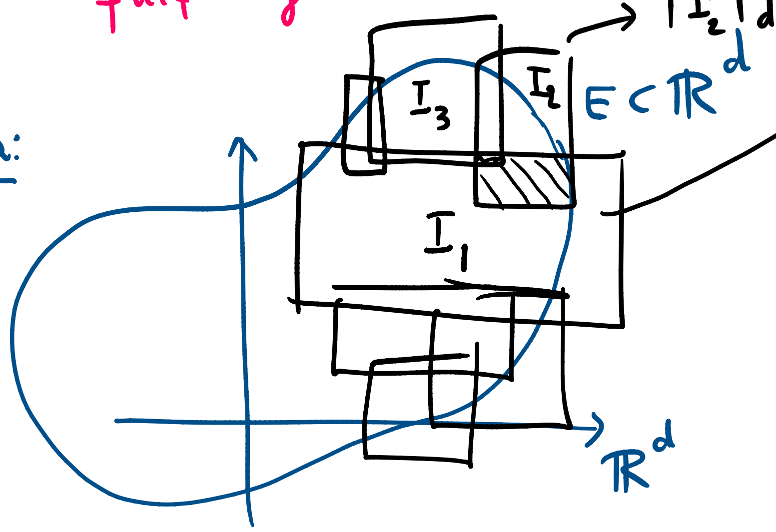
martedì 1 ottobre 2019 12:48

Outer Measure

Goal: To def. $\lambda_d : P(\mathbb{R}^d) \longrightarrow [0,+\infty]$

fulfilling certain conds.

Idea:



$\cancel{\lambda_d(I_1)}$ $|I_1|_d$

Suppose

$$E \subset \bigcup_{k=1}^{\infty} I_k$$

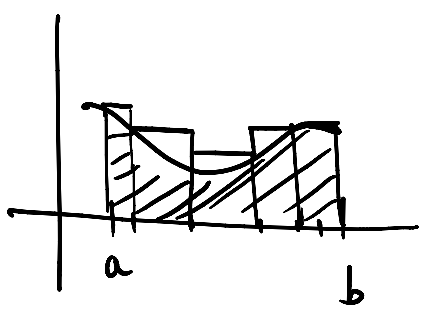
where I_k are intervals in $\mathbb{R}^d$

(not necess. disjoint)

$$|I|_d := (b_1-a_1)(b_2-a_2) \cdots (b_d-a_d)$$

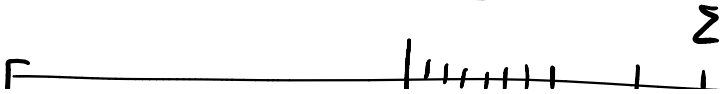
$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$$

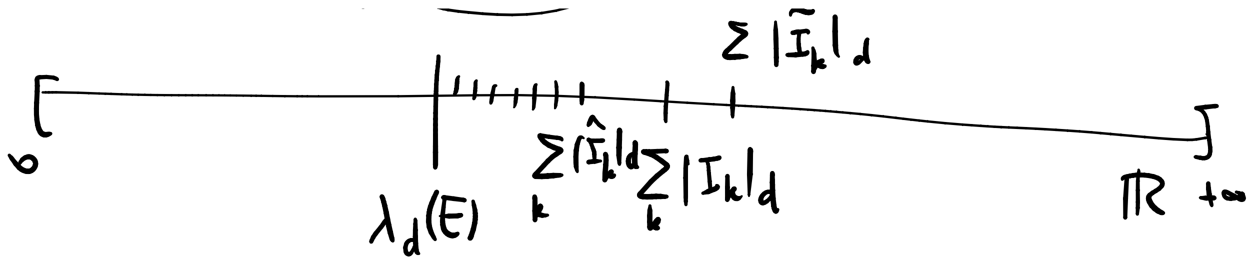
Naturally $\sum_k |I_k|_d$ is an approx by exass of my goal $\lambda_d(E)$



$$\left\{ \underbrace{\sum_{k=0}^{\infty} |I_k|_d} : (I_k) \text{ s.t. } E \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

$\sum |\tilde{I}_k|_d$





$$\boxed{\lambda_d^*(E) := \operatorname{Inf}\left\{ \overset{0\,\le}{\sum_k |I_k|_d} : E \subset \bigcup_{k=1}^{\infty} I_k \right\}}$$

$\forall\, E \in P(\mathbb{R}^d).$

outer measure

$$\lambda_d^* : P(\mathbb{R}^d) \longrightarrow [0, +\infty].$$

Pb: Is λ_d^* a sol of our pb?

