lunedì 2 dicembre 2019 12:38

Convolution

Def: Let $f, g \in L^1(\mathbb{R})$. We define

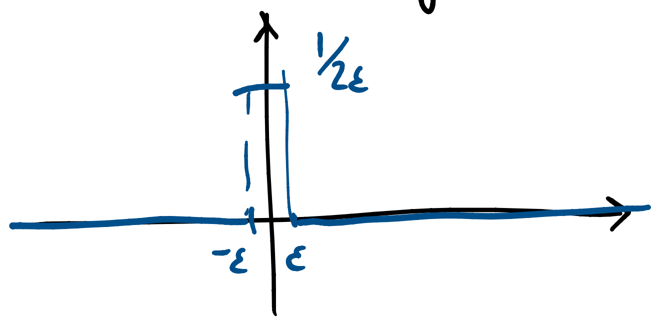
$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)\, g(y)\, dy$$

convolution of f and g

$$\left(\begin{matrix} z = x - y \\ y = x - z \end{matrix} \quad = \int_{\mathbb{R}} f(z)\, g(x-z)\, dz \right)$$

$g * f$

Example

$g_\varepsilon(x) =$ 

$$= \frac{1}{2\varepsilon} \mathbb{1}_{[-\varepsilon, \varepsilon]}$$

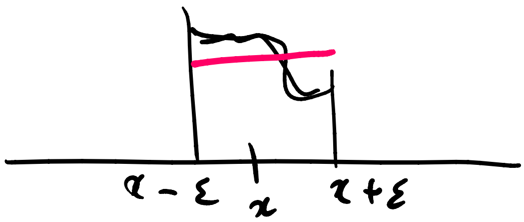
$$f * g_\varepsilon(x) = \int_{\mathbb{R}} f(x-y)\, g_\varepsilon(y)\, dy$$

$$= \int_{-\varepsilon}^{\varepsilon} f(x-y) \frac{1}{2\varepsilon}\, dy = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(x-y)\, dy$$

$f \in \tau$

$\downarrow$

$f(x)$



$\blacksquare$

Prop: $f, g \in L^1 \implies f * g \in L^1$

Proof: We just check $\displaystyle\int_{\mathbb{R}} |f * g| < +\infty$

$$\int_{\mathbb{R}} |f*g(x)|\,dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-y)\,g(y)\,dy \right| dx$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)\,g(y)|\,dy\,dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)|\,|g(y)|\,dx\,dy$$

$$= \int_{\mathbb{R}} \left(|g(y)| \int_{\mathbb{R}} |f(x-y)|\,dx \right) dy$$

$z = x - y$

$$\left(\int_{\mathbb{R}} |f(z)| \, dz \right) = \|f\|_1$$

$$= \int_{\mathbb{R}} |g(y)| \, \|f\|_1 \, dy$$

$$= \|f\|_1 \cdot \int_{\mathbb{R}} |g| \, dy = \|f\|_1 \|g\|_1$$

$$\Rightarrow \quad \boxed{\|f * g\|_1 \leq \|f\|_1 \|g\|_1}$$

(Young inequality)

Prop: $f, g \in L^1$ $\quad (\Rightarrow f * g \in L^1)$

Then

$$\widehat{f * g} = \hat{f} \cdot \hat{g}$$

$$\left(\widehat{f * g}(\xi) = \hat{f}(\xi) \hat{g}(\xi) \quad \forall \xi \in \mathbb{R} \right)$$

Proof:

$$\widehat{f * g}(\xi) = \int (f * g)(x) \, e^{-i2\pi\xi x} \, dx$$

$$\widehat{f*g}(\xi) = \int_{\mathbb{R}} (f*g)(x)\, e^{-i2\pi\xi x}\, dx$$

$$= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)\, g(y)\, dy \right) e^{-i2\pi\xi x}\, dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y)\, g(y)\, e^{-i2\pi\xi x}\, dx\, dy$$

$$= \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x-y)\, e^{-i2\pi\xi(x-y+y)}\, dx\, dy$$

$$\| $$

$$e^{-i2\pi\xi(x-y)}\, e^{-i2\pi\xi y}$$

$$= \int_{\mathbb{R}} g(y)\, e^{-i2\pi\xi y} \underbrace{\left(\int_{\mathbb{R}} f(z)\, e^{-i2\pi\xi z}\, dz \right)}_{\hat{f}(\xi)} dy$$

