lunedì 18 novembre 2019 12:32

Ex 4.5.4 Solve

$$\min_{a,b\in\mathbb{R}} \int_0^{+\infty} \left| e^{-x} - (a e^{-2x} + b e^{-3x}) \right|^2 dx$$

Sol: We may notice that

$$\int_0^{+\infty} \left| \qquad \right|^2 dx = \| e^{-\#} - \underbrace{(a e^{-2\#} + b e^{-3\#})}\|^2_{L^2([0,+\infty[)}$$

$$\left\{ a e^{-2\#} + b e^{-3\#} \;:\; a,b\in\mathbb{R} \right\}_{a,b\in\mathbb{R}} =$$

$= \text{vector space gen by } e^{-2\#}, e^{-3\#}$

$$= \operatorname{Span}\langle e^{-2\#}, e^{-3\#} \rangle =: U$$

To $\displaystyle\min_{a,b\in\mathbb{R}} \| e^{-\#} - (a e^{-2\#} + b e^{-3\#}) \|_2^2 \equiv \min_{u\in U} \| \underset{\substack{\shortparallel \\ f}}{e^{-\#}} - u \|_2^2$

So, if U is closed, the sol of this pb would be $u = \Pi_U f$.

Because we don't have yet a formula for Π_U, we find the sol treating the pb as an ordinary calc. pb:

$$F(a,b) := \int_0^{+\infty} \left| e^{-x} - \left(a e^{-2x} + b e^{-3x}\right) \right|^2 dx$$

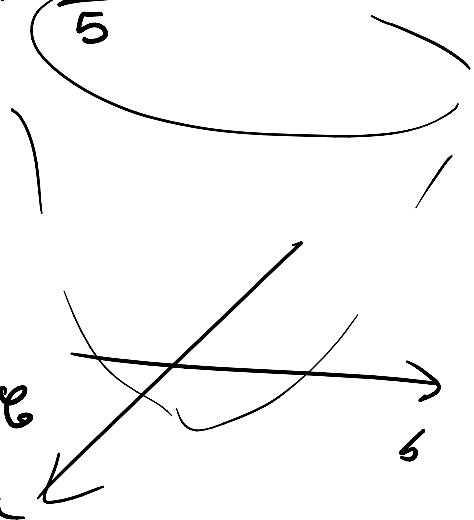
$$F : \mathbb{R}^2 \longrightarrow [0,+\infty[\qquad \left(e^{-x} - a e^{-2x} - b e^{-3x}\right)^2$$

$$= \int_0^{+\infty} \left(e^{-2x} + a^2 e^{-4x} + b^2 e^{-6x} - 2a e^{-3x} - 2b e^{-4x} + 2ab\, e^{-5x} \right) dx$$

$$\int_0^{+\infty} e^{-kx}\, dx = \left[\frac{e^{-kx}}{-k} \right]_{x=0}^{x=+\infty} = \frac{1}{k}.$$

$k > 0$

$$= \frac{1}{2} + \frac{a^2}{4} + \frac{b^2}{6} - \frac{2a}{3} - \frac{2b}{4} + \frac{2ab}{5}$$



We can see that

$$\lim_{(a,b)\to\infty_2} F(a,b) = +\infty$$

Once we know this, being $F \in \mathcal{C}$

$\Rightarrow$ (Weierstrass) $\exists$ global min for F

$$(\hat{a},\hat{b}) : \quad F(\hat{a},\hat{b}) \leq F(a,b) \quad \forall (a,b) \in \mathbb{R}^2$$

$$a/2 = \rho \cos\theta \qquad \left(\frac{a}{2}\right)^2 + \left(\frac{b}{\sqrt{6}}\right)^2 = \rho^2$$

$$\frac{a}{2} = \rho\cos\theta \qquad \left(\frac{a}{2}\right)^2 + \left(\frac{b}{\sqrt6}\right)^2 = \rho^2$$

$$\frac{b}{\sqrt6} = \rho\sin\theta \qquad \frac{a^2}{4} + \frac{b^2}{6} = \rho^2$$