Prop 1:

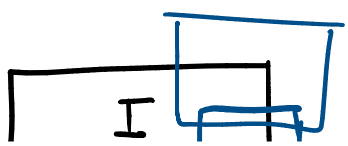
$$\lambda_d^*(I) = |I|_d. \quad \left(= (b_1 - a_1)\cdot(b_2 - a_2)\cdots(b_d - a_d) \right)$$

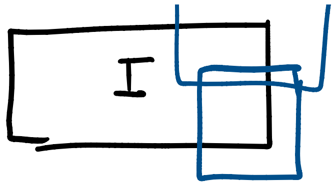
interval $[a_1, b_1] \times \cdots \times [a_d, b_d]$

Proof: We've to check that

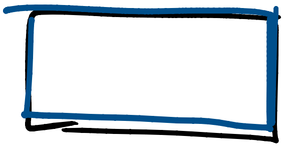
$$\lambda_d^*(I) = \operatorname{Inf}\left\{ \sum_k |I_k|_d : I \subset \bigcup I_k \right\}$$

$$\stackrel{?}{=} |I|_d$$





$$\lambda_d^*(I) \le |I|_d$$



$I_1 = I$

$I_2 = \emptyset$

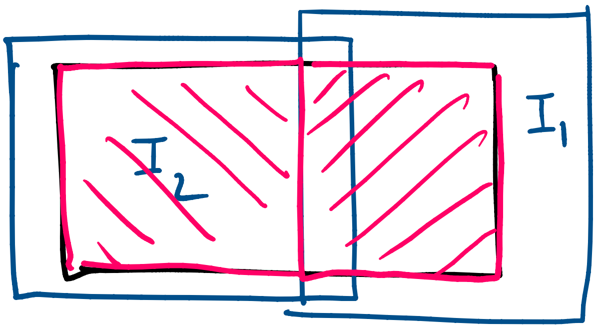
$$\Downarrow$$

$$\sum |I_k|_d = |I|_d$$

$$\Rightarrow \quad |I|_d \in \Big\{ \sum |I_k|_d \ : \ I \subset \bigcup_k I_k \Big\}$$

$$\Rightarrow \quad |I|_d \ge \inf \downarrow = \lambda_d^*(I).$$

Now: how do we prove that $\lambda_d^*(I) \ge |I|_d$?



$$\underbrace{|I_1|_d + |I_2|_d}_{\Downarrow} \ge |I|_d$$

$|I|_d$ is a bound for $\sum |I_k|_d$

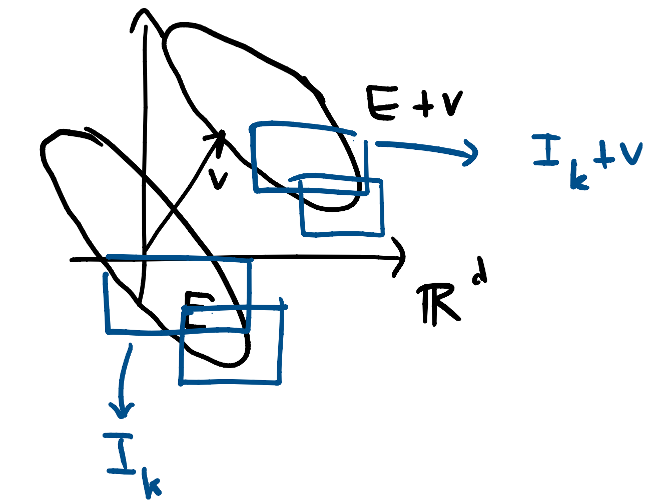
$$\Downarrow$$

$$\lambda_d^*(I) \ge |I|$$

λ^* is transl. invariant

Prop 2: λ_d^* is transl. invariant

$$\lambda_d^*(E+v) = \lambda_d^*(E) \qquad \forall E \in P(\mathbb{R}^d),\ \forall v \in \mathbb{R}^d.$$