$z = x - y$

$y = x - z$

$$= \hat{f}(\xi) \int_{\mathbb{R}} g(y)\, e^{-i2\pi\xi y}\, dy = \hat{f}(\xi)\, \hat{g}(\xi) \quad \blacksquare$$

Ex. 6.5.4 Solve the eqn

$$u'' - u = e^{-|x|} \qquad x \in \mathbb{R}.$$

($u = u(x)$ $x \in \mathbb{R}$ $\uparrow$ 2nd order diff eqn)

Sol: Idea: let $u = u(x)$ be the sol.

$\hat{u} = \hat{u}(\xi)$ be its FT.

Notice that

$$u'' - u = e^{-|x|} \overset{FT}{\Longrightarrow} \widehat{u'' - u}(\xi) = \widehat{e^{-|\#|}}(\xi) \qquad (*)$$

$$\widehat{u''-u} = \widehat{u''} - \hat{u} = (i2\pi\xi)^2 \hat{u} - \hat{u}$$

$$= (-4\pi^2\xi^2 - 1)\hat{u}$$

$$= -(1 + 4\pi^2\xi^2)\hat{u}$$

$$\widehat{e^{-|\#|}}(\xi) = \frac{2}{1 + 4\pi^2\xi^2}$$

$$\Rightarrow (*) \Longleftrightarrow \quad -(1 + 4\pi^2\xi^2)\hat{u} = \frac{2}{1 + 4\pi^2\xi^2}$$

$$\Leftrightarrow \quad \underset{\uparrow}{\hat{u}}(\xi) = -\frac{2}{(1+4\pi^2\xi^2)^2} = v(\xi)$$

We could now invoke the inv formula

$(u \in L^1, \hat{u} \in L^1)$ 　 <u>Pb:</u> given $v(\xi) = -\dfrac{2}{(1+4\pi^2\xi^2)^2}$

$$? \exists u \in L^1 : v = \hat{u}$$

Γ $v \in L^1$, $\boxed{\hat{v} \in L^1} \underset{\text{INV F.}}{\Longrightarrow} v(\xi) = \hat{\hat{v}}(-\xi)$ 　 $\dfrac{C}{1+\xi^4}$

$(v, v', v'' \in L^1)$ 　 $\widehat{(\hat{v})}$ 　 $\dfrac{\xi^3}{(1+\xi^4)^2}$

$$v'' \in L^1 \Rightarrow \widehat{v''} = (i2\pi\xi)^2 \hat{v}$$

$$\Rightarrow \hat{v} = \frac{\widehat{v''}}{(i2\pi\xi)^2} \qquad |\hat{v}(\xi)| = \frac{|\widehat{v''}(\xi)|}{4\pi^2 |\xi|^2}$$

$$|\hat{v}(\xi)| \le \frac{\|v''\|_1}{4\pi^2|\xi|^2} \Rightarrow \int |\hat{v}(\xi)| < +\infty$$

int at $\pm\infty$

$\Rightarrow$ given $v(\xi) = -\dfrac{2}{(1+4\pi^2\xi^2)^2}$ $\exists\, u \in L^1$:

$v = \hat{u}$

$$= -\frac{2}{4}\,\frac{2}{1+4\pi^2\xi^2}\,\frac{2}{1+4\pi^2\xi^2}$$

$$\qquad\quad \widehat{e^{-|\#|}} \qquad \widehat{e^{-|\#|}}$$

$$= -\frac{1}{2}\,\widehat{e^{-|\#|}}\,\widehat{e^{-|\#|}}$$

$$\hat{u}(\xi) = -\frac{1}{2}\,\widehat{e^{-|\#|} * e^{-|\#|}}$$

$$\Rightarrow \quad u(x) = -\frac{1}{2}\, e^{-|\#|} * e^{-|\#|}(x). \qquad \boxtimes$$

