martedì 12 novembre 2019 12:33

Exercise: Show that if $f_n \xrightarrow{H} f$

$(\|f_n - f\| \to 0) \quad \Rightarrow \quad \langle f_n, g\rangle \to \langle f, g\rangle \ \forall g \in H.$

Sol:

$$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle|$$

$$\overset{CS}{\leq} \|f_n - f\| \|g\| \to 0$$

$$\downarrow$$

$$0$$

$$\Downarrow$$

$$\langle f_n, g\rangle \to \langle f, g\rangle$$

$\blacksquare$

Ex 4.5.1 $V = \{ f \in \mathcal{C}^1([0,1]) : f(0) = 0 \}$

$$\langle f, g\rangle := \int_0^1 f' g' \, dx$$

i) $\langle \, , \, \rangle$ is a scalar prod

ii) Discuss if V is an H. sp.

Sol: First, because $f, g \in V \Rightarrow f, g \in \mathcal{C}^1$

$\Rightarrow f', g' \in \mathcal{C}$

$\Rightarrow f'g' \in \mathcal{C}$

$\Rightarrow \exists \int_0^1 f'g' \Rightarrow \langle f, g \rangle$ is well def'd $\forall f, g \in V$

Second, to check that $\langle\, , \rangle$ is a scalar prod we've to check the characteristic props:

i) pos $\langle f, f \rangle \geq 0 \quad \forall f \in V$

$$\langle f, f \rangle = \int_0^1 (f')^2 \quad \text{evident}$$

ii) vanish: $\langle f, f \rangle = 0 \Rightarrow \underbrace{\int_0^1 (f')^2}_{\in \mathcal{C}} = 0$

$\Rightarrow (f')^2 \equiv 0$ on $[0,1]$

$\Rightarrow f' \equiv 0$ on $[0,1]$

$\Rightarrow f$ constant on $[0,1]$

and because $f \in V$, in part $f(0) = 0$

$\Rightarrow f \equiv 0$ on $[0,1]$.

iii) linearity: $\langle \alpha f + \beta g, h \rangle = \int_0^1 (\alpha f + \beta g)' \cdot h'$

$$(\alpha f + \beta g)' \cdot h' = (\alpha f' + \beta g') \cdot h'$$

$= \alpha \int_0^1 f'h' + \beta \int_0^1 g'h' =$

$= \alpha \langle f,h\rangle + \beta \langle g,h\rangle .$

iv) symm.: $\langle f,g\rangle = \int_0^1 f'g' = \int_0^1 g'f' = \langle g,f\rangle$
$\forall f,g \in V.$

Q2: Is V Hilbert?

The question asks if V is Banach sp. resp to

$$\|f\| = \sqrt{\langle f,f\rangle} = \left(\int_0^1 (f')^2\right)^{1/2}.$$

$\Leftrightarrow \forall (f_n)$ Cauchy seq $\Rightarrow (f_n)$ is conv in V.

$$\boxed{\begin{array}{c} f_n \to f \\ \Updownarrow \\ f_n - f \to 0 \end{array}}$$