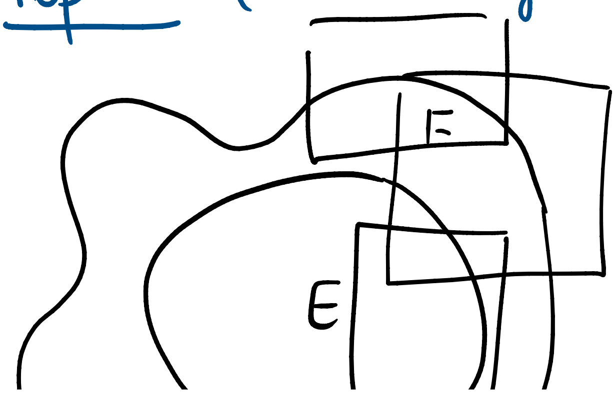
$$E \subset \bigcup_k I_k$$
$$\Downarrow$$
$$E+v \subset \bigcup_k (I_k + v)$$
$$\Downarrow$$

$$E \subset \bigcup_k J_k - v \qquad \Uparrow \qquad E+v \subset \bigcup_k J_k \quad \leftarrow \text{inf.}$$

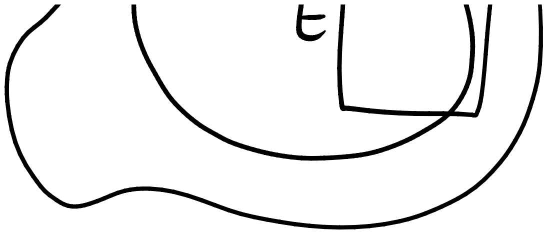
$$\lambda_d^*(E) = \inf \left\{ \sum |I_k|_d \right\}$$
$$\left\{ \sum_k |I_k + v|_d \right\} \geq \lambda_d^*(E+v)$$

$$\lambda_d^*(E+v) \geq \lambda_d^*(E) \implies = . \qquad \square$$

Prop 3: (monotonicity) $E \subset F \Rightarrow \lambda_d^*(E) \leq \lambda_d^*(F)$



$$E \subset F \subset \bigcup_k I_k$$
$$\Downarrow$$
$$\lambda^*(E) \leq \sum |I_k|_d \qquad \forall (I_k)$$



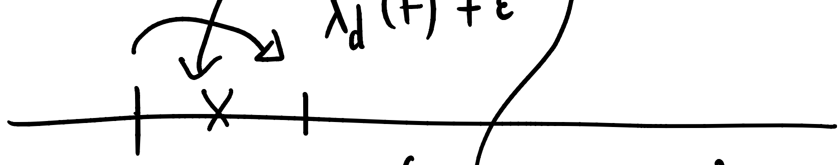
$$\lambda(E) \leq \underbrace{\sum |I_k|_d}_{} \qquad v(I_k)$$

$$\operatorname{Inf} = \lambda_d^*(F) \qquad F \subset \bigcup_k I_k$$

If $\lambda_d^*(F) = +\infty$ nothing to prove

$\underline{\lambda_d^*(F)} \leq +\infty$

$\lambda_d^*(F) + \varepsilon$



$$\lambda_d^*(F) = \operatorname{Inf}\{ \sim \}$$

$$\forall \varepsilon > 0 \quad \exists \sum |I_k|_d \quad \bigcup I_k \supset F \supset E$$

