Examples of Hermitian products

- $\mathbb{C}^d = \{ (z_1, \dots, z_d) : z_j \in \mathbb{C} \}$ on $\mathbb{R}$ (or $\mathbb{C}$)

$$(z_1 \dots z_d) \cdot (w_1 \dots w_d) := \sum_{j=1}^{d} z_j \bar{w}_j$$

- $L^2_{\mathbb{C}}(X) = \{ f : X \to \mathbb{C} : f \text{ meas}, \int_X |f|^2 < +\infty \}$ on $\mathbb{R}$ ($\mathbb{C}$). We define

$$\langle f, g \rangle_{L^2_{\mathbb{C}}} := \int_X f \bar{g} \, d\mu .$$

Prop: (Cauchy - Schwarz ineq)
If $\langle , \rangle$ is a scalar/hermitian prod on V
then

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\| \quad \forall f, g \in V$$

where

$$\|f\| := \sqrt{\langle f, f \rangle}$$

In part $\|\cdot\|$ is a norm on V.

Rmk: . On $V = L^2_{\mathbb{R}}(X)$, $\langle f, g \rangle = \int_X f g \, d\mu$

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_X f^2 d\mu \right)^{1/2} \equiv \|f\|_{L^2}$$

- On $V = \mathbb{R}^d$, $\langle f, g \rangle = \sum f_j g_i$

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\sum f_j^2} = \text{euclidean norm.}$$

Other natural norms are not associated to any scalar prod. For example:

$V = \mathbb{R}^d$, $\|f\|_1 = \sum_{j=1}^{d} |f_j|$ are not induced

$\|f\|_\infty = \max_{1 \leq j \leq d} |f_j|$ by any scalar prod.

$V = L^1(X)$ $\|f\|_1 = \int_X |f| \, d\mu$.

<u>Proof of Prop:</u>

<u>CS ineq:</u> Take $f - \lambda g \in V$ $\lambda \in \mathbb{R}$

Because $\langle \, , \, \rangle$ is positive

$$\varphi(\lambda) := \langle f - \lambda g, f - \lambda g \rangle \geq 0 \quad \forall \lambda \in \mathbb{R}$$

$$\varphi : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\varphi(\lambda) \underset{\text{linearity}}{=} \langle f, f \rangle - \lambda \underbrace{\langle f, g \rangle - \lambda \langle g, f \rangle}_{= \text{ by symmetry}} + \lambda^2 \langle g, g \rangle$$

$$= \langle f,f\rangle - 2\lambda \langle f,g\rangle + \lambda^2 \langle g,g\rangle$$

If $\langle g,g\rangle = 0 \underset{\text{vanish}}{\Rightarrow} g = 0 \Rightarrow \varphi(\lambda) = \langle f,f\rangle$

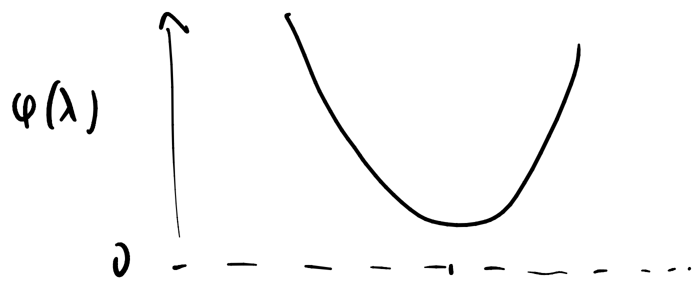
In this case CS is trivial because:

$$\underset{\substack{\parallel \\ 0}}{|\langle f,g\rangle|} \leq \|f\| \underset{\substack{\parallel \\ 0}}{\|g\|} = \sqrt{\underset{\substack{\parallel \\ 0}}{\langle g,g\rangle}} \quad g=0$$

$0 \leq 0$ true.

If $\langle g,g\rangle \neq 0 \Rightarrow \varphi$ is a 2nd degree polynomial

and because $\langle g,g\rangle \geq 0$



φ has a minimum at $\lambda_0 = \dfrac{\langle f,g\rangle}{\langle g,g\rangle}$

$$\varphi'(\lambda) = -2\langle f,g\rangle + 2\lambda \langle g,g\rangle = 0$$

$$\Leftrightarrow \quad \lambda = \frac{\langle f,g\rangle}{\underset{\neq 0}{\langle g,g\rangle}}$$

$$\Rightarrow \quad \varphi(\lambda) \geq \boxed{\varphi(\lambda_0) \geq 0}$$

$$\langle f,f\rangle - 2\underbrace{\frac{\langle f,g\rangle}{\langle g,g\rangle}}_{\lambda_0}\langle f,g\rangle + \underbrace{\left(\frac{\langle f,g\rangle}{\langle g,g\rangle^{\cancel{2}}}\right)^2}_{\lambda_0^2} \cdot \cancel{\langle g,g\rangle} \geq 0$$

$$\Leftrightarrow \quad \langle g,g\rangle\langle f,f\rangle - \cancel{2}\langle f,g\rangle^2 + \cancel{\langle f,g\rangle^2} \geq 0$$

$$\Leftrightarrow \quad \langle f,g\rangle^2 \leq \langle f,f\rangle\langle g,g\rangle$$

$$\Leftrightarrow \quad |\langle f,g\rangle| \leq \sqrt{\langle f,f\rangle}\sqrt{\langle g,g\rangle} \qquad \text{C.S.}$$

