martedì 19 novembre 2019 12:33

Do 4.5.2, 4.5.5, 4.5.10

Gram - Schmidt orthogonalization

Assume H be Hilbert,

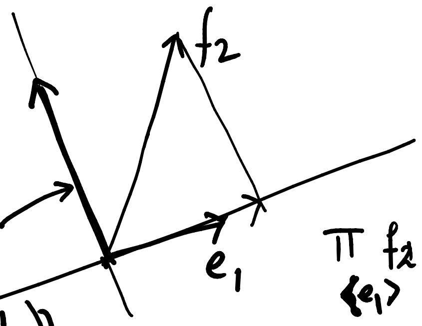
$$U = \mathrm{Span}\,\langle f_n \rangle_{\substack{n\in\mathbb{N} \\ \neq 0}} = \left\{ \sum_{k=0}^{\infty} \alpha_k f_k \ : \ \text{the series conv. in } H \right\}$$

where f_n are linearly indep (in part no f_n is lin comb of f_k $k \neq n$)

Pb: How do we do to det an orthonormal base (for U) by (f_k)?

Idea: take f_1, $\quad e_1 = \dfrac{f_1}{\|f_1\|} \quad (\|e_1\| = 1)$

take f_2

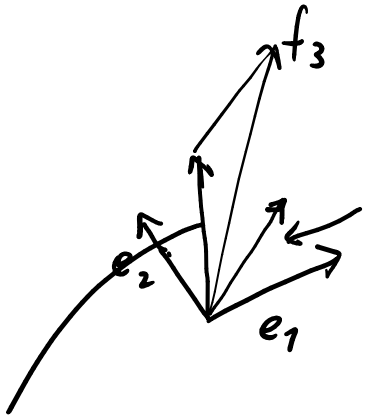


$$\frac{f_2 - \pi_{\langle e_1 \rangle} f_2}{\|f_2 - \pi_{\langle e_1 \rangle} f_2\|} = e_2$$

$$\pi_{\langle e_1 \rangle} f_2 = \langle f_2, e_1 \rangle e_1$$

e_2

f_3



$\Pi_{\mathrm{Span}\langle e_1, e_2\rangle} f_3$

$\| $

$\langle f_3, e_1\rangle e_1 + \langle f_3, e_2\rangle e_2$

$f_3 \notin \mathrm{Span}\langle e_1, e_2\rangle$

$\| $

$\mathrm{Span}\langle f_1, f_2\rangle$

$f_3 - \Pi_{\mathrm{Span}\langle e_1, e_2\rangle} f_3 \neq 0$ (otherwise $f_3 \in \mathrm{Span}\langle e_1, e_2\rangle = \mathrm{Span}\langle f_1, f_2\rangle$

$\Downarrow$

$$e_3 := \frac{f_3 - \Pi_{\mathrm{Span}\langle e_1, e_2\rangle} f_3}{\| \qquad \|}$$

By constr e_3 is s.t.

- $\|e_3\| = 1$
- $e_3 \perp \mathrm{Span}\langle e_1, e_2\rangle \Rightarrow e_3 \perp e_1, e_2$
- $\mathrm{Span}\langle e_1, e_2, e_3\rangle = \mathrm{Span}\langle f_1, f_2, f_3\rangle$

In general you iterate this porocedure to produce

$\vdots$

$$e_n := \frac{f_n - \Pi_{\mathrm{Span}\langle e_1 - - e_{n-1}\rangle} f_n}{\| \qquad // \qquad \|}$$

<u>Example</u> (Hermite polynomials)

$$H = L^2(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} : \int |f(x)|^2 e^{-\frac{x^2}{2}} dx < +\infty \right\}$$

$$H = L^2_{\mathcal{N}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}} |f(x)|^2 e^{-x^2/2} dx < +\infty \right\}$$

$\uparrow$ gaussian normal

Rmk: $1, x, x^2, x^3, \dots \; x^n, \dots \in L^2_{\mathcal{N}}(\mathbb{R}) \quad \forall n \in \mathbb{N}$.

$$\left(f(x) = x^n \Rightarrow \int_{\mathbb{R}} |f|^2 e^{-x^2/2} dx \right.$$

$$\left. = \int_{\mathbb{R}} |x|^{2n} e^{-x^2/2} dx < +\infty \right)$$

In this case $H = \mathrm{Span} \langle 1, x, x^2, \dots \rangle$

However, $f_n(x) = x^n$, (f_n) is not ortho-normal respect to the natural scalar prod:

$$\langle f, g \rangle_{L^2_{\mathcal{N}}} := \int_{\mathbb{R}} f(x) g(x) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

(which is well def'd $\forall f, g \in L^2_{\mathcal{N}}$).