$$\lambda_d^*(E) \leq \sum |I_k|_d < \lambda_d^*(F) + \varepsilon$$

$$\Downarrow$$

$$\lambda^*(E) \leq \lambda^*(F) + \varepsilon \qquad \forall \varepsilon > 0$$

$$\Downarrow$$

$$\lambda^*(E) \leq \lambda^*(F) \qquad \blacksquare$$

<u>Thm:</u> T inv matrix

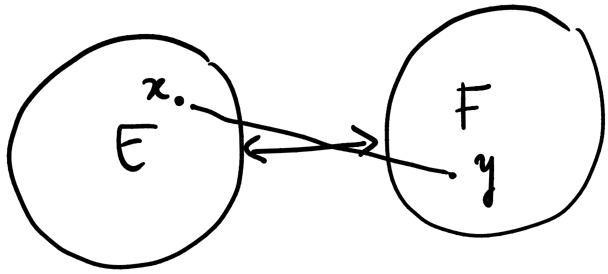
$$\lambda_d^*(TE+v) = |\det T| \, \lambda_d^*(E) \qquad \forall E \subset \mathcal{P}(\mathbb{R}^d) \quad \forall v \in \mathbb{R}^d.$$

$$\operatorname{dist}(E,F) \mathrel{\overset{..}{=}} \operatorname{Inf}\{ \|x-y\| : x \in E, y \in F \}$$

Prob: Let $E, F \in \mathcal{P}(\mathbb{R}^d)$ s.t.

Prop: Let $E, F \in \mathcal{P}(\mathbb{R}^d)$ s.t. $\inf\{\|x-y\| : x \in E, y \in F\}$

$$\mathrm{dist}(E,F) > 0$$

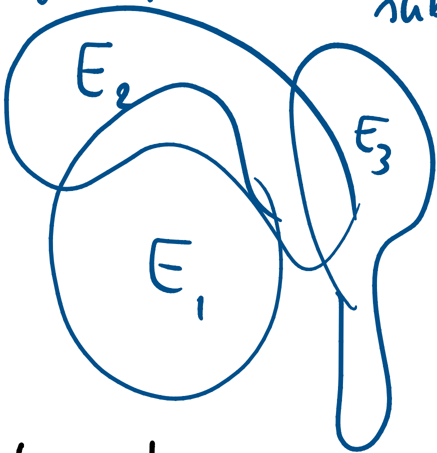


Then

$$\lambda_d^*(E \cup F) = \lambda_d^*(E) + \lambda_d^*(F) \quad \text{(additivity)}$$

Prop: $(E_n)_{n \in \mathbb{N}} \subset \mathcal{P}(\mathbb{R}^d)$ (not nec. disj.)

$$\lambda_d^*\left(\bigcup E_n\right) \leq \sum_n \lambda_d^*(E_n) \quad \text{(countable sub-additivity)}$$



Essentially, up to this pt it remains count add to be checked.

Thm (Vitali) $\nexists\ \lambda_d : \mathcal{P}(\mathbb{R}^d) \longrightarrow [0, +\infty]$ s.t.

i) $\lambda_d(I) = |I|_d$

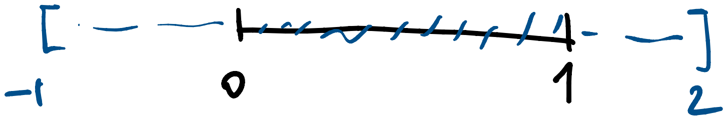
ii) λ_d be transl. inv.

iii) λ_d be countably additive.

$\exists\ S : [-1,2] \supset S + q,\ S + \tilde{q},\ \cap = \emptyset$, $q \in \mathbb{Q}$, $q \neq \tilde{q}$

$d=1$

$\exists\ S$: $[-1,2] \supset S+\tilde{q}$, $\cap = \phi$ $\forall \tilde{q}$



$q \in [0,1]$

$$[-1,2] \supset \bigcup_q (S+q) \supset [0,1]$$

$$\text{i)} \qquad \Downarrow \lambda_1 \qquad \text{i)}$$

$$3 \geq \lambda_1\left(\bigcup S+q\right) \geq 1$$

$$\overset{\text{iii)}}{=} \sum \lambda_1(S+q)$$

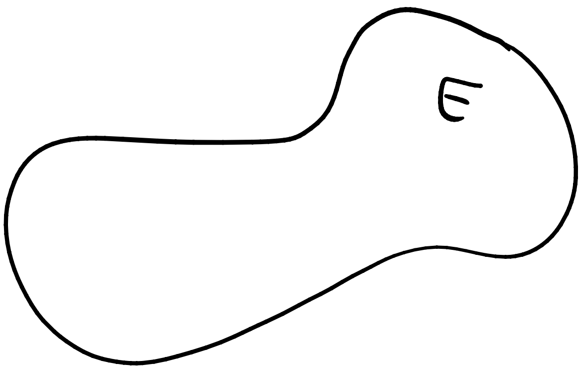
$$\overset{\text{ii)}}{=} \sum_q \lambda_1(S)$$