Rmk: Here we're using the following fact

$$\hat{f} = \hat{g} \Longrightarrow f = g$$

This follows by inv formula

$$\hat{f} - \hat{g} = 0 \Longleftrightarrow \widehat{f-g} = 0$$

$$\hat f - \hat g = 0 \iff \widehat{f-g} = 0$$

$$f-g \in L^1, \quad \widehat{f-g} \in L^1 \xrightarrow{\text{INV F}} (f-g)(x) = \widehat{\widehat{f-g}}(-x)$$

$(f, g \in L^1)$ $\qquad$ (because $\equiv 0$)

$$= \hat 0(-x)$$

$$= 0$$

$$\Rightarrow f - g \equiv 0$$

$$\Rightarrow f = g. \qquad \blacksquare$$

Example (Heat Eqn)

$u = u(t, x)$

temperature, time, pt x

<u>Typical pb</u>: Given initial temp $u(0,x) = \varphi(x)$

φ known, det the future temp

$u(t,x) \quad \forall t > 0, \ x \in \mathbb{R}.$

It can be proved that $u = u(t,x)$ solves the following PDE

$$\begin{cases} \partial_t u(t,x) = \dfrac{\sigma^2}{2} \partial_{xx} u(t,x) & t>0 \\ & x \in \mathbb{R} \\ u(0,x) = \varphi(x) \in L^1(\mathbb{R}) & x \in \mathbb{R}. \end{cases}$$

Idea: Def:

$$v(t,\xi) = \int_{\mathbb{R}} u(t,x)\, e^{-i2\pi\xi x} dx$$

$$= \widehat{u(t,\#)}(\xi).$$

Because $u(0,x) = \varphi(x)$

$$\Downarrow$$

$$v(0,\xi) = \widehat{u(0,\#)}(\xi) = \hat{\varphi}(\xi)$$

$\Rightarrow v(0,\xi)$ is known.

Then we take the PDE and apply both sides

FT:

$$\partial_t u(t,x) = \frac{\sigma^2}{2}\partial_{xx} u(t,x)$$

$$\Downarrow$$

$$\widehat{\partial_t u(t,\#)}(\xi) = \int_{\mathbb{R}} \partial_t u(t,x)\, e^{-i2\pi\xi x}\, dx$$

$$\int_{\mathbb{R}} \partial_t \left[u(t,x)\, e^{-i2\pi\xi x} \right]$$

$$= \partial_t \int_{\mathbb{R}} u(t,x)\, e^{-i2\pi\xi x}\, dx$$

$$= \partial_t v(t,\xi)$$

$$\widehat{\frac{\sigma^2}{2}\partial_{xx} u(t,\#)} = \frac{\sigma^2}{2}\widehat{\partial_{xx} u} = \frac{\sigma^2}{2}(i2\pi\xi)^2\, \widehat{u(t,\#)}$$

$$= -2\pi^2\sigma^2\xi^2\, v(t,\xi)$$

So, $\boxed{\partial_t v(t,\xi) = -2\pi^2\sigma^2\xi^2 v(t,\xi)}$

We look at this as an ordinary diff of type

$$\boxed{y'(t) = C\, y(t)} \qquad C = -2\pi^2\sigma^2\xi^2$$

$$y(t) = v(t,\xi)$$

$$\Longrightarrow \quad y(t) = k\, e^{ct}$$

$$\Rightarrow \quad \boxed{v(t,\xi) = k\, e^{-2\pi^2\sigma^2\xi^2 t}}$$

and because $v(0,\xi) = \hat{\varphi}(\xi)$

$$\Downarrow$$

$$\hat{\varphi}(\xi) = k \quad \Longrightarrow \quad k = \hat{\varphi}(\xi)$$

$$\boxed{\widehat{u(t,\#)}(\xi) = v(t,\xi) = \hat{\varphi}(\xi)\, e^{-2\pi^2\sigma^2\xi^2 t}}$$