$$F = \frac{1}{2} + \rho^2 - \frac{2}{3}2\rho\cos\theta - \frac{1}{2}\sqrt6\rho\sin\theta + \frac{2}{5}\left(2\sqrt6\,\rho^2\cos\theta\sin\theta\right)$$

$$= \rho^2 + \frac{2\sqrt6}{5}\rho^2\underbrace{\sin(2\theta)}_{\sin 2\theta} - \frac{4}{3}\rho\overset{1}{\overset{\wedge}{\cos\theta}} - \frac{\sqrt6}{2}\rho\overset{1}{\overset{\wedge}{\sin\theta}} + \frac{1}{2}$$

$$= \rho^2\left(1 + \frac{2\sqrt6}{5}\underset{-1}{\underset{\vee}{\sin(2\theta)}}\right) \geq \left(1 - \frac{2\sqrt6}{5}\right) > 0$$

$$\frac{2\sqrt6}{5} \overset{?}{<} 1 \iff 2\sqrt6 < 5 \iff 4\cdot6 < 25$$

$$\geq \rho^2\left(1 - \frac{2\sqrt6}{5}\right) - \left(\frac{4}{3} + \frac{\sqrt6}{2}\right)\rho + \frac{1}{2} \xrightarrow[\rho\to+\infty]{} +\infty$$

$$\Downarrow$$

$$\lim_{(a,b)\to\infty_2} F = +\infty.$$

Now by min F ∃. To find it we look for st pts for F:

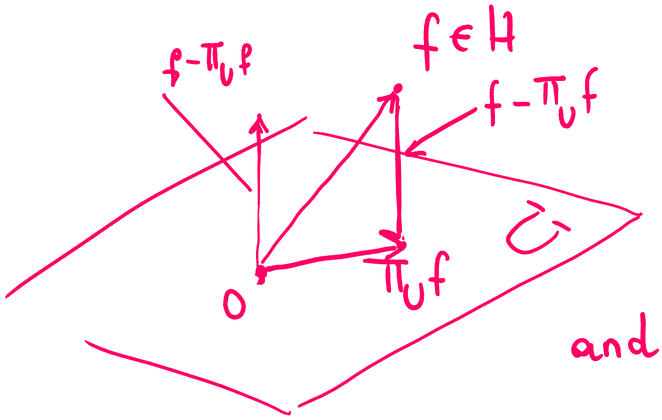
$$\nabla F = 0 \iff \begin{cases} \partial_a & \frac{a}{2} - \frac{2}{3} + \frac{2b}{5} = 0 \\ \partial_b & \frac{b}{3} - \frac{1}{2} + \frac{2a}{5} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \dots \\ b = \dots \end{cases}$$

$\blacksquare$

We will return on this type of pbs after some remarks on orthogonal proj. Let's first refresh this concept:

Thm: Let H be an Hilbert space, $U \subset H$ be a closed subspace of H. Then,



$\forall f \in H \ \exists ! \, \pi_U f \in U$:

$$\|f - \pi_U f\| = \min_{u \in U} \|f - u\|$$

and

$$\langle f - \pi_U f, u \rangle = 0 \quad \forall u \in U.$$

$$\left[\; f = \underbrace{\pi_U f} + \underbrace{(f - \pi_U f)}_{\perp U} \; \right]$$

Corollary: $U = \operatorname{Span} \langle e_n \rangle_{n \in J}$ $\quad J \subset \mathbb{N}$, (e_n) orthonormal vectors

$$= \Big\{ \sum_{k \in J} \overbrace{f_k e_k}^{v_k} \in H \Big\} \qquad \langle e_n, e_m \rangle = \delta_{nm}$$