Rmk: $=$ holds $\Leftrightarrow$ $\varphi(\lambda_0) = 0$

$\Leftrightarrow$ $\langle f - \lambda_0 g, f - \lambda_0 g\rangle = 0$

$\underset{\text{vanish}}{\Leftrightarrow}$ $f - \lambda_0 g = 0$ $\Leftrightarrow$ f, g lin dep

$\Leftrightarrow$ $f = \lambda_0 g$

$\Leftrightarrow$ $f /\!/ g$.

∎

Let's now check that $\|f\| = \sqrt{\langle f,f\rangle}$ is a norm on V. We have to check:

- vanishing: $\|f\| = 0 \iff f = 0$

$$\left(\|f\| = 0 \iff \sqrt{\langle f, f\rangle} = 0 \iff \langle f, f\rangle = 0 \iff f = 0 \text{ by vanish of } \langle\ ,\ \rangle. \right)$$

- homogenuity: $\| \lambda f \| = \sqrt{\langle \lambda f, \lambda f\rangle}$

$$\overset{\text{scalar prod}}{=} \sqrt{\lambda^2 \langle f, f\rangle} = |\lambda| \sqrt{\langle f, f\rangle} = |\lambda| \, \|f\|.$$

$$\overset{\text{herm.}}{=} \sqrt{\underbrace{\lambda \bar{\lambda}}_{= |\lambda|^2} \langle f, f\rangle} = |\lambda| \, \|f\|.$$

- Δ ineq: $\|f+g\| \leq \|f\| + \|g\| \quad \forall f, g.$

$$\|f+g\|^2 = \langle f+g, f+g\rangle$$

$$= \langle f, f\rangle + \underbrace{\langle f, g\rangle + \langle g, f\rangle}_{=} + \langle g, g\rangle$$

$$= \langle f, f\rangle + 2\langle f, g\rangle + \langle g, g\rangle$$

$$= \|f\|^2 + \|g\|^2 + 2 \underset{\|f\| \, \|g\|}{\underset{\wedge \text{ CS}}{\langle f, g\rangle}}$$

$$\leq \|f\|^2 + \|g\|^2 + 2\|f\|\|g\| = (\|f\| + \|g\|)^2$$

$$\Rightarrow \quad \|f+g\| \leq \|f\| + \|g\|$$

Rmk: $=$ holds true $\Leftrightarrow$ $f /\!/ g$.

Def: We say that H is an Hilbert space if H is a Banach space respect to the norm induced by its scalar / hermitian prod.

Examples:

- $\mathbb{R}^d$, $\mathbb{C}^d$ are H. spaces
- $L^2_{\mathbb{R}}(X)$ / $L^2_{\mathbb{C}}(X)$ are H. spr.
- $\ell^2_{\mathbb{R}}$ $\quad \langle (f_n), (g_n) \rangle = \sum_n f_n g_n$

  $\ell^2_{\mathbb{C}}$ $\quad \langle (f_n), (g_n) \rangle = \sum f_n \overline{g_n}$

$\ulcorner$ $\ell^2_{\mathbb{R}} = L^2_{\mathbb{R}}(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu = \text{counting meas})$ $\lrcorner$

Orthonormal Base

A base of vectors in an infinite dim space

A base of vectors in an infinite dim space is something as a set of vectors $(f_n)_{n\in\mathbb{N}}$ s.t.

$$\forall f \in V \quad , \qquad f = \sum_{n=0}^{\infty} \cancel{\alpha_n} f_n .$$

<u>Def</u>: Let V be a normed space, $(f_n) \subset V$

We say that

$$\sum_{n=0}^{\infty} f_n \in V$$

if

$$\exists \lim_{N\to+\infty} s_N := \sum_{n=0}^{\lim_{N\to\infty} N} f_n \in V$$

that is $\exists\, s \in V \quad : \quad \| s_N - s \| \to 0.$

In principle, to discuss conv of $\sum_n f_n \quad (f_n) \subset V$ is "simple": just compute $\quad s_N = \sum_{n=0}^{N} f_n$, then compute $\lim_N s_N$ in V.

The pb is that in practice is almost never possible to get useful expressions for s_N.