Notice that $f_n \xrightarrow{V} 0$

$\Updownarrow$

$f_n' \xrightarrow{L^2} 0 \Leftrightarrow \int_0^1 f_n'(x)^2 \, dx \to 0$

$\ulcorner \int_0^1 f_n(x)^2 \, dx \to 0 \lrcorner$

C. seq
$\forall \varepsilon > 0$
$\|f_n - f_m\| \leq \varepsilon \quad \forall n,m \geq N$

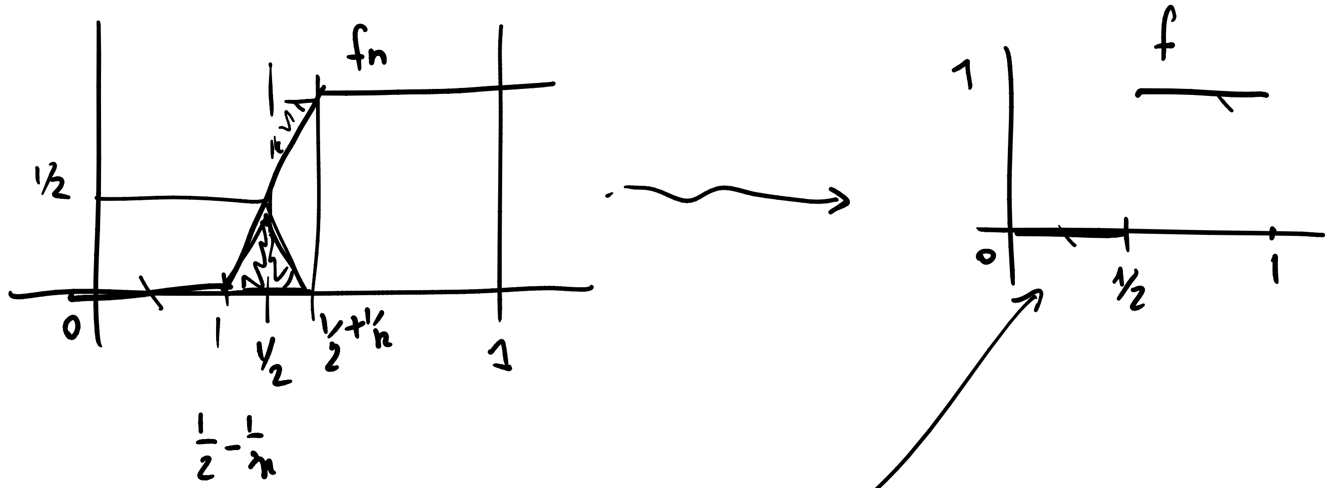
$\Updownarrow$

$\int_0^1 [(f_n - f_m)']^2 \leq \varepsilon$

$\int_0^1 (f_n' - f_m')^2 \leq \varepsilon$

$$\tilde{V} = \{ f \in \mathcal{C}([0,1]) \} \qquad f_n \to f \iff \int_0^1 (f_n - f)^2 \to 0$$

$$\langle f, g \rangle = \int_0^1 fg \qquad \|f\| = \|f\|_{L^2}$$



<u>Claim 1:</u> $f_n \xrightarrow{L^2} f$

$$\|f_n - f\|_{L^2}^2 = \int_0^1 |f_n - f|^2$$

$$= \int_0^1 |f_n - f| \underbrace{|f_n - f|}_{\leq 1}$$

$$\leq 1 \cdot \int_0^1 |f_n - f|$$

$$= 1 \cdot \frac{1}{\cancel{2}} \frac{\cancel{2}}{n} \cdot \frac{1}{2} = \frac{1}{2n} \longrightarrow 0$$

$$\Rightarrow f_n \xrightarrow{L^2} f \Rightarrow (f_n) \subset \tilde{V}, \text{ is } C \text{ resp } \|\cdot\|_2$$

<u>Claim 2:</u> $\nexists \lim_n f_n$ on $\tilde{V}$

Indeed, if $g = \lim f_n \in \tilde{V} \subset L^2$, Indeed, if $g = \text{unif th} \ldots$,

$$f_n \xrightarrow{L^2} f$$

$\Rightarrow$ $g = f$ a.e. this is imposs for $g \in \mathcal{C}$

$$\overset{\shortparallel}{\mathbb{1}_{[1/2,1]}}$$

Let's return to our Pb:

$$V = \{ f \in \mathcal{C}^1([0,1]) : f(0) = 0 \}$$

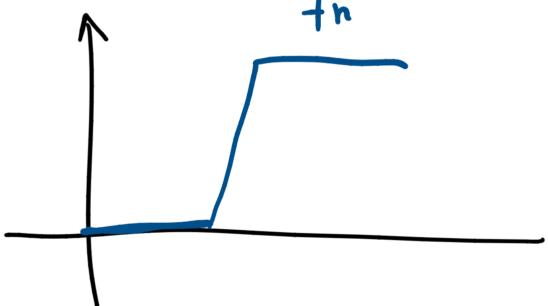
$$\langle f, g \rangle = \int_0^1 f' g'$$

$$g_n \to g \iff \int_0^1 (g_n' - g')^2 \longrightarrow 0$$

$$\boxed{\begin{array}{l} g_n' = f_n \\ g' = f \end{array}}$$

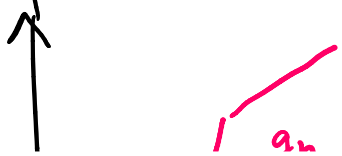
(assume for a second $\exists (g_n) \subset V$: $g_n' = f_n$

