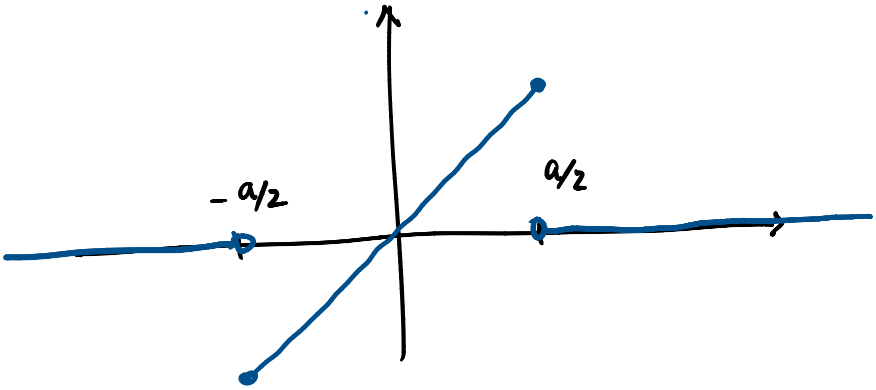
Exam: Dec. 12

Ex 6.7.1 Compute FT of

1. $x \, \text{rect}_a(x) = x \, \mathbb{1}_{[-a/2, a/2]}(x)$



$$\widehat{\#\text{rect}_a}(\xi) = \int_{\mathbb{R}} x \, \text{rect}_a(x) \, e^{-i2\pi\xi x} \, dx$$

$$e^{\alpha x}$$

$$= \int_{-a/2}^{a/2} x \underbrace{e^{-i2\pi\xi x}}_{\left(\frac{e^{-i2\pi\xi x}}{-i2\pi\xi} \right)'} dx$$

$$= \begin{cases} \xi = 0 & \displaystyle\int_{-a/2}^{a/2} x \cdot \underset{1}{e^{0}} \, dx = 0 \\ \xi \neq 0 & \displaystyle\int_{-a/2}^{a/2} x \, \partial_x \left(\frac{e^{-i2\pi\xi x}}{-i2\pi\xi} \right) dx \end{cases}$$

$$= \left[x \frac{e^{-i2\pi\xi x}}{-i2\pi\xi} \Bigg|_{x=-a/2}^{x=a/2} - \int_{-a/2}^{a/2} 1 \, \frac{e^{-i2\pi\xi x}}{-i2\pi\xi} dx \right]$$

$$= \left[x \frac{e^{-i2\pi\xi x}}{-i2\pi\xi} \Bigg|_{x=-a/2}^{x=a/2} - \int_{-a/2}^{a/2} 1 \frac{e^{-i2\pi\xi x}}{-i2\pi\xi} dx \right]$$

$$= \frac{a}{2\pi\xi}\left[\frac{e^{-i\cancel{2}\pi\xi\frac{a}{\cancel{2}}}}{\enclose{circle}{-}i2} + \frac{e^{+i2\pi\xi a/2}}{-i2} \right] + \frac{1}{i2\pi\xi} \cdot \frac{\sin(\pi a\xi)}{\pi\xi}$$

$$= i \frac{a}{2\pi\xi} \cos(\pi a\xi) - i \frac{1}{2\pi^2\xi^2} \sin(\pi a\xi) .$$ ☒

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta \qquad \frac{1}{-i} = i$$

Ex. 6.7.2 $f: \mathbb{R} \to \mathbb{R}$ even $f(-x) = f(x)$

$\Rightarrow \hat{f} \in \mathbb{R}$

odd $\Rightarrow \hat{f} \in i\mathbb{R}$

$$\overline{\hat{f}(\xi)} = \overline{\int_{\mathbb{R}} f(x)\, e^{-i2\pi\xi x} dx}$$

$$= \int_{\mathbb{R}} \underset{\substack{\shortparallel \\ f(x)}}{\overline{f(x)}} \cdot \underset{\substack{\shortparallel \\ e^{i2\pi\xi x}}}{\overline{e^{-i2\pi\xi x}}} dx = \int_{\mathbb{R}} f(x) e^{-i2\pi\xi(-x)} dx$$

$$\overset{y=-x}{=} \int_{\mathbb{R}} \underset{\substack{\shortparallel \\ f(y)}}{f(-y)}\, e^{-i2\pi\xi y} dy = \hat{f}(\xi)$$