For inst $\langle 1, x^2 \rangle \neq 0$, $\langle x, x^3 \rangle \neq 0 \dots$

Applying the G.S. algorithm we may def an ortho-normal base:

$f_0 = 1 \qquad e_0 = \dfrac{f_0}{\|f_0\|_{L^2_{\mathcal{N}}}}$

for

$$\|f_0\|^2_{L^2_N} = \int_{-\infty}^{+\infty} 1 \cdot e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 1$$

$e_0 \equiv 1.$

$$f_1 = x \quad , \qquad e_1 = \frac{f_1 - \overset{0}{\overset{\|}{\langle f_1, e_0\rangle}} e_0}{\| \quad \|} = \frac{f_1}{\|f_1\|}$$

$$\langle f_1, e_0 \rangle = \int_{-\infty}^{+\infty} x \cdot 1 \quad e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 0$$

$$\|f_1\|^2_{L^2_N} = \int_{-\infty}^{+\infty} x^2 e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$x \cdot \left(x e^{-x^2/2} \right)$$

$$\|$$

$$\left(-e^{-x^2/2}\right)'$$

$$= \Big[\underbrace{-x e^{-x^2/2} \Big|_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}}_{=0} - \int_{-\infty}^{+\infty} 1 \cdot \left(-e^{-x^2/2}\right) \frac{dx}{\sqrt{2\pi}} \Big]$$

$$= \int_{-\infty}^{+\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 1$$

$e_1 = x.$

$$e_2 = \frac{f_2 - \left(\langle f_2, e_0\rangle e_0 + \langle f_2, e_1\rangle e_1 \right)}{}$$

$$\| \qquad \|$$

$$\langle f_2, e_0 \rangle = \int_{-\infty}^{+\infty} x^2 \cdot 1 \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 1$$

$$\langle f_2, e_1 \rangle = \int_{-\infty}^{+\infty} \underbrace{x^2 \cdot x}_{= x^3} \, e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 0$$

$$= \frac{x^2 - 1 \cdot}{\| x^2 - 1 \|_{L^2_N}}$$

$$\| x^2 - 1 \|^2_{L^2_N} = \int_{-\infty}^{+\infty} \underbrace{(x^2-1)^2}_{= x^4 - 2x^2 + 1} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{+\infty} \underbrace{x^4}_{= x^3 \cdot \left(x e^{-x^2/2} \right)} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \underbrace{- 2 \cdot 1 + 1}$$

$$x e^{-x^2/2} = \left(- e^{-x^2/2} \right)'$$

$$= \Big[\underbrace{- x^3 e^{-x^2/2} \Big|_{-\infty}^{+\infty}}_{= 0} \cdot \frac{1}{\sqrt{2\pi}} + 3 \underbrace{\int_{-\infty}^{+\infty} x^2 e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}}_{= 1} - 1$$

$$= 2$$

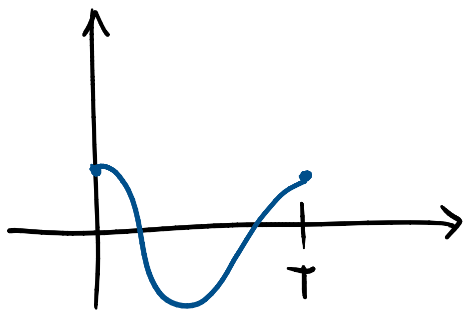
$$e_2 = \frac{x^2 - 1}{\sqrt{2}}.$$

Functs e_n are called Hermite polynomials. ☒

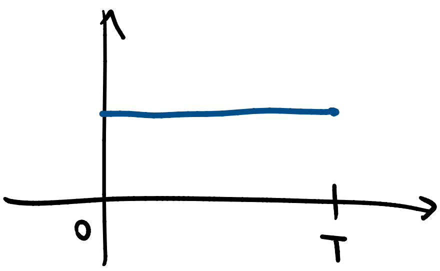
Fourier Analysis

Fourier Series : a crash course

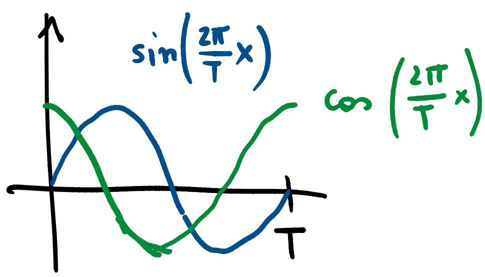
Pb: Suppose $f : [0,T] \to \mathbb{R}$ periodic $f(0) = f(T)$