$\Rightarrow$ g_n

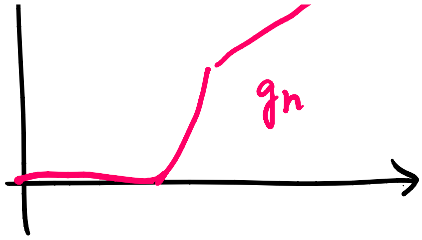


$$g_n(x) = \underbrace{\int_0^x f_n(y)\,dy}_{\mathcal{C}}$$

$\rightarrow \| g_n - g_m \|^2 =$



$$\|g_n - g_m\|^2 =$$



$$= \int_0^1 (g_n' - g_m')^2$$

$$= \int_0^1 (f_n - f_m)^2 = \|f_n - f_m\|_{L^2} \le \varepsilon \quad \forall n,m \geqslant N$$

(because (f_n) conv. in L^2 $\Rightarrow$ is C in L^2 norm)

$g_n \in V$ $\quad$ ($g_n \in \mathcal{C}$, $g_n' = f_n \in \mathcal{C} \Rightarrow g_n \in \mathcal{C}^1$

$g_n(0) = 0$)

but $g_n \not\to g \in V$.

$$\Updownarrow$$

$$\|g_n - g\| \to 0 \iff \int_0^1 (g_n' - g')^2 \to 0$$

$$\iff \begin{tikzcd} g_n' \arrow[r, "L^2"] \arrow[d, equal] & g' \arrow[dd, equal] \\ f_n \arrow[d, "L^2"'] & \\ 1_{[1/2,1]} & \end{tikzcd}$$

$g' = 1_{[1/2,1]}$

$\uparrow$

$\mathcal{C}$ a.e.

impossible!

$$\Downarrow$$

$\nexists g \in V.$ $\blacksquare$

We introduced the concept of orthonormal base for an Hilbert sp. H. If $(e_n)_{n\in\mathbb{N}}$ is an O.B. $\rightarrow$

$$\underbrace{f = \sum_{n=0}^{\infty} \langle f, e_n \rangle e_n}_{\text{abstract Fourier series.}}$$

There're two major questions we wish to discuss:

Q1: (e_n) be an orthonormal system in H

when this is a base ?

Q2: Given an Hilb. sp H how do we construct an orthonormal base?

Example

$$\ell^2_{\mathbb{R}} = \left\{ (f_n)_{n\in\mathbb{N}} : \sum f_n^2 < +\infty \right\}$$

$$\langle f, g \rangle = \sum_{n\geq 0}^{\infty} f_n g_n .$$

An example of orthonormal base is

$$e_n = (0,0,\dots,0,\overset{n}{1},0,\dots\quad 0\dots) \quad n\in\mathbb{N}$$

$$= (\delta_{kn})_{k\in\mathbb{N}} \in \ell^2_{\mathbb{R}}$$

It's easy to check that $(e_n)_{n\in\mathbb{N}} \subset \ell^2_{\mathbb{R}}$

is an orthonormal system:

$$\langle e_n, e_m\rangle = \delta_{nm}. \qquad \forall n,m\in\mathbb{N}$$

$$\Downarrow$$

$$\|e_n\| = 1$$

Let's check that (e_n) is a base:

$f\in\ell^2$, $f = (f_n)$.