$\mathbb{R}$

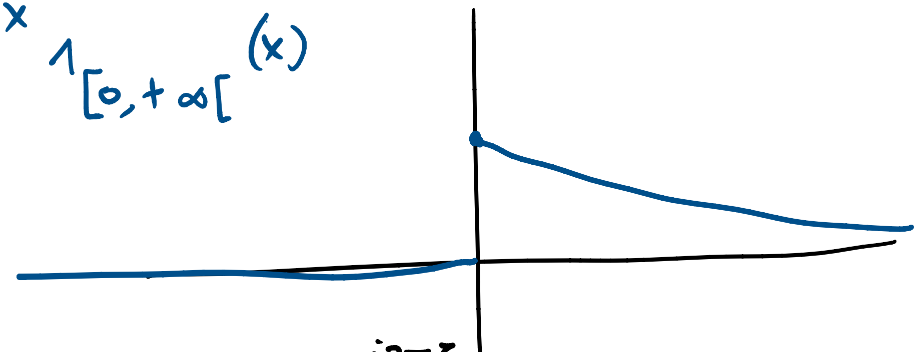
$$\overset{\shortparallel}{f(y)}$$

$$\Rightarrow \overline{\hat{f}(\xi)} = \hat{f}(\xi) \Rightarrow \hat{f}(\xi) \in \mathbb{R}$$

If $f(-y) = -f(y)$

$$\overline{\hat{f}(\xi)} = -\hat{f}(\xi) \Rightarrow \hat{f}(\xi) \in i\mathbb{R}$$ ∎

5. $f(x) = e^{-x}\mathbb{1}_{[0,+\infty[}(x)$



$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-x}\mathbb{1}_{[0,+\infty[}(x)\, e^{-i2\pi\xi x}\, dx$$

$$= \int_0^{+\infty} e^{-(1 \overset{\overset{0}{\neq}}{+} i2\pi\xi)x}\, dx$$

$$= \left[\frac{e^{-(1+i2\pi\xi)x}}{-(1+i2\pi\xi)} \right]_{x=0}^{x=+\infty} = \frac{1}{1+i2\pi\xi}.$$

$$e^{-(1+i2\pi\xi)x} = \underset{\underset{0}{\downarrow}}{e^{-x}}\ \underset{\text{bded}}{e^{-i2\pi\xi x}} \xrightarrow[x\to+\infty]{} 0$$ ∎

$\downarrow$ valu

0

∎

Elementary properties of FT.

$$\widehat{f(\# + T)}(\xi) = \int_{\mathbb{R}} f(x+T)\, e^{-i2\pi\xi x}\, dx$$

$$\underset{y=x+T}{=} \int_{\mathbb{R}} f(y)\, e^{-i2\pi\xi(y-T)}\, dy$$

$$= e^{i2\pi\xi T} \int_{\mathbb{R}} f(y) e^{-i2\pi\xi y}\, dy$$

$$\widehat{f(\# + T)}(\xi) = e^{i2\pi\xi T}\, \hat{f}(\xi)$$

$$\widehat{f(\lambda \#)}(\xi) = \int_{\mathbb{R}} f(\lambda x)\, e^{-i2\pi\xi x}\, dx$$

for inst.

