\frac{|f|}{\|f\|_2}}_{\alpha} \cdot \underbrace{\frac{|g|}{\|g\|_2}}_{\beta} \, d\mu \leq 1$$

This is true because $\alpha\beta \leq \frac{1}{2}(\alpha^2 + \beta^2)$

$$\int_X \frac{|f|}{\|f\|_2} \cdot \frac{|g|}{\|g\|_2} \, d\mu \leq \frac{1}{2} \underbrace{\int \Big(\underbrace{\frac{|f|^2}{\|f\|_2^2}}_{=1} + \frac{|g|^2}{\|g\|_2^2} \Big)}_{=1} \, d\mu$$

$$\int \frac{|f|^2}{\|f\|_2^2} \, d\mu = \frac{1}{\|f\|_2} \int_X |f|^2 \, d\mu = \frac{1}{\|f\|_2^2} \cdot \|f\|_2^2 = 1$$

$$= \frac{1}{2}(1+1) = 1$$

$$\Rightarrow \int |fg| \, d\mu < +\infty \Rightarrow fg \in L^1 \Rightarrow \exists \int_X fg \, d\mu \in \mathbb{R}$$

Notice that for standard L^2 scalar prod

$$\langle f, f \rangle = 0 \iff \int_X f^2 \, d\mu = 0$$

$$\iff f^2 = 0 \text{ a.e. } X$$

$$\iff f = 0 \text{ a.e. } X.$$

$$\ell^2 := \Big\{ \underset{\substack{\| \\ (f_n)_{n \in \mathbb{N}}}}{(f_1, f_2, f_3, \ldots)} : \sum_{n=0}^{\infty} f_n^2 < +\infty \Big\}$$

Prop: ℓ^2 is a vector space with

Prop: ℓ^2 is a vector space with

$$(f_n) + (g_n) = (f_n + g_n)$$

$$\lambda (f_n) = (\lambda f_n) \qquad \lambda \in \mathbb{R}$$

Check $+$ is well defined (do as we did for similar check on L^2)

On ℓ^2 we define

$$\langle (f_n), (g_n) \rangle_{\ell^2} := \sum_{n=0}^{\infty} f_n g_n .$$

Prop: $\langle \, , \, \rangle_{\ell^2}$ is a well defined scal. prod on ℓ^2.

(we need a CS like ineq

$$\sum |f_n| |g_n| \leq \left(\sum |f_n|^2 \right)^{1/2} \left(\sum |g_n|^2 \right)^{1/2}$$

When the scalars are $\mathbb{C}$ we need a slightly diff. def:

Def: (hermitian product)

that

Def: (hermitian product)

V be a Vect sp on $\mathbb{C}$, we say that

$$\langle \, , \rangle : V \times V \longrightarrow \mathbb{C}$$

is an hermitian prod if

i) pos: $\langle f, f \rangle \geq 0 \quad \forall f \in V$

ii) vanish. $\langle f, f \rangle = 0 \iff f = 0$

iii) linearity in the first factor: $\langle \alpha f + \beta g, h \rangle =$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$$\forall \alpha, \beta \in \mathbb{C}, \ \forall f, g, h \in V$$

iv) antisymmetry: $\langle f, g \rangle = \overline{\langle g, f \rangle} \quad \forall f, g \in V$

$$\langle h, \alpha f + \beta g \rangle = \overline{\langle \alpha f + \beta g, h \rangle}$$

$$= \overline{\alpha \langle f, h \rangle + \beta \langle g, h \rangle}$$

$$= \bar{\alpha} \, \overline{\langle f, h \rangle} + \bar{\beta} \, \overline{\langle g, h \rangle}$$

$$= \bar{\alpha} \langle h, f \rangle + \bar{\beta} \langle h, g \rangle$$