So, in part, λ_d^* won't be the sol of our pb as measure on $\mathcal{P}(\mathbb{R}^d)$. However, it turns out that restricting a bit the class $\mathcal{P}(\mathbb{R}^d)$, λ_d^* has all the required properties.

How this family is defined?

Def: (Lebesgue class)

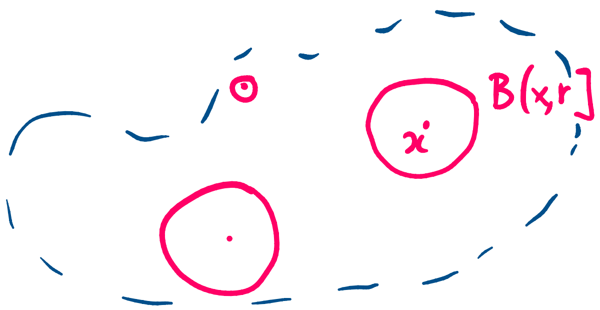
Def: We say that $\Omega \subset \mathbb{R}^d$ is open



$O \subset \mathbb{R}^d$ is <u>open</u>

if

$\forall x \in O \quad \exists \; B(x,r] = \{y \in \mathbb{R}^d : \|y-x\| \leq r\} \subset O$

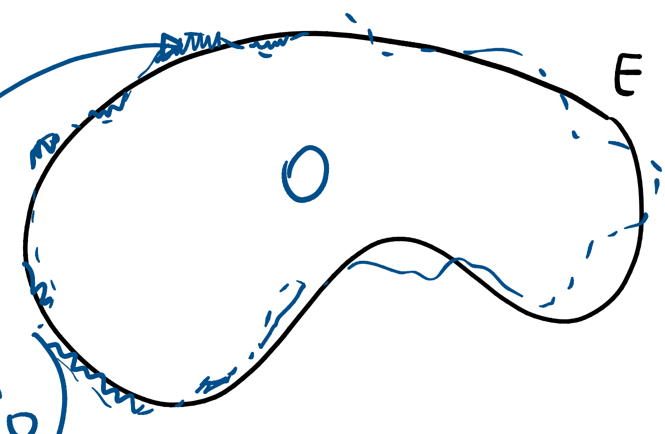


$x^2 + y^2 + z^2 \leq r^2$

<u>Rmk:</u> A set of type $O = \{x \in \mathbb{R}^d : f(x) > 0\}$ with $f \in \mathcal{C}(\mathbb{R}^d) \Rightarrow O$ is open.

Furthermore, unions and intersections of a finite number of open sets is open.

<u>Def:</u> (Leb class)
We say that $E \subset \mathbb{R}^d$ is ($\mathcal{L}$eb) meas. if it <u>differs</u> by an open set for a $\lambda^* = 0$ meas set



$E \Delta O$
(excess between E and O)

$\ulcorner \; A \Delta B = A \setminus B \cup B \setminus A \; \lrcorner$

$\ulcorner A \triangle B = A \setminus B \cup B \setminus A \lrcorner$

Formally: we say $E \in \mathcal{M}_d$ (Leb. class) if

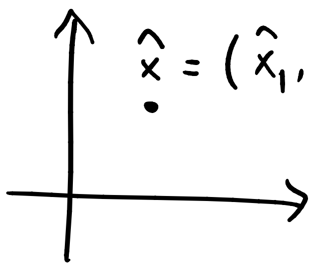
$$\boxed{\exists\, O \text{ open} : \lambda_d^*(E \triangle O) = 0.}$$

We say that a set $N \subset \mathbb{R}^d$ is negligible / null set if $\lambda_d^*(N) = 0$.