$f(-\#)$

$$\underset{y=\lambda x}{=} \int_{\mathbb{R}} f(y)\, e^{-i2\pi \boxed{\xi \frac{y}{\lambda}}}\, \frac{dy}{|\lambda|}$$

$x = \frac{y}{\lambda}$

$dx = \frac{dy}{|\lambda|}$

$$= \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

$$\widehat{f(\lambda \#)}(\xi) = \frac{1}{|\lambda|}\hat{f}\left(\frac{\xi}{\lambda}\right)$$

So for inst

$$\widehat{f(- \#)}(\xi) = \hat{f}(-\xi).$$

We proved that there's a certain connection between regularity of f and how $\hat{f}$ goes to 0 at ∞

and also a specific rel. between

$$\widehat{\partial_x f} = (i2\pi\xi)\hat{f}$$

A converse is also true:

regularity on $\hat{f}$ $\Longleftrightarrow$ how f goes to 0 at ∞

$$\partial_\xi \hat{f} = \underline{\widehat{-i2\pi\# f}}$$

Prop: let $f \in L^1$ s.t. $\# f(\#) \in L^1$

Rmk: f could be L^1 without $x f(x)$

Prop: Let $f \in L^1$ s.t. $\# f(\#) \in L^1$

$(x f(x) \in L^1)$ $\Downarrow$

$$\exists\, \partial_\xi \hat{f}(\xi) = \widehat{-i2\pi \# f(\#)}(\xi)$$

L^1 without $x f(x)$ $\in L^1$.

$f(x) = \frac{1}{1+x^2} \in L^1$

$x f(x) = \frac{x}{1+x^2} \notin L^1$

$\underset{\pm\infty}{\sim} \frac{1}{x}$

Proof: If we want to derive $\hat{f}(\xi)$

$$\partial_\xi \hat{f}(\xi) = \partial_\xi \int_{\mathbb{R}} \underbrace{f(x)\, e^{-i2\pi\xi x}}\, dx$$

$$\overset{?}{=} \int_{\mathbb{R}} f(x)\, \partial_\xi \underbrace{\left(e^{-i2\pi\xi x}\right)}_{= -i2\pi x\, e^{-i2\pi\xi x}} dx$$

$$= \int_{\mathbb{R}} \left[-i2\pi x f(x)\right] e^{-i2\pi\xi x}\, dx$$

$$= \widehat{-i2\pi \# f(\#)}(\xi)$$

To justify the passage of ∂_ξ under $\int_{\mathbb{R}}$ we apply the deriv thm:

$$\partial_\xi \int_{\mathbb{R}} F(x,\xi)\, dx = \int_{\mathbb{R}} \partial_\xi F(x,\xi)\, dx$$

This is possible if

- $F(\cdot, \xi) \in L^1(\mathbb{R}) \quad \forall \xi \in \mathbb{R}$

  In our case

  $$F(x,\xi) = f(x)\, e^{-i2\pi\xi x} \in L^1(\mathbb{R}) \Leftrightarrow f \in L^1$$

- $\exists\, \partial_\xi F(x,\xi) \quad \forall \xi \in \mathbb{R}, \quad \text{a.e. } x \in \mathbb{R}$

  In our case

  $$\partial_\xi F = \partial_\xi \left(f(x)\, e^{-i2\pi\xi x} \right)$$

  $$= -i2\pi x f(x)\, e^{-i2\pi\xi x} \quad \forall \xi \ \forall x.$$

- $\exists\, g \in L^1 \ : \quad |\partial_\xi F(x,\xi)| \leq g(x) \quad \forall \xi \ \text{a.e. } x.$

  In our case

  $$|\partial_\xi F(x,\xi)| = \underbrace{2\pi\, |x f(x)|}_{\substack{g(x) \in L^1 \\ \text{by assumptions.}}} \ \underbrace{\left| e^{-i2\pi\xi x} \right|}_{= 1}$$

$\Rightarrow$ we can ∂_ξ under int. ☒

Corollary: $f, \cancel{xf}, \cancel{x^2 f}, \cancel{x^3 f}, \cancel{\ldots}\ x^k f \in L^1$