Natural T - periodic functs are



- constants



- $\sin\left(\frac{2\pi}{T}x\right)$, $\cos\left(\frac{2\pi}{T}x\right)$

- $\sin\left(\frac{2\pi}{T}kx\right)$ $\cos\left(\frac{2\pi}{T}kx\right)$

$k = 0, 1, 2, \ldots \in \mathbb{N}$

An ancien question was the following:

If f is a generic T- per. funct is it possible to represent f as (infinite) linear combination of $\sin\left(\frac{2\pi}{T}kx\right)$, $\cos\left(\frac{2\pi}{T}kx\right)$ $k \in \mathbb{N}$?

$$f(x) = \sum_{k=0}^{\infty}\left[a_k \cos\left(\frac{2\pi}{T}kx\right) + b_k \sin\left(\frac{2\pi}{T}kx\right)\right]$$

$$= a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\left(\frac{2\pi}{T}kx\right) + b_k \sin\left(\frac{2\pi}{T}kx\right)\right)$$

Recall

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= a_0 + \sum_{k=1}^{\infty}\left(a_k \frac{e^{i\frac{2\pi}{T}kx} + \boxed{e^{-i\frac{2\pi}{T}kx}}}{2} + -i b_k \frac{e^{i\frac{2\pi}{T}kx} - e^{-i\frac{2\pi}{T}kx}}{2\cancel{i}}\right)$$

$$= a_0 + \underbrace{\sum_{k=1}^{\infty}\left[\frac{a_k - ib_k}{2} e^{i\cdot\frac{2\pi}{T}kx}\right.}_{} + \frac{\overset{a_{-(-k)} + ib_{-(-k)}}{\overset{\parallel}{a_k + ib_k}}}{2} \left. e^{-i\frac{2\pi}{T}kx}\right]$$

$$\parallel$$

$$i\frac{2\pi}{T}(-k)x$$

$$c_k := \frac{a_k - i b_k}{2} \quad \Big| \quad k = 1, \dots \in \mathbb{N}$$

$$j<0 \quad c_j = \frac{a_{-j} + i b_{-j}}{2}$$

$$\sum_{k=1}^{\infty} c_k e^{i\frac{2\pi}{T}kx} + \sum_{k=1}^{\infty} c_{-k} e^{i\frac{2\pi}{T}(-k)x}$$

$$c_0 = a_0$$

$$= \sum_{k=-\infty}^{+\infty} c_k \underbrace{e^{i\frac{2\pi}{T}kx}}_{=: e_k(x)}$$

Let

$$H = L^2_{\mathbb{C}}([0,T]) = \left\{ f : [0,T] \to \mathbb{C} : \int_0^T |f|^2 < +\infty \right\}$$

with

$$\langle f, g \rangle = \int_0^T f(x)\, \overline{g(x)}\, \frac{dx}{T}$$

Then

$$\langle e_n, e_m \rangle = \int_0^T e_n(x)\, \overline{e_m(x)}\, \frac{dx}{T}$$

$$= \int_0^T e^{i\frac{2\pi}{T}nx}\,\overline{e^{i\frac{2\pi}{T}mx}}\,\frac{dx}{T}$$

$$\overline{e^{i\frac{2\pi}{T}mx}} = e^{-i\frac{2\pi}{T}mx}$$

$$= \int_0^T e^{i\frac{2\pi}{T}(n-m)x}\,\frac{dx}{T} \qquad e^{\alpha x} = \left(\frac{e^{\alpha x}}{\alpha}\right)'$$

$$= \begin{cases} n = m & \displaystyle\int_0^T \underset{\overset{\|}{1}}{e^{i\frac{2\pi}{T}\cdot 0\cdot x = 0}}\,\frac{dx}{T} = \int_0^T 1\,\frac{dx}{T} = 1 \\ & \left(\text{in part } \|e_n\|^2_{L^2} = \langle e_n, e_n\rangle = 1\right) \end{cases}$$

$$n \neq m \qquad e^{i\frac{2\pi}{T}(n-m)x} = \left(\frac{e^{i\frac{2\pi}{T}(n-m)x}}{i\frac{2\pi}{T}(n-m)}\right)'$$

$$\Rightarrow \int_0^T e^{i\frac{2\pi}{T}(n-m)x}\,\frac{dx}{T} = \frac{1}{\cancel{T}}\left[\frac{e^{i\frac{2\pi}{T}(n-m)x}}{i\frac{2\pi}{\cancel{T}}(n-m)}\right]_{x=0}^{x=T}$$

$$= \frac{1}{i2\pi(n-m)}\left[e^{i\frac{2\pi}{\cancel{T}}(n-m)\cancel{T}} - e^{i\frac{2\pi}{T}(n-m)\cdot 0}\right]$$

$$e^{i2\pi\cdot k} = 1 \quad - \quad e^0 = 1$$

$$e^{i2\pi\cdot k} = \cos(2\pi\cdot k) + i\sin(2\pi k)$$

$= 0 \quad \Rightarrow \quad \langle e_n, e_m \rangle = 0 \quad n \neq m$

<u>Prop:</u> $e_n(x) := e^{i \frac{2\pi}{T} nx} \quad n \in \mathbb{Z}, \quad x \in [0,T]$

$(e_n)_{n\in\mathbb{Z}} \subset L^2_{\mathbb{C}}([0,T])$ is an orthonormal system of vectors of $L^2_{\mathbb{C}}([0,T])$.