$\Updownarrow$

$$\sum_{k\in\mathcal{J}} f_k^2 < +\infty$$

$\ulcorner$ If vectors v_k are orthogonal $\sum v_k$ conv in H

$$\Updownarrow$$

$$\sum \|v_k\|^2 < +\infty.$$

Then

$$\pi_U f = \sum_{n\in\mathcal{J}} \langle f, e_n\rangle e_n$$

<u>Proof:</u> First $\Pi_U f \in U \Rightarrow \Pi_U f = \sum f_k e_k$

for suitable $(f_k) \subset \mathbb{R}$. Now because

$$\langle f - \Pi_U f, u\rangle = 0 \quad \forall u \in U$$

$$\Downarrow$$

$$\langle f - \Pi_U f, e_n\rangle = 0 \quad \forall k \in \mathcal{J}$$

$$\|$$

$$\langle f, e_n\rangle - \langle \Pi_U f, e_n\rangle = 0$$

$$\|$$

$$\left\langle \sum_{k\in\mathcal{J}} f_k e_k, e_n\right\rangle$$

$$\|$$

$$\lim \sum_{\text{finite}}$$

$$\left\langle \sum_{k\in J} f_k e_k , e_n \right\rangle = \lim \sum_{\text{finite}}$$

$$\langle f, e_n \rangle - \sum_k f_k \underbrace{\langle e_k, e_n \rangle}_{= \delta_{kn}} = 0$$

$$\Updownarrow$$

$$\langle f, e_n \rangle - f_n \cdot 1 = 0 \Rightarrow f_n = \langle f, e_n \rangle$$

$$\Rightarrow \quad \Pi_U f = \sum_{k\in J} \langle f, e_k \rangle e_k . \qquad \blacksquare$$

Corollary: (test for orthonormal bases)
Let H be an Hilbert space, (e_n) be an orthonormal system of vectors $\langle e_n, e_m \rangle = \delta_{nm}$
(e_n) is a base for $H \iff$ the following holds:

$$\boxed{\langle f, e_n \rangle = 0 \ \forall n \implies f = 0.} \quad (*)$$

Proof:
$\Rightarrow$ Hp (e_n) orthonormal base
Th: $(*)$ holds true.
Supp. $\langle f, e_n \rangle = 0 \ \forall n$

Supp. $\langle f, e_n \rangle = 0 \quad \forall n$

By Hyp

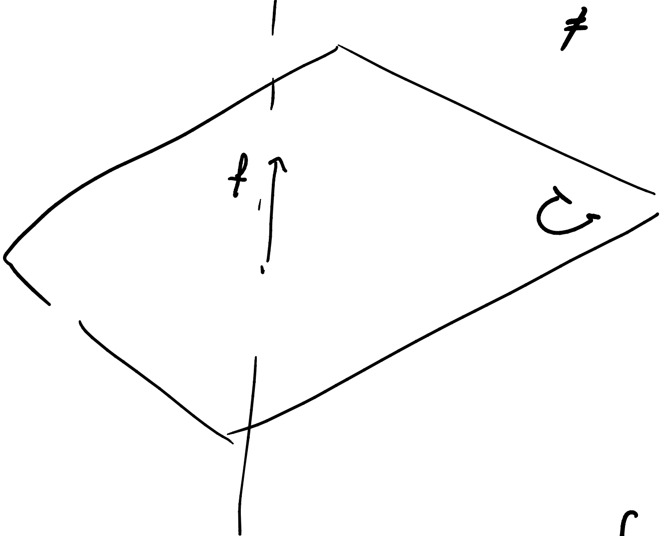
$$f = \sum \langle f, e_n \rangle e_n = \sum 0 e_n = 0.$$

$\Leftarrow$ Hp: $(*)$ true $\left(f \perp e_n \ \forall n \Rightarrow f = 0 \right)$

Th: (e_n) is a base.

$\overline{U} = \mathrm{Span}\langle e_n \rangle$. The thesis consists in proving that $\overline{U} = H$

Assume $U \subsetneq H$,



$$f = \pi_U f + \underbrace{(f - \pi_U f)}$$

$$\langle f - \pi_U f, e_n \rangle = 0 \quad \forall n$$

$$\Downarrow$$

$$f - \pi_U f \perp e_n \quad \forall n$$

$$\Downarrow \text{Hyp}$$

$$f - \pi_U f = 0$$

$$\Downarrow$$

$$f = \pi_U f \in U$$

$$\Rightarrow H \subset U \Rightarrow H = \overline{U} \quad \blacksquare$$

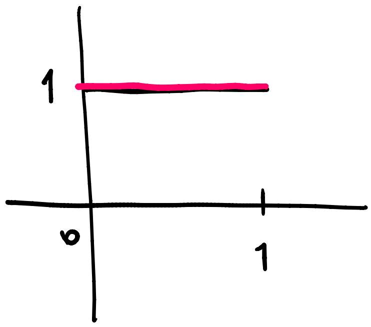
Example (The Haar base)