[**Rmk:** actually we just need to check

Rmk: actually we just need to

$$f \in L^1, \quad x^k f \in L^1 \longleftarrow$$

Why? $$\int |x^j f| = \int |x|^j |f|$$

$$j < k \qquad |x|^j \overset{?}{\leq} |x|^k$$

$$= \int_{|x|>1} |x|^j |f| + \int_{|x|\leq 1} \underset{\overset{\wedge\!\!\shortparallel}{1}}{|x|^j} |f|$$

$$\leq \int_{|x|\geq 1} |x|^k |f| + \underbrace{\int_{|x|<1} |f|}$$

$$\leq \int_{\mathbb{R}} \underset{|x^k f|}{|x|^k |f|} + \int_{\mathbb{R}} |f| < +\infty$$

Then

$$\exists\ \partial_\xi^k \hat{f}(\xi) = \widehat{(-i2\pi x)^k f}(\xi) \quad \forall \xi \in \mathbb{R}.$$

$$\widehat{\partial_x^j f} = (i2\pi\xi)^j \hat{f}$$

$$(i2\pi\xi)^j \underbrace{\partial_\xi^k \hat{f}} = (i2\pi\xi)^j \widehat{(-i2\pi x)^k f}$$

$$= \widehat{\partial_x^j\left[(-i2\pi\#)^k f\right]}$$

$$\begin{tikzcd}
\hat{f} \arrow[r, "\partial_\xi^k"] & \partial_\xi^k \hat{f} \arrow[d, "(i2\pi\xi)^j"] \\
& (i2\pi\xi)^j \partial_\xi^k \hat{f}
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[d, "(-i2\pi\#)^k"] & \\
(-i2\pi\#)^k f \arrow[r, "\partial_x^j"'] & \partial_x^j[\ ] \arrow[l, bend right]
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto, "\wedge"] & \hat{f} \arrow[r, mapsto, "\partial_\xi^k"] & \partial_\xi^k \hat{f} \arrow[r, mapsto, "(i2\pi\xi)^j"] & (i2\pi\xi)^j \partial_\xi^k \hat{f} \arrow[d, equals] \\
f \arrow[r, mapsto, "(-i2\pi\#)^k"] & (-i2\pi\#)^k f \arrow[r, mapsto, "\partial_x^j"] & \partial_x^j[\quad] \arrow[r, "\wedge"] & \widehat{[\quad]}
\end{tikzcd}$$

Thm: (Inv. formula)

$f \in L^1$, $\hat{f} \in L^1$ $\Longrightarrow$ $f(x) = \hat{\hat{f}}\,{}^{-y}{}^{y}(-x)$ a.e. $x \in \mathbb{R}$

Why are we interested in the inv formula?

Pb: Given $g = g(\xi)$ we ask under which conds $\exists f$: $\hat{f} = g$

(f is called also the original of g)

If such f exists

If such f exists

$$\hat{f}(\xi) = g(\xi) \Longrightarrow \hat{g}(y) = \hat{\hat{f}}(y) \overset{inv}{=} f(-y)$$

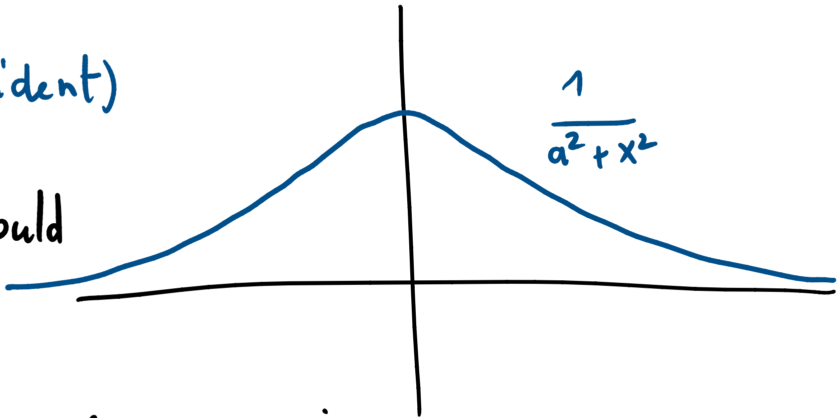
$$\Rightarrow \boxed{f(y) = \hat{g}(-y)}$$

Example (FT of Cauchy distribution)

Let $f(x) = \dfrac{1}{a^2+x^2}$ $\quad a>0$

$f \in L^1(\mathbb{R})$ (evident)

Directly we should compute



$$\hat{f}(\xi) = \int_{\mathbb{R}} \frac{1}{a^2+x^2} e^{-i2\pi\xi x} dx$$ not easy.