<u>Thm:</u> $(e_n)_{n\in\mathbb{Z}}$ is an othonormal base for $L^2_{\mathbb{C}}([0,T])$.

In part.

$$\forall f \in L^2_{\mathbb{C}}([0,T]) \qquad f = \sum_{n\in\mathbb{Z}} \langle f, e_n \rangle e_n$$

$$= \sum_{n\in\mathbb{Z}} \underbrace{\langle f, e_n \rangle}_{\text{Fourier coeff}} \cdot e^{i \frac{2\pi}{T} nx}$$

where

$$\langle f, e_n \rangle = \int_0^T f(x) \overline{e_n(x)} \frac{dx}{T}$$

$$= \int_0^T f(x) e^{-i \frac{2\pi}{T} nx} \frac{dx}{T} . =: \hat{f}(n)$$

Conclusion: $\forall f \in L^2_{\mathbb{C}}$

$$= \quad \hat{\ } \qquad e^{i \frac{2\pi}{T} nx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e_n(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{i\frac{2\pi}{T}nx}$$

$$f(x) = \sum_{n\in\mathbb{Z}} \left(\int_0^T f(y)\, e^{-i\frac{2\pi}{T}ny} \frac{dy}{T} \right) \cdot e^{i\frac{2\pi}{T}nx} .$$

$$\|$$

$$f(x) = \sum_{n\in\mathbb{Z}} \cancel{\frac{1}{T}} \left(\int_{-T/2}^{T/2} f(y)\, e^{-i2\pi\frac{n}{T}y}\, dy \right) e^{i2\pi\frac{n}{T}x}\, \frac{1}{T}$$

$$\downarrow$$

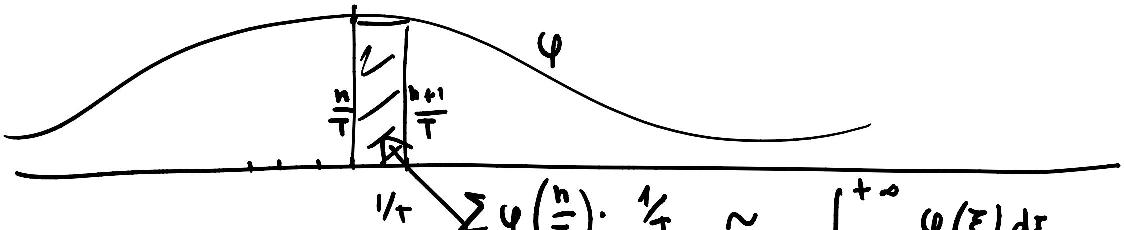
$$\int_{-\infty}^{+\infty} \qquad T \to +\infty$$

$-T/2$ $\qquad\qquad\qquad$ $T/2$

$$\sim \underbrace{\sum_n \left[\int_{-\infty}^{+\infty} f(y)\, e^{-i2\pi\frac{n}{T}y}\, dy \right]}_{F\left(\frac{n}{T}\right)} \underbrace{e^{i2\pi\frac{n}{T}x}}_{g\left(\frac{n}{T}\right)} \frac{1}{T}$$

$$F(\xi) = \int_{-\infty}^{+\infty} f(y)\, e^{-i2\pi\xi y}\, dy \qquad g(\xi) = e^{i2\pi\xi x}$$

$$\sum_n F\left(\frac{n}{T}\right) g\left(\frac{n}{T}\right) \cdot \frac{1}{T} \sim \int_{-\infty}^{+\infty} F(\xi)\, g(\xi)\, d\xi$$



$$\sum_{1} \varphi\left(\frac{n}{T}\right) \cdot \frac{1}{T} \sim \int_{-\infty}^{+\infty} \varphi(\xi)\, d\xi$$

$$f(x) = \int_{-\infty}^{+\infty} F(\xi)\, e^{i2\pi\xi x}\, d\xi$$

where $$F(\xi) = \int_{-\infty}^{+\infty} f(y)\, e^{-i2\pi\xi y}\, dy$$