Now: RHS $\hat{\varphi}(\xi)\, e^{-2\pi^2\sigma^2\xi^2 t}$

$$\|$$

$$\hat{\varphi}(\xi) \cdot \widehat{\frac{e^{-\frac{\#^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}}(\xi)$$

$$\widehat{u(t,\#)}(\xi) = \widehat{\varphi * \frac{e^{-\frac{\#^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}}(\xi)$$

$$\Big\Downarrow \text{ INV F.}$$

$$u(t,x) = \varphi * \frac{e^{-\frac{\#^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}(x)$$

$$= \int_{\mathbb{R}} \varphi(x-y) \frac{e^{-\frac{y^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}\, dy$$

$$u(t,x) := \int_{\mathbb{R}} \varphi(y) \frac{e^{-\frac{(x-y)^2}{2\sigma^2 t}}}{\sqrt{2\pi\sigma^2 t}}\, dy \qquad \boxtimes$$

Do pbs 6.7.20 → 6.7.24. / 6.7.9/11/$\underbrace{15/16}_{(*)}$

L^2 - F.T.

L^1 FT has some limitations/difficulties.

For inst consider the inv pb:

given $g = g(\xi)$ under which conds

$$g = \hat{f} \ ?$$

A suff cond says:

if $g \in L^1$, $\hat{g} \in L^1$ $\Rightarrow$ $g = (\hat{\hat{g}})(-\#)$

$\Downarrow$

$(g, g', g'' \in L^1 \Uparrow)$

This solution is not satisfactory because we don't have a simple and gen cond on g to have

$$g = \hat{f}.$$

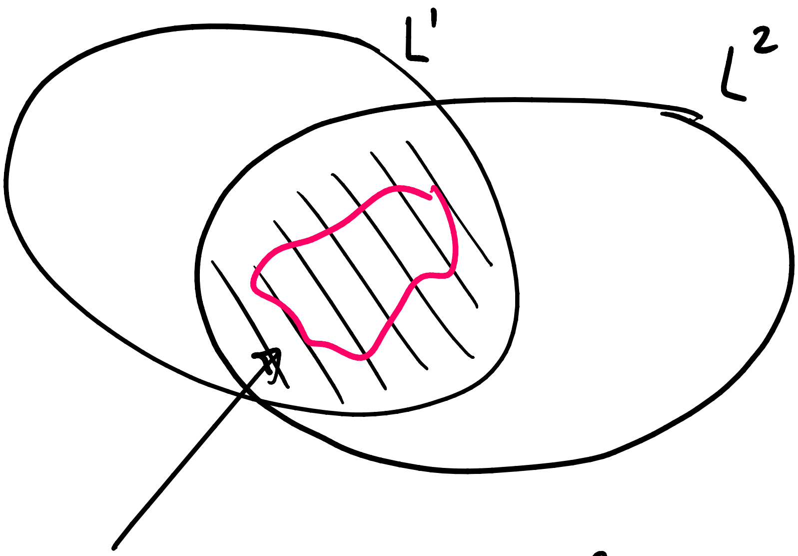
The sol to this Pb is the L^2 FT. There's however a difficulty with the def of L^2 FT:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-i2\pi\xi x}\, dx$$

is well defd $\Leftrightarrow$ $f(\#)\, e^{-i2\pi\xi \#} \in L^1(\mathbb{R})$

$\Leftrightarrow \quad f \in L^1(\mathbb{R})$

The difficulty is that



we consider "good functs"

Def: (Schwarz space)

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}) : \quad x^n \partial_x^k f(x) \xrightarrow[x \to \pm\infty]{} 0 \quad \forall n, \forall k \right\}$$

f and its derivatives go to 0 at ∞ faster than any power

Ex. $f(x) = e^{-x^2} \in \mathcal{C}^\infty$

$$\partial_x^0 f = e^{-x^2}$$

$$\partial_x^1 f = e^{-x^2}(-2x)$$

$$\partial_x^2 f = e^{-x^2}\left[4x^2 - 2\right]$$

$$\partial_x^k f = e^{-x^2} p_k(x)$$

polynomial deg $= k$

polynomial deg $= k$

$$\partial_x f = e^{-x^2}\left[4x^2 - 2\right]$$

$$\partial_x^3 f = e^{-x^2}\left[-2x(4x^2-2) + 8x\right]$$

Then

$$x^n \partial_x^k f = \underbrace{x^n \cdot p_k(x)} e^{-x^2} \xrightarrow[|x|\to+\infty]{} 0$$