Recall

$$\widehat{e^{-b|t|}}(\xi) = \frac{2b}{b^2+4\pi^2\xi^2} \in L^1$$

$$\Downarrow$$

$$L^1 \ni \widehat{\widehat{e^{-b|\#|}}}(x) = \widehat{\frac{2b}{b^2+4\pi^2\#^2}}(x)$$

$\|$ inv form

$$e^{-b|-x|} = e^{-b|x|}$$

$$\Rightarrow \widehat{\frac{2b}{b^2+4\pi^2\#^2}}(\overset{\xi}{\cancel{x}}) = e^{-b|\overset{\xi}{x}|}$$

$$\Rightarrow \widehat{\frac{2b}{b^2+4\pi^2\#^2}}(\xi) = e^{-b|\xi|}.$$

Then:

$$\widehat{\frac{1}{a^2+\#^2}} = \frac{\cancel{4}^{2}\pi}{\cancel{4}\pi a}\widehat{\frac{\cancel{1}\,4\pi a}{\underset{(2\pi a)^2}{4\pi^2a^2} + 4\pi^2\#^2}} = \frac{\pi}{a}\widehat{\frac{2\cdot 2\pi a}{(2\pi a)^2+4\pi^2\#^2}}$$

$$= \frac{\pi}{a}e^{-2\pi a|\xi|}$$

Rmk: FT is linear

$$\widehat{cf} = c\hat{f} \qquad \widehat{\alpha f+\beta g} = \alpha\hat{f}+\beta\hat{g}$$

$\alpha,\beta \in \mathbb{C}$

$^{-i2\pi\xi x}$

$$= \int_{\mathbb{R}} (\alpha f + \beta g)\, e^{-i2\pi \xi x}\, dx \qquad \blacksquare$$

$$\Rightarrow \quad \widehat{\frac{1}{a^2 + \#^2}}(\xi) = \frac{\pi}{a} e^{-2\pi a |\xi|}.$$

Ex 1 Let $f(\xi) = \int_{\mathbb{R}} \frac{\sin(\xi x)}{x(1+x^2)}\, dx$

i) f well defd $\forall \xi \in \mathbb{R}$, $f \in \mathcal{C}$

ii) $\exists\ \partial_\xi f$ and compute it connecting to some FT.

iii) Det f.

Sol: Fix $\xi \in \mathbb{R}$ we have to check

$$x \longmapsto \frac{\sin(\xi x)}{x(1+x^2)} \in L^1(\mathbb{R})$$

$$\Updownarrow$$

$$\int_{\mathbb{R}} \left| \frac{\sin(\xi x)}{x(1+x^2)} \right| < +\infty$$

I could say $|\sin| \leq 1$

$$\int \left| \qquad \right| \leq \int_{\mathbb{R}} \frac{1}{|x|(1+x^2)}$$ not int at 0

$|\sin t| \leq |t|$

$$\leq \int_{\mathbb{R}} \boxed{\frac{|\xi x|}{|x|(1+x^2)}} = |\xi| \int \frac{1}{1+x^2} < +\infty$$

(in alternative

$$F(\xi,x) = \frac{\sin \xi x}{x(1+x^2)} \sim_0 \frac{\xi \cancel{x}}{\cancel{x}(1+x^2)} \sim_0 \xi \Rightarrow F(\xi,\#)$$

is int at 0

$\sin t \sim_0 t$

$$|F(\xi,x)| \leq \frac{1}{|x|(1+x^2)} \sim_{+\infty} \frac{1}{|x|^3} \quad \text{int at } \pm\infty$$

$$\Rightarrow \exists \int_{\mathbb{R}} |F(\xi,x)|\,dx \quad \forall \xi \in \mathbb{R} \;)$$