The FS of f is $\sum_{n=0}^{\infty} \langle f, e_n\rangle e_n$

$$\langle f, e_n\rangle = f_n \qquad = \sum_{n=0}^{\infty} f_n e_n.$$

Is

$$f = \sum_{n=0}^{\infty} f_n e_n \ ?$$

$$f = \overset{\ell^2}{\lim_{N\to\infty}} \sum_{n=0}^{N} f_n e_n$$

$$\iff \quad \left\| f - \sum_{n=0}^{N} f_n e_n \right\|_{\ell^2}^2 \longrightarrow 0$$

$$\|$$

$$\left\| (f_n) - (f_0, f_1, f_2, \dots, f_N, 0, 0, 0, \dots) \right\|_{\ell^2}^2$$

$$= \left\| (\overbrace{0, 0, 0, \dots, 0}^{N}, \overset{N+1}{f_{N+1}}, f_{N+2}, \dots) \right\|_{\ell^2}^2$$

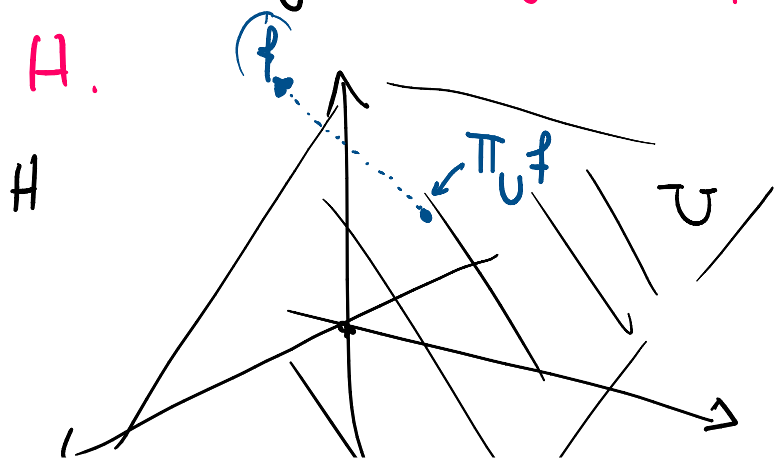
$$= \sum_{n=N+1}^{+\infty} f_n^2 \xrightarrow[N \to +\infty]{} 0 \quad \text{because}$$

$\uparrow$ is the tail of the serie $\sum_{n=0}^{\infty} f_n^2 < +\infty$

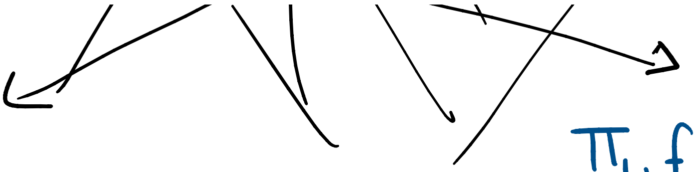
$(f \in \ell^2)$

∎

Both Q1, Q2 are solved by a fundamental result concerning orthogonal projection on subspaces of H.



U linear subsp. of H



$\Pi_U f$ = orthogonal proj of f on U

is the element at min dist of U to f.

To ensure existence, uniqueness of the $\Pi_U f$ we need a minimal ass.

Def: Let V be a normed space. $S \subset V$. We say that S is <u>closed</u> if

$$\forall (f_n) \subset S : f_n \to f \Rightarrow f \in S.$$

Thm: Let H be an Hilbert space, $U \subset H$ be a <u>closed</u> $\underbrace{\text{subspace}}$ of H

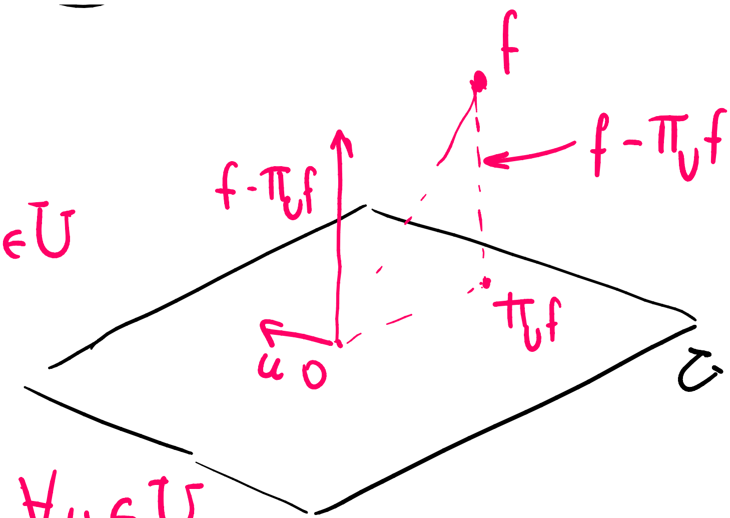
$$\left(f, g \in U \Rightarrow \alpha f + \beta g \in U \quad \forall \alpha, \beta \in \mathbb{R}(\mathbb{C}) \right)$$