Examples:

i) $\emptyset$ is a null set

ii) $N = \{\hat{x}\}$ where $\hat{x} \in \mathbb{R}^d$ is null set



$\hat{x} = (\hat{x}_1, \hat{x}_2, \dots \hat{x}_d)$

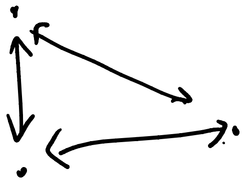
$$I = [\hat{x}_1, \hat{x}_1] \times \dots \times [\hat{x}_d, \hat{x}_d] = \{\hat{x}\}$$

$$\lambda_d^*(\{\hat{x}\}) = \lambda_d^*(I) = |I|_d = \underbrace{(\hat{x}_1 - \hat{x}_1)}_{0} \cdots = 0.$$

iii) $N = \{\underset{\mathbb{R}^d}{\hat{x}_1}, \underset{\mathbb{R}^d}{\hat{x}_2}, \dots, \underset{\mathbb{R}^d}{\hat{x}_k}\}$ (finite set)

$$\lambda_d^*(N) = 0.$$

$$0 \le \lambda_d^*(\{\hat{x}_1 \dots \hat{x}_k\}) = \lambda_d^*\left(\bigcup_{j=1}^{k} \{\hat{x}_j\}\right)$$



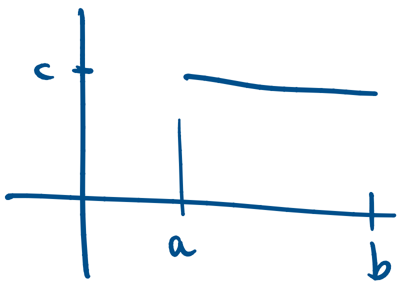
$$\underset{\text{sub-add}}{\leqslant} \sum_{j=1}^{k} \underbrace{\lambda_d^*(\{\hat{x}_j\})}_{=0} = 0$$

$$\Rightarrow \lambda_d^*(N) = 0.$$

iii) Countable set are null sets.

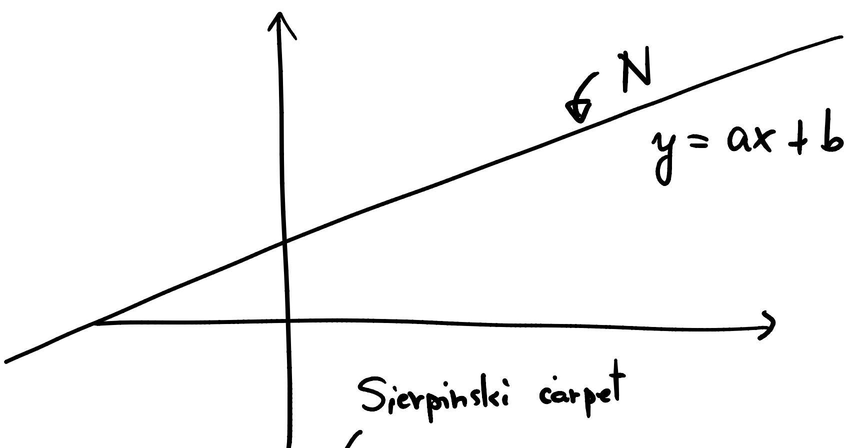
$$N = \{\hat{x}_1, \dots \hat{x}_n, \dots\}$$

$$\lambda_d^*(N) = \lambda_d^*\left(\bigcup_{j=1}^{+\infty} \{\hat{x}_j\}\right) \underset{\text{sub add}}{\leqslant} \sum_{j=1}^{+\infty} \underbrace{\lambda_d^*(\{\hat{x}_j\})}_{=0} = 0.$$

iv) 

$[a,b] \times [c,c]$

$$\lambda_2^*([a,b]\times[c,c]) = 0$$



$\lambda_2^*(N) = 0.$

$y = ax + b$ Try to prove

Sierpinski carpet

Do ex 1.9.1 / 1.9.2 / 1.9.3 (*) / 1.9.4

λSMMF2019)

Cantor set

(moodle passwd: ASMME2019)