To check cont of $f(\xi) = \int_{\mathbb{R}} F(\xi,x)\,dx$ we apply the cont thm for integrals dep on param:

- $F(\xi,\#) \in L^1 \quad \forall \xi \in \mathbb{R}$
- $F(\#,x) \in \mathcal{C} \quad \forall x \in \mathbb{R}$
- $\exists g \in L^1 : \quad |F(\xi,x)| \leq g(x) \quad \forall \xi, \text{ a.e } x \in \mathbb{R}$

$$F(\xi,x) = \frac{\sin(\xi x)}{x(1+x^2)}$$

$$|F(\xi,x)| \leq \frac{|\xi \cancel{x}|}{\cancel{|x|}(1+x^2)} = \frac{|\xi|}{1+x^2} \leq \frac{R}{1+x^2} = g_R(x)$$

$\xi \in [-R,R]$

$\Downarrow$

$$f \in \mathscr{C}([-R,R]) \quad \forall R>0$$

$$\Downarrow$$

$$f \in \mathscr{C}(]-\infty,+\infty[).$$

ii) $$\exists\, \partial_\xi f(\xi) = \partial_\xi \int_{\mathbb{R}} F(\xi,x)\,dx \overset{?}{=} \int_{\mathbb{R}} \partial_\xi F(\xi,x)\,dx$$

We need

- $F(\xi,\#)\in L^1 \quad \forall \xi$ (already chucked)
- $\exists\, \partial_\xi F = \dfrac{\cancel{x}\cos(\xi x)}{\cancel{x}(1+x^2)} = \dfrac{\cos(\xi x)}{1+x^2} \quad \forall \xi,\ \forall x$
- $\exists\, g\in L^1(\mathbb{R}) : \quad |\partial_\xi F(\xi,x)| \leq g(x) \quad \forall \xi \ \forall x\in\mathbb{R}$

But $$|\partial_\xi F| = \left|\frac{\cos(\xi x)}{1+x^2}\right| \leq \frac{1}{1+x^2} =: \underset{\overset{\uparrow}{L^1(\mathbb{R})}}{g(x)}$$

$$\Rightarrow \quad \exists\, \partial_\xi f(\xi) = \int_{\mathbb{R}} \frac{\cos(\xi x)}{1+x^2}\,dx$$

$$\cos\theta \underset{\text{Euler}}{=} \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$= \frac{1}{2}\int_{\mathbb{R}} \left(\frac{1}{1+x^2}e^{i\xi x} + \underset{\uparrow}{\frac{1}{1+x^2}}e^{-i\xi x}\right)dx$$

$x - 2\pi k$

$$\mathbb{R} \qquad\qquad \uparrow x = 2\pi y$$

$$x = -2\pi y$$

$$= \frac{1}{2}\left[\int_{\mathbb{R}} \frac{1}{1+4\pi^2 y^2} e^{-i2\pi\xi y}\, dy + \int_{\mathbb{R}} \frac{1}{1+4\pi^2 y^2} e^{-i2\pi\xi y}\, dy \right.$$

$$\|$$

$$\widehat{\frac{1}{1+4\pi^2 \#^2}}(\xi) + \widehat{\frac{1}{1+4\pi^2 \#^2}}(\xi)$$

$$= \boxed{\tfrac{1}{2}}\, \widehat{\frac{2}{1+4\pi^2 \#^2}}(\xi) = \frac{1}{2}\, \widehat{\widehat{e^{-|\#|}}}(\xi)$$

$$\underset{L^1}{\Uparrow} \qquad \|\ \text{inv.}$$

$$\widehat{e^{-|\#|}} = \frac{2}{1+4\pi^2 \#^2} \in L^1$$

$$= \frac{1}{2} e^{-|-\xi|} = \frac{1}{2} e^{-|\xi|}$$

$$\Rightarrow \quad \partial_\xi f = \frac{1}{2} e^{-|\xi|}.$$

iii) $\partial_\xi f = \frac{1}{2} e^{-|\xi|}$

$\xi > 0 \qquad \partial_\xi f = \frac{1}{2} e^{-\xi} \Rightarrow f(\xi) = -\frac{e^{-\xi}}{2} + c_1$