We need some test that might be easy

to use :

Thm: (normal conv. test)

Let V be a Banach sp. Suppose that

$$\sum_{n=0}^{\infty} \|f_n\| < +\infty.$$

Then $\sum f_n$ conv. in V.

Proof: Because $\sum f_n$ conv $\overset{\text{by def}}{\iff}$ $(s_N)_N \subset V$ conv

$$s_N = \sum_{n=0}^{N} f_n$$

V is B. $\iff$ (s_N) is C. in V

$$\iff \forall \varepsilon > 0 \ \exists \hat{N} : \underbrace{\|s_N - s_M\|}_{} \leq \varepsilon, \ \forall N, M \geq \hat{N}.$$

$$\|s_N - s_M\| = \left\| \sum_{n=0}^{N} f_n - \sum_{n=0}^{M} f_n \right\|$$

$N \geq M$

$$= \left\| \sum_{n=M+1}^{N} f_n \right\|$$

$\wedge$ inea N

$$\overset{\Delta \text{ ineq}}{\leq} \sum_{n=M+1}^{N} \|f_n\| = \underbrace{s_N - s_M}_{+}$$

$$S_N \neq s_N = \sum_{n=0}^{N} \|f_n\|$$

Now: because, by ass, $\sum \|f_n\| < +\infty$

$$\Updownarrow$$

$$\exists \lim_{N \to +\infty} s_N \in \mathbb{R}$$

$$\Updownarrow$$

(s_N) is C. in $\mathbb{R}$

$$\Updownarrow$$

$$\forall \varepsilon > 0 \quad \exists \hat{N} : \quad |s_N - s_M| \leq \varepsilon \quad \forall N, M \geq \hat{N}$$

Therefore, if $N > M \geq \hat{N}$

$$\|S_N - S_M\| \leq \underbrace{s_N - s_M}_{+} = |s_N - s_M| \leq \varepsilon \quad \forall N, M \geq \hat{N}$$

$\Rightarrow \|S_N - S_M\| \leq \varepsilon \quad \forall N, M \geq \hat{N}$. $\boxtimes$

In part, if we're in an Hilbert space (a Banach space with norm induced by an inner [scalar/hermitian] prod) by applying this test

if $\sum_n \|\underbrace{\alpha_n e_n}_{f_n}\| = \sum |\alpha_n| \|e_n\| = \sum |\alpha_n| < +\infty$

$(\|e_n\| = 1)$

$\Downarrow$

$\sum \alpha_n e_n$ is conv.

In the case of an H. sp this is $\Longleftrightarrow$ provided e_n are orthogonal:

Def: We say f, g are orthogonal $(f \perp g)$ if

$$\langle f, g \rangle = 0.$$

Proposition: Let $(e_n)_{n \in \mathbb{N}} \subset H$ H. space be an orthonormal system of vectors

$$\Big(\langle e_n, e_m \rangle = 0 \quad \forall n, m \in \mathbb{N}, \ n \neq m$$

$$\|e_n\| = 1 \Longleftrightarrow \langle e_n, e_n \rangle = 1$$

$$\Longleftrightarrow \quad \langle e_n, e_m \rangle = \delta_{n,m} \quad \text{(Kronecker } \delta) \Big)$$

$\Leftrightarrow \quad \langle e_n, e_m \rangle = \delta_{n,m}$ (Kronecker δ)

$$= \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Then:

$$\sum \alpha_n e_n \text{ conv (in } H) \Leftrightarrow \sum \alpha_n^2 \text{ conv. (in } \mathbb{R})$$

Proof: $\sum \alpha_n e_n$ conv $\Leftrightarrow$ (s_N) is C. seq

$$s_N = \sum_{n=0}^{N} \alpha_n e_n$$

$$\Leftrightarrow \forall \varepsilon > 0 \ \exists \hat{N} : \ \|s_N - s_M\| \leq \varepsilon \quad \forall N, M \geq \hat{N}$$

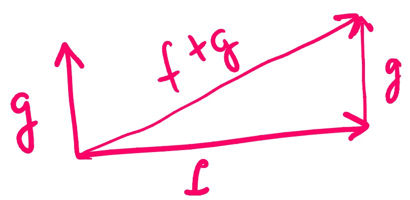
$$\|s_N - s_M\|^2 = \left\| \sum_{n=0}^{N} - \sum_{n=0}^{M} \right\|^2 \underset{N > M}{=} \left\| \sum_{n=M+1}^{N} \alpha_n e_n \right\|^2.$$

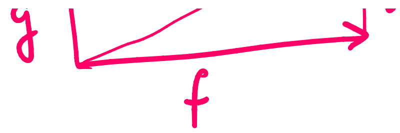
Lemma (Pythagorean thm)

Let $\langle , \rangle$ be a scalar/herm prod on V.