Then, $\forall f \in H$ $\exists!$ $\Pi_U f \in U$:

unique

$$\| f - \Pi_U f \| = \text{min} \{ \| f - u \| : u \in U \}$$

Moreover



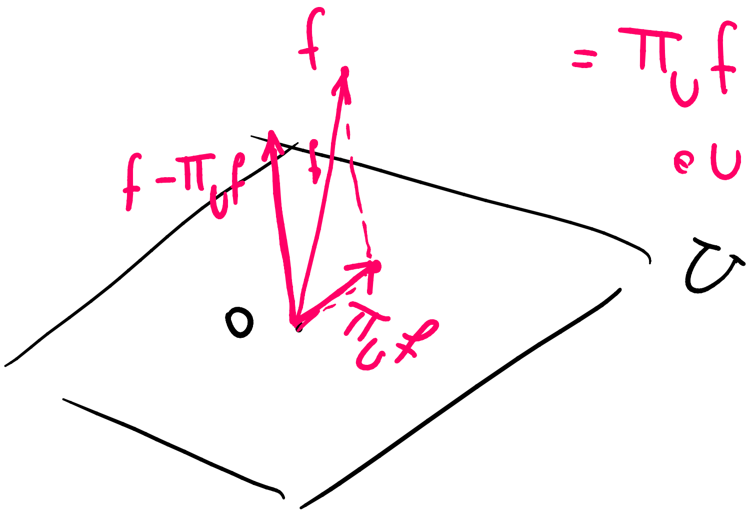
$$f - \pi_U f \perp u, \ \forall u \in U$$

$$\Updownarrow$$

$$\to \langle f-\pi_U f, u\rangle = 0 \quad \forall u \in U.$$

In part: $\forall f \in H, \quad f = f - \pi_U f + \pi_U f$

$$= \underbrace{\pi_U f}_{\in U} + (f - \pi_U f)$$

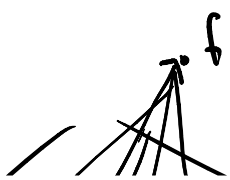


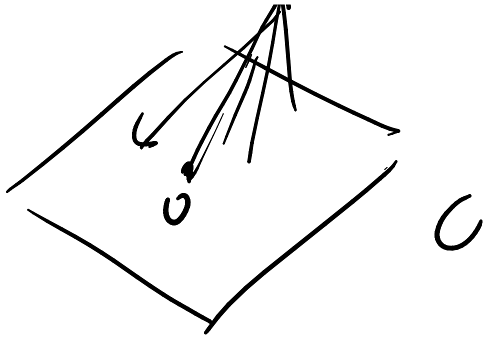
Proof: In gen min set doesn't exist but

$\alpha := \operatorname{Inf}\{\|f-u\| : u \in U\} \leq \|f\| < \infty$ is well. defd.

Goal: prove that Inf = min.

$\Leftrightarrow \ \exists\, \hat{u} \in U : \quad \operatorname{Inf}\{\|f-u\| : u \in U\} = \|f-\hat{u}\|.$

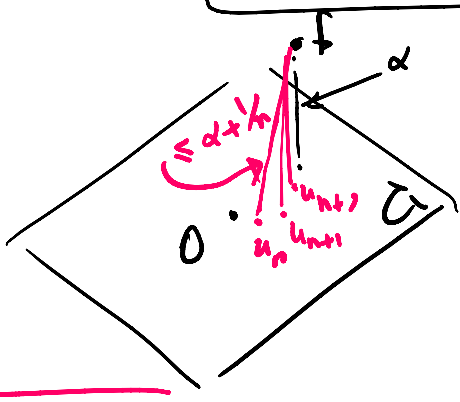




$$\forall \varepsilon>0 \quad \exists u \in U : \quad \|f-u\| \leq \alpha+\varepsilon$$

$$\Downarrow$$

$$\varepsilon = 1/n : \quad \exists u_n \in U : \quad \boxed{\|f-u_n\| \leq \alpha + 1/n}$$