$\xi < 0 \qquad \partial_\xi f = \frac{1}{2} e^{\xi} \Rightarrow f(\xi) = \frac{e^{\xi}}{2} + c_2$

If $\xi = 0$

$$f(0) = \int_{\mathbb{R}} \frac{\overset{0}{\overset{\|}{\sin(0x)}}}{x\cdot(1+x^2)} = 0$$

$$\Rightarrow \quad f(0+) = -\frac{1}{2} + c_1 = 0 \underset{\substack{\uparrow \\ \text{by cont}}}{=} f(0)$$

$$\Rightarrow \quad c_1 = 1/2$$

$$f(0-) = \frac{1}{2} + c_2 \underset{\text{by cont}}{=} f(0) = 0$$

$$c_2 = -1/2$$

$$f(\xi) = \begin{cases} -\dfrac{e^{\boxed{-\xi}}}{2} + 1/2 & \xi \geq 0 \\[2ex] \dfrac{e^{\boxed{\xi}}}{2} - 1/2 & \xi < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left(1 - e^{-|\xi|}\right) & \xi \geq 0 \\ -\frac{1}{2}\left(1 - e^{-|\xi|}\right) & \xi < 0 \end{cases}$$

$$= \frac{1}{2}\left(1 - e^{-|\xi|}\right)(\operatorname{sgn} \xi). \qquad \blacksquare$$

<u>Ex</u> Let $f(x) = \dfrac{1}{(1+x^2)^2}$.

i) Use multipl - deriv duality to compute

$\widehat{\# f(\#)}$ (hint : $x f(x) = \partial_x \dots$)

ii) Use i) to det $\hat{f}$

iii) Compute $\int_0^{+\infty} \frac{1}{(1+x^2)^2}\,dx$, $\int_0^{+\infty} \frac{\sin x}{(1+x^2)^2}\,dx$.

i) Let $g(x) = x f(x) = -\frac{2x}{(1+x^2)^2}\frac{(-1)}{2} = -\frac{1}{2}\partial_x \frac{1}{1+x^2}$

$$\partial_x \left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2}$$

$$\partial_x \frac{1}{\varphi} = -\frac{\varphi'}{\varphi^2}$$

$$\Rightarrow \widehat{\# f(\#)}(\xi) = -\frac{1}{2}\widehat{\partial_x \frac{1}{1+\#^2}}(\xi)$$

$$\widehat{\partial_x \varphi} = (i2\pi\xi)\,\hat{\varphi}$$

$$= -\frac{1}{2}(i2\pi\xi)\,\widehat{\frac{1}{1+\#^2}}(\xi)$$

$$\shortparallel$$

$$\widehat{\frac{1}{a^2+\#^2}} = \frac{\pi}{a} e^{-2\pi a|\xi|}$$

$$= -\frac{1}{2}(i2\pi\xi)\,\pi\, e^{-2\pi|\xi|}$$

$$= -\frac{1}{\cancel{2}}(i\cancel{2}\pi\xi)\,\pi\, e^{-2\pi|\xi|}$$

$$\Rightarrow \quad \widehat{\# f(\#)}(\xi) = -i\pi^2\xi\, e^{-2\pi|\xi|}$$

ii) Compute $\hat{f}$

$$\widehat{-i2\pi \# f(\#)} = \partial_\xi \hat{f}$$

$$\Downarrow$$

$$\partial_\xi \hat{f} = -i2\pi\, \widehat{\# f(\#)} = -i2\pi\left(-i\pi^2\xi\, e^{-2\pi|\xi|}\right)$$

$$= -2\pi^3\xi\, e^{-2\pi|\xi|} = \begin{cases} -2\pi^3\xi\, e^{-2\pi\xi} & \xi \geq 0 \\ -2\pi^3\xi\, e^{2\pi\xi} & \xi < 0 \end{cases}$$