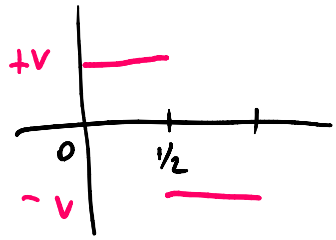
This provides a model for a base of

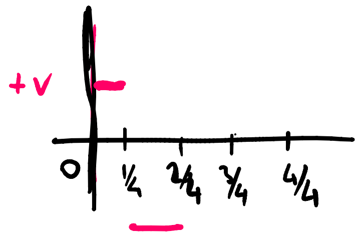
$L^2(I) \qquad I \subset \mathbb{R}^d$ is an interval

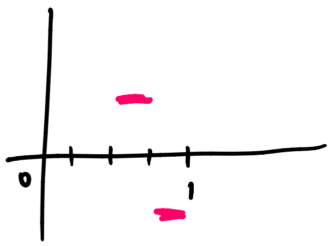
$I = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$

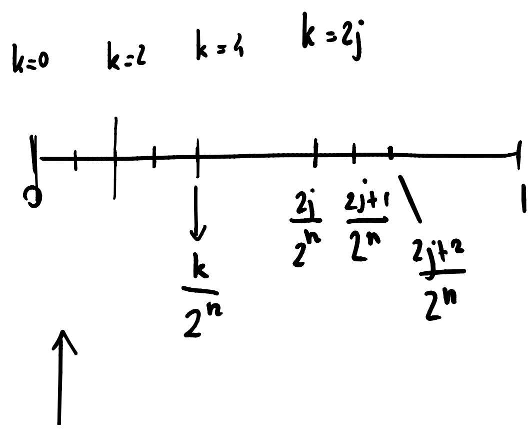
We show the idea on $I = [0, 1]$.



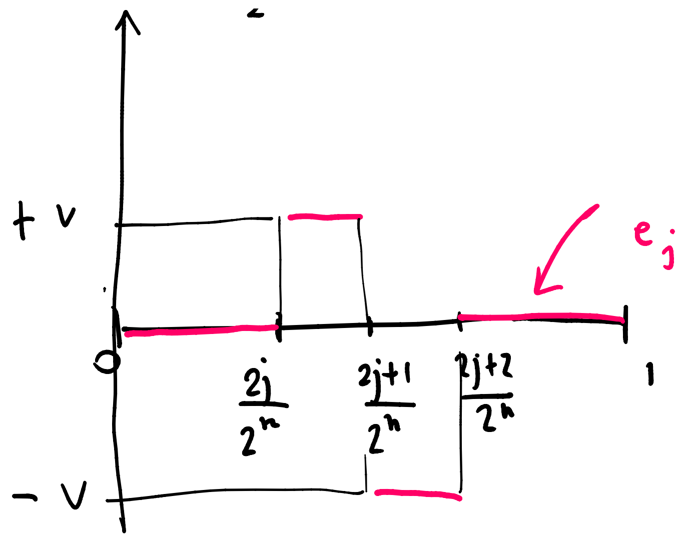








$k = 0, \ldots, 2^n$



$$e_{j,n}(x) = +v\,\mathbb{1}_{\left[\frac{2j}{2^n},\frac{2j+1}{2^n}\right]} - v\,\mathbb{1}_{\left[\frac{2j+1}{2^n},\frac{2j+2}{2^n}\right]}$$

where v is imposed by $\|e_{j,n}\|_2 = 1$

$$1 = \int_0^1 |e_{j,n}|^2\,dx = 2v^2\frac{1}{2^n} = \frac{v^2}{2^{n-1}} \qquad v = 2^{\frac{n-1}{2}}$$

$$e_{j,n}(x) = +2^{\frac{n-1}{2}}\,\mathbb{1}_{\left[\frac{2j}{2^n},\frac{2j+1}{2^n}\right]} - 2^{\frac{n-1}{2}}\,\mathbb{1}_{\left[\frac{2j+1}{2^n},\frac{2j+2}{2^n}\right]}$$