$$\forall n \in \mathbb{N}, \ \forall k \in \mathbb{N}$$

$$\Rightarrow \quad e^{-x^2} \in \mathcal{S}(\mathbb{R}).$$

$$e^{-x} \notin \mathcal{S}(\mathbb{R}) \qquad e^{-x} \xrightarrow[x\to-\infty]{} +\infty$$

$$e^{-x^3} \notin \mathcal{S}(\mathbb{R}) \qquad e^{-x^3} \in \mathcal{C}^\infty \text{ but } e^{-x^3} \xrightarrow[x\to-\infty]{} +\infty$$

$$\frac{1}{1+x^2} \notin \mathcal{S}(\mathbb{R}) \qquad \frac{1}{1+x^2} \in \mathcal{C}^\infty$$

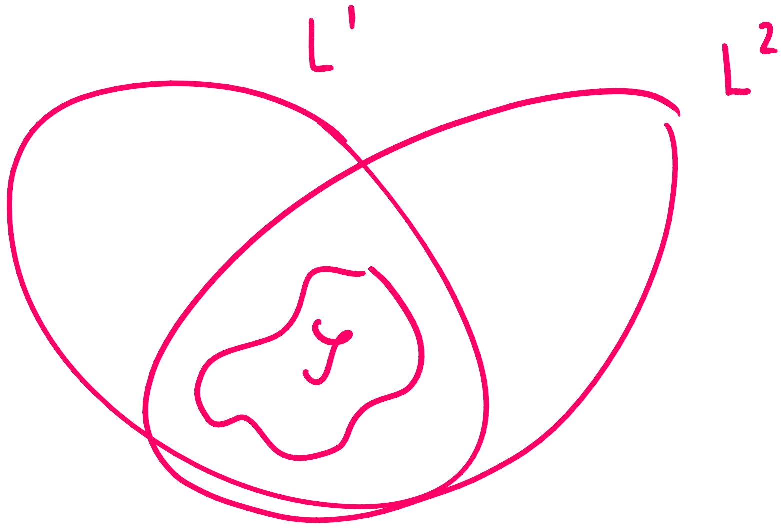
$$x^n \frac{1}{1+x^2} \xrightarrow[|x|\to+\infty]{} \infty \qquad n \geq 3$$

$$e^{-|x|} \notin \mathcal{S}(\mathbb{R}) \qquad x^n \partial_x^k f(x) \xrightarrow[|x|\to+\infty]{} 0 \quad \forall n \ \forall k$$

but $\quad f \notin \mathcal{C}^\infty \quad (\nexists f'(0))$

$\blacksquare$

Prop:



It's just a remark:

we know $\quad x^n \partial_x^0 f = x^n f(x) \xrightarrow[|x|\to+\infty]{} 0 \quad \forall n \in \mathbb{N}$

$x f(x) \longrightarrow 0 \quad \Longrightarrow \quad |x f(x)| \leqslant C \quad |x| \geqslant R.$

$\Longrightarrow \quad |f(x)| \leqslant \dfrac{C}{|x|} \leftarrow$ not int at $\pm\infty$

$x^2 f(x) \longrightarrow 0 \quad \Longrightarrow \quad |x^2 f(x)| \leqslant C \quad |x| \geqslant R$

$\Longrightarrow \quad |f(x)| \leqslant \dfrac{C}{x^2} \quad$ int at $\pm\infty$

$$\Gamma \int_{-\infty}^{+\infty} |f| = \int_{-\infty}^{-R} |f| + \underbrace{\int_{-R}^{R} |f|} + \int_{R}^{+\infty} |f| \quad \leftarrow \int_R^{+\infty} \frac{C}{x^2} < +\infty$$

$\wedge$

$\int_R \frac{}{x^2} < +\infty$

$$\int_{-\infty}^{-R} \frac{C}{x^2} \geq$$

$f \in \mathcal{C} \Rightarrow$ int. on $[-R,R]$

$\wedge$

$+\infty$

Check that $f \in \mathcal{S} \Rightarrow f \in L^2$. $\blacksquare$