$$\Rightarrow \hat{f}(\xi) = \int -2\pi^3\xi \underset{\left(\frac{e^{-2\pi\xi}}{-2\pi}\right)'}{\underset{\shortparallel}{e^{-2\pi\xi}}}\, d\xi + c \qquad \xi > 0$$

$$= -2\pi^3\left[\xi\,\frac{e^{-2\pi\xi}}{-2\pi} - \int \frac{e^{-2\pi\xi}}{-2\pi}\right] + c$$

$$= \pi^2\xi e^{-2\pi\xi} + \pi^{\cancel{2}}\,\frac{e^{-2\pi\xi}}{+2\cancel{\pi}} + c$$

$$= \pi^2\xi e^{-2\pi\xi} + \frac{\pi}{2}e^{-2\pi\xi} + c$$

$$\hat{f}(\xi) = \int -2\pi^3 \xi \underset{\overset{\|}{\left(\frac{e^{2\pi\xi}}{2\pi}\right)'}}{e^{2\pi\xi}} \, d\xi + \tilde{c} \qquad \xi < 0$$

$$= -2\pi^3 \left[\xi \frac{e^{2\pi\xi}}{2\pi} - \int \frac{e^{2\pi\xi}}{2\pi} \, d\xi \right] + \tilde{c}$$

$$= -\pi^2 \left[\xi e^{2\pi\xi} - \frac{e^{2\pi\xi}}{2\pi} \right] + \tilde{c}$$

$$= -\pi^2 \xi e^{2\pi\xi} + \frac{\pi}{2} e^{2\pi\xi} + \tilde{c}$$

$c, \tilde{c}$?

$$f, f' \in L^1 \implies |\hat{f}(k)| \leqslant \frac{\|f'\|}{2\pi|k|} \xrightarrow{k \to +\infty} 0$$

$\left(f' = \frac{4x^3}{(1+x^4)^2} \sim_{\pm\infty} 4 \frac{x^3}{x^8} = \frac{4}{x^5} \right.$ integrable at $\left. \pm\infty \right)$

In particular $\hat{f}(k) \underset{|k| \to \infty}{\longrightarrow} 0$

Thus

for $\xi > 0$: $\hat{f}(\xi) = \underset{\substack{\downarrow \\ 0}}{\pi^2 \xi e^{-2\pi\xi}} + \underset{\substack{\downarrow \\ 0}}{\frac{\pi}{2} e^{-2\pi\xi}} + c$

$\xi \to +\infty$ $\Downarrow$

$$\hat{f}(\xi) \underset{\xi \to +\infty}{\longrightarrow} c \Rightarrow \boxed{c = 0}$$

Similarly $\tilde{c} = 0 \implies$

$$\hat{f}(\xi) = \begin{cases} \pi^2 \xi e^{-2\pi\xi} + \frac{\pi}{2} e^{-2\pi\xi} & \xi \geqslant 0 \\ -\pi^2 \xi e^{+2\pi\xi} + \frac{\pi}{2} e^{+2\pi\xi} & \xi < 0 \end{cases}$$

$$= \pi^2 |\xi| e^{-2\pi|\xi|} + \frac{\pi}{2} e^{-2\pi|\xi|}$$

iii) Compute $\displaystyle\int_0^{+\infty} \frac{1}{(1+x^2)^2} dx$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2} e^{-i2\pi 0 \cdot x} dx = \frac{1}{2} \hat{f}(0) = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{+\infty} \frac{\cos x}{(1+x^2)^2} dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\overset{\text{Re } e^{ix}}{\overset{\parallel}{\cos x}}}{(1+x^2)^2}$$

$$= \frac{1}{2} \text{Re} \int_{-\infty}^{+\infty} \frac{e^{ix}}{(1+x^2)^2}$$

$$= \frac{1}{2} \text{Re} \hat{f}\left(-\frac{1}{2\pi}\right) = \frac{1}{2}\left[\frac{\pi}{2} e^{-1} + \frac{\pi}{2} e^{-1}\right]$$

$$= \frac{\pi}{2} \frac{1}{e}.$$

$\boxtimes$