$j = 0, 1, \dots, 2^{n-1} - 1$

$n = 1, \dots$

$\rightarrow e_{0,0} \equiv 1$

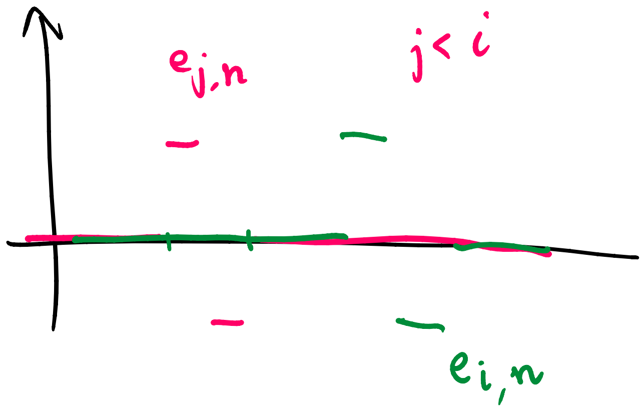
Thm: $(e_{j,n})_{\substack{n=0,\dots \\ j=0,\dots,2^{n-1}-1}} \cup \{e_{0,0}\}$ is an orthonormal base for $L^2([0,1])$.

Proof: We proved $\|e_{j,n}\|_2 = 1 \quad \forall j, n$

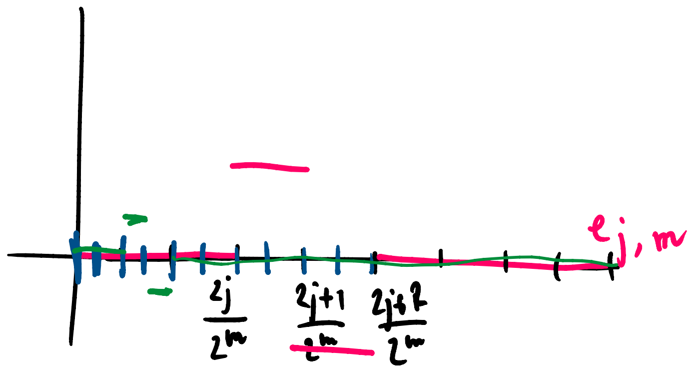
$$\langle e_{j,n}, e_{i,m} \rangle = 0 \qquad \begin{array}{l} \forall\, n \neq m \\ \forall\, n = m \quad j \neq i \end{array}$$

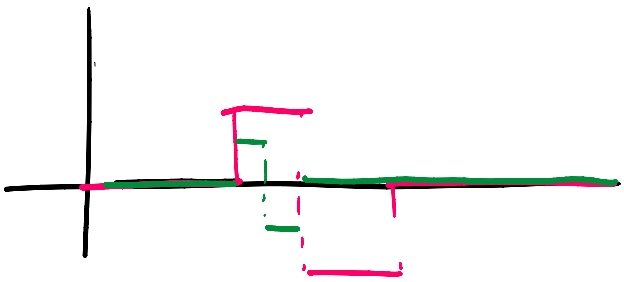
For example

$n = m \quad j \neq i$



$n \neq m \qquad n = m+1 \qquad e_{i,n} = e_{i,m+1}$

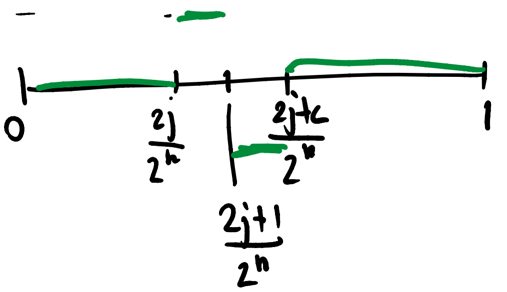




To check $(e_{j,n})$ is a base suppose now $f \in L^2([0,1])$:

$$\Rightarrow \langle f, e_{j,n} \rangle = 0 \quad \forall j,n$$

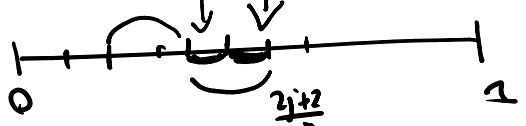
$$0 = \int_0^1 f e_{j,n} = \int_{\frac{2j}{2^n}}^{\frac{2j+2}{2^n}} f \cdot e_{j,n}$$



$$= \int_{\frac{2j}{2^n}}^{\frac{2j+1}{2^n}} f \cdot \left(\cancel{2^{\frac{n-1}{2}}}\right) + \int_{\frac{2j+1}{2^n}}^{\frac{2j+2}{2^n}} f \cdot \left(\cancel{-2^{\frac{n-1}{2}}}\right)$$

$$\Downarrow$$

$$\int_{\frac{2j}{2^n}}^{\frac{2j+1}{2^n}} f = \int_{\frac{2j+1}{2^n}}^{\frac{2j+2}{2^n}} f$$



$$2 \int_{\frac{2j}{2^n}}^{\frac{2j+2}{2^n}} f$$

$$2^n \int_{\frac{2j}{2^n}}^{\frac{2j+1}{2^n}} f = \int_0^1 f = 0 \leftarrow \left(\langle f, e_{0,0} \rangle = 0 \right)$$

$$\Rightarrow \int_{\frac{2j}{2^n}}^{\frac{2j+1}{2^n}} f = 0 \qquad \forall j,n$$

$$\overline{2^n}$$

$$\Rightarrow \quad \int_{\frac{k}{2^n}}^{\frac{h}{2^n}} f = 0 \qquad \forall\, k, h, \quad \forall n$$

$$\Rightarrow \quad \int_I f = 0 \qquad I = [a, b] \quad a, b \text{ dydic numbers}$$

$$a = \frac{k}{2^n} \qquad b = \frac{h}{2^n}$$

$$\Rightarrow \quad \int_a^b f = 0 \qquad \forall\ a, b \in [0,1]$$

$$\Rightarrow \quad \int_E f = 0 \qquad \forall\ E \subset [0,1] \text{ meas.}$$

$$\Rightarrow \quad f = 0 \quad \text{a.e. on } [0,1]$$

$\Rightarrow$ the concl follows by density test. $\boxtimes$

<u>Ex 4.5.3</u> Solve

$$\min_{a,b,c \in \mathbb{R}} \int_{-1}^{1} |x^3 + ax^2 + bx + c|^2 \, dx$$

<u>Sol:</u> Let $H = L^2([-1,1])$,

$$\langle f, g \rangle = \int_{-1}^{1} fg \, dx$$

... consists in finding

H is an Hilb space. The pb consists in finding

$$\min_{a,b,c} \| x^3 - \underbrace{(ax^2 + bx + c1)} \|_2^2$$

$$= \min_{u \in \mathrm{Span}\langle 1, x, x^2 \rangle} \| x^3 - u \|_2^2$$

Once $\mathrm{Span}\langle 1, x, x^2 \rangle =: U$ is closed, the sol of this pb is

$$\Pi_{\bar{U}} x^3$$

It is clear that $1, x, x^2$ are lin indep

$$(ax^2 + bx + c = 0 \text{ a.e. } \Rightarrow a, b, c = 0)$$

$\Rightarrow$ $\mathrm{Span}\langle 1, x, x^2 \rangle$ has dim 3.

Unfortunately $(1, x, x^2)$ is not nec. an ortho-normal base:

1 $\quad \| 1 \|_2^2 = \int_{-1}^{1} |1|^2 \, dx = 2$

$\Rightarrow \underset{e_0}{\underset{\shortparallel}{\frac{1}{\sqrt{2}}}}$ it's a unit vect in L^2.