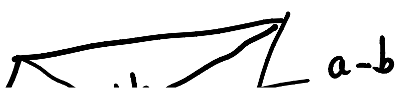
Guess: $\boxed{u_n \to \hat{u}}$ Let's check (u_n) is C.

Estimate of $\|u_n - u_m\|$

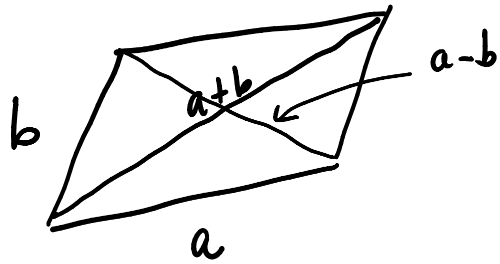
$$= \|(u_n - f) + (f - u_m)\| \overset{\Delta}{\leq} \overbrace{\|u_n - f\|}^{\leq \alpha + 1/n} + \overbrace{\|u_m - f\|}^{\leq \alpha + 1/m}$$

$$\underline{2\alpha} + \frac{1}{n} + \frac{1}{m}$$

Parallelogram id:



Parallelogram id:



$$\|a+b\|^2 + \|a-b\|^2 =$$

$$= \langle a+b, a+b\rangle + \langle a-b, a-b\rangle$$

$$= \|a\|^2 + \|b\|^2 \quad + \quad \|a\|^2 + \|b\|^2$$

$$\cancel{2\langle a,b\rangle} \qquad - \cancel{2\langle a,b\rangle}$$

$$= 2\left(\|a\|^2 + \|b\|^2\right)$$

$$\Rightarrow \|a+b\|^2 = 2\left(\|a\|^2 + \|b\|^2\right) - \|a-b\|^2$$

$$\Rightarrow \|u_n - u_m\|^2 = \|\overbrace{(u_n - f)}^{a} + \overbrace{(f - u_m)}^{b}\|^2$$

$$= 2\Big(\underbrace{\|u_n - f\|^2}_{\le (\alpha + 1/n)^2} + \underbrace{\|u_m - f\|^2}_{\le (\alpha + 1/m)^2}\Big) - \|u_n + u_m - 2f\|^2$$

$$\le 2\left[\left(\alpha + \frac{1}{n}\right)^2 + \left(\alpha + \frac{1}{m}\right)^2\right] - 4\left\|\frac{u_n + u_m}{2} - f\right\|^2$$

$\cancel{2\alpha^2}$

$$\alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} + \alpha^2 + \frac{2\alpha}{m} + \frac{1}{m^2}$$

$$\leq 2\left[\left(\alpha + \tfrac{1}{n}\right)^2 + \left(\alpha + \tfrac{1}{m}\right)^2\right] - \cancel{4\alpha^2}$$

$$\left\| \underbrace{\frac{u_n + u_m}{2}}_{\in U} - f \right\|^2 \geq \alpha^2$$

$$\leq \frac{1}{n}\left(2\alpha + \frac{1}{n}\right) + \frac{1}{m}\left(2\alpha + \frac{1}{m}\right) \leq \varepsilon \qquad n, m \geq N$$

$\Rightarrow$ (u_n) is C $\Rightarrow$ u_n conv

$$\downarrow$$
$$\hat{u}$$

and because

$$\alpha \leq \|u_n - f\| \leq \alpha + \tfrac{1}{n}$$

$$\alpha \leq \|\hat{u} - f\| \leq \alpha$$

$$\Rightarrow \quad \|\hat{u} - f\| = \alpha = \operatorname{Inf}\{\|u - f\| : u \in U\}$$

$\Rightarrow$ Inf = min.

$\boxtimes$