If $f \perp g$

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2$$





Proof:

$$\|f+g\|^2 = \langle f+g, f+g\rangle$$

$$= \langle f,f\rangle + 2\underbrace{\langle f,g\rangle}_{=0} + \langle g,g\rangle$$

$$= \langle f,f\rangle + \langle g,g\rangle = \|f\|^2 + \|g\|^2 \qquad \blacksquare$$

More in general: if $f_1, \dots, f_n \in V$, $f_i \perp f_j$ $i \neq j$

$$\Rightarrow \|f_1 + \dots + f_n\|^2 = \|f_1\|^2 + \dots + \|f_n\|^2. \qquad \blacksquare$$

Noticed that $\alpha_n e_n \perp \alpha_m e_m$ $\forall n \neq m$

$$\Rightarrow \|s_N - s_M\|^2 = \left\| \sum_{n=M+1}^{N} \alpha_n e_n \right\|^2$$

$$\overset{\text{Pyth.}}{=} \sum_{n=M+1}^{N} \|\alpha_n e_n\|^2$$

$$= \sum_{n=M+1}^{N} \alpha_n^2 \cdot 1$$

$$= \sum_{n=M+1}^{N} \alpha_n^2 = s_N - s_M$$

where $s_N = \sum_{n=0}^{N} \alpha_n^2$. We proved that

$$s_N = \sum_{n=0}^{N} \alpha_n e_n \in H \qquad \Big| \qquad s_N = \sum_{n=0}^{N} \alpha_n^2 \in \mathbb{R}$$

$$\|s_N - s_M\|^2 = \underbrace{s_N - s_M}_{+} \qquad N > M$$

$$\Rightarrow \quad (s_N) \text{ is } C \iff (s_N) \text{ is } C$$

$$\Updownarrow H \text{ Hilbert} \qquad\qquad \Updownarrow \mathbb{R} \text{ is B. sp.}$$

$$s_N \text{ conv} \qquad\qquad s_N \text{ conv}$$

$\blacksquare$

<u>Def</u>: Let H be an Hilb. space, (e_n) be an orthonormal system of vectors in H.
We say that (e_n) is an <u>orthonormal base for H</u> if

$$\forall f \in H \quad \exists (\alpha_n) \subset \mathbb{R} \ (\mathbb{C}) : \quad f = \sum_{n=0}^{\infty} \alpha_n e_n .$$

<u>Prop</u>: Assume (e_n) be orthonormal base for H Hilb. Then

Prop: Assume ...

Hilbert. Then

$$f = \sum_{n=0}^{\infty} \langle f, e_n \rangle e_n \qquad \forall f \in H$$

(is the unique representation of f in the base (e_n))

Rmk:

$H = \mathbb{R}^d \qquad \langle f, g \rangle = \sum f_j g_j \doteq f \cdot g$

$$f = (f_1, \dots, f_d)$$

$$= f_1 \underbrace{(1, 0, \dots, 0)}_{e_1} + f_2 \underbrace{(0, 1, 0, \dots, 0)}_{e_2} + \dots + f_d \underbrace{(0, \dots, 0, 1)}_{e_d}$$

$$f_j = f \cdot e_j = (f_1, \dots, f_d) \cdot (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$$

$$= \sum_{j=1}^{d} (f \cdot e_j) e_j$$

$\boxtimes$

Proof: Because (e_n) is a base

$$\exists (\alpha_n) \subset \mathbb{R} \quad : \quad f = \sum_{n=0}^{\infty} \alpha_n e_n$$

$\Rightarrow$

$\Rightarrow$

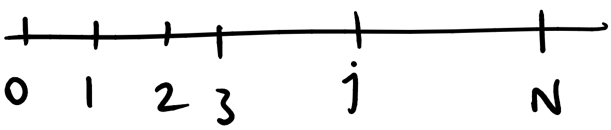
$$\langle f, e_j\rangle = \langle \sum_{n=0}^{\infty} \alpha_n e_n , e_j \rangle$$

$$= \langle \lim_{N \to +\infty} \sum_{n=0}^{N} \alpha_n e_n , e_j \rangle$$

$$\overset{?}{=} \lim_{N \to +\infty} \langle \sum_{n=0}^{N} \alpha_n e_n , e_j \rangle$$

lin

$$= \lim_{N \to +\infty} \sum_{n=0}^{N} \alpha_n \underbrace{\langle e_n , e_j \rangle}_{= \delta_{nj}} = \alpha_j$$



$\Downarrow$

$$\alpha_j = \langle f, e_j \rangle$$

$\blacksquare$

Prop:

$$f_n \xrightarrow{H} f \quad \Rightarrow \quad \langle f_N, g\rangle \longrightarrow \langle f, g\rangle$$

$$\lim_{N} \langle f_N, g \rangle = \langle \lim_{N} f_N , g \rangle$$

use (CS)

Do pbs 4.5.1