$x \quad \langle x, \underset{e_0}{\underset{\shortparallel}{\frac{1}{\sqrt{2}}}} \rangle_{L^2} = \int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}} \, dx = 0$

$\acute{e}_0$

$$\|x\|_2^2 = \int_{-1}^{1} |x|^2\, dx = 2\int_0^1 x^2\, dx = 2\left.\frac{x^3}{3}\right|_0^1$$

$$= \frac{2}{3}$$

$$\|x\|_2 = \sqrt{\frac{2}{3}} \qquad e_1 = \sqrt{\frac{3}{2}}\, x.$$

x^2 $\quad \langle x^2, e_0\rangle = \int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}} \neq 0.$

$$\langle x^2, e_1\rangle = \int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{2}}\, x = \sqrt{\frac{3}{2}} \underbrace{\int_{-1}^{1} x^3}_{=0}$$

How do we adjust $\underbrace{x^2 + \alpha x + \beta}$ $\perp e_0, e_1$?

Span $\langle x^2 + \alpha x + \beta, x, 1\rangle$ = Span $\langle 1, x, x^2\rangle$

To find α, β

$$\begin{cases} \langle x^2 + \alpha x + \beta, \frac{1}{\sqrt{2}}\rangle = 0 \\ \langle x^2 + \alpha x + \beta, \sqrt{\frac{3}{2}}\, x\rangle = 0 \end{cases}$$

$$\Leftrightarrow \quad \frac{1}{\sqrt{2}}\int_{-1}^{1} x^2 + \frac{\alpha}{\sqrt{2}}\underbrace{\int_{-1}^{1} x\, dx}_{=0} + \frac{\beta}{\sqrt{2}}\int_{-1}^{1} 1 = 0$$

$$\int_{-1}^{1} x^2 = 2\left.\frac{x^3}{3}\right|_0^1 = 2/3$$

$$\frac{\sqrt{2}}{3} + 0.\alpha + \sqrt{2}\,\beta = 0 \implies \beta = -\frac{1}{3}$$

$$\frac{\sqrt{2}}{3} + 0.\alpha + \sqrt{2}\beta = 0 \implies \beta = -\frac{1}{3}$$

$$\iff \sqrt{\tfrac{3}{2}} \underbrace{\int_{-1}^{1} x^3}_{=0} + \alpha\sqrt{\tfrac{3}{2}} \int_{-1}^{1} x^2 + \beta\sqrt{\tfrac{3}{2}} \underbrace{\int_{-1}^{1} x}_{=0} = 0$$

$$\iff \alpha = 0$$

$$x^2 - \tfrac{1}{3} \perp e_0, e_1$$

$$\|x^2 - \tfrac{1}{3}\|_2^2 = \int_{-1}^{1} \left(x^2 - \tfrac{1}{3}\right)^2 dx = \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)$$

$$= 2\left[\frac{x^5}{5}\Big|_0^1 - \frac{2}{3}\,\frac{x^3}{3}\Big|_0^1 + \frac{1}{9}\cdot 1\right]$$

$$= 2\left[\frac{1}{5} \underbrace{- \frac{2}{9} + \frac{1}{9}}_{-1/9}\right]$$

$$= 2\,\frac{4}{45} = \frac{8}{45}$$

$$e_2 = \sqrt{\frac{45}{8}}\left(x^2 - \tfrac{1}{3}\right).$$

Now $\langle e_0, e_1, e_2 \rangle$ is an ortho-normal base for U.

$$\Pi_U x^3 = \sum_{j=0}^{2} \langle x^3, e_j \rangle e_j \qquad 2\int_0^1$$

$$= \underbrace{\left(\int_{-1}^{1} t^3 \cdot \frac{1}{\sqrt{2}}\,dt\right)}_{=0} \cdot \frac{1}{\sqrt{2}} + \left(\int_{-1}^{1} t^3 \sqrt{\tfrac{3}{2}}\, t\, dt\right) \sqrt{\tfrac{3}{2}}\, x$$

$$+ \left(\overset{0}{\int_{-1}^{1}} t^3 \cdot \sqrt{\frac{45}{8}} \left(t^2 - \tfrac{1}{3}\right) dt \right)^{\!=0} \sqrt{\frac{45}{8}} \left(x^2 - \tfrac{1}{3}\right)$$

$$= \left(\frac{3}{\cancel{2}} \cancel{2} \, \frac{t^5}{5} \Big|_0^1 \right) x = \frac{3}{5} x. \qquad \blacksquare$$