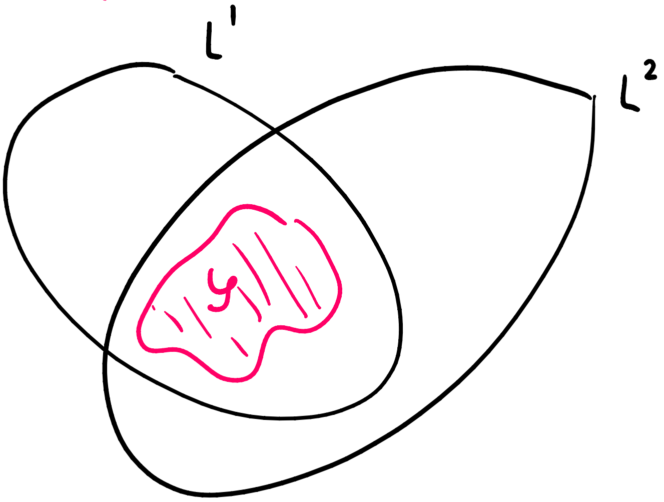
martedì 3 dicembre 2019 12:34

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}) \; : \; x^n \partial_x^k f(x) \xrightarrow[|x|\to+\infty]{} 0 \quad \forall k, \forall n \right\}$$



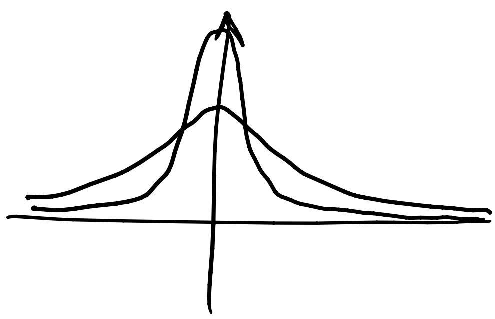
Thm: $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$

that is:

$$\forall f \in \underset{L^2}{L^1}(\mathbb{R}) \quad \exists (f_n) \subset \mathcal{S}(\mathbb{R}) \; : \; f_n \xrightarrow{\|\cdot\|_1} f, \quad f_n \xrightarrow{\|\cdot\|_2} f$$

Idea: $f_n = f * g_{1/n}(x) = \int_{\mathbb{R}} f(y)\, g_{1/n}(x-y)\, dy \in \mathcal{S}(\mathbb{R})$

where $$g_{\varepsilon}(x) = \frac{e^{-\frac{x^2}{2\varepsilon^2}}}{\sqrt{2\pi\varepsilon^2}}$$



$$\times \; f_n(x) = \int_{\mathbb{R}} f(y) \times \frac{e^{-\frac{(x-y)^2}{2\, 1/n}}}{\sqrt{2\pi\, 1/n}}\, dy$$

$$\partial_x f_n(x) = \partial_x \int_{\mathbb{R}}$$

$$= \int_{\mathbb{R}} f(y) \left(-\frac{(x-y)}{1/n}\right) \frac{e^{-\frac{(x-y)^2}{2\,1/n}}}{\sqrt{2\pi\, 1/n}}\, dy$$

$\boxtimes$

Prop: FT applies $\mathcal{S}(\mathbb{R})$ into itself, that

$$f \xmapsto{FT} \hat{f}$$

$$\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$$

and FT is a bijection on $\mathcal{S}(\mathbb{R})$.

$$L^1 \longrightarrow L^1$$

$$\mathrm{rect}_1 \longrightarrow \frac{\sin(\pi \#)}{\pi \#} \notin L^1$$

Proof: $f \in \mathcal{S} \underset{?}{\Rightarrow} \hat{f} \in \mathcal{S}$ $\nearrow$ $\boxed{\hat{f} \in \mathcal{C}^\infty}$

$\searrow \partial_\xi^k \hat{f} \to 0$ faster than any power ξ^n?

Recall that $\partial_\xi^k \hat{f} = \widehat{(-i2\pi\#)^k f}$

(precisely: if $(-i2\pi\#)^k f \in L^1 \Rightarrow \exists\, \partial_\xi^k \hat{f}$

true because $f \in \mathcal{S}$ $\quad |x^{k+2} f(x)| \leq C$

true because

$$\boxed{x^{k+2} f(x) \to 0} \qquad |x^{k+2} f(x)| \le C$$

$$\Downarrow$$

$$|x^k f(x)| \le \frac{C}{x^2}$$

int at $\pm\infty$.

$\forall k \Rightarrow \hat{f} \in \mathcal{C}^\infty$.

IInd: $\xi^n \hat{f}(\xi) \to 0$ as $|\xi| \to \infty$

$\hat{g}(\xi) \to 0$ as $|\xi| \to \infty$

$g, g' \in L^1$

$$\widehat{(\partial_x^n f)} = (i2\pi\xi)^n \hat{f}$$

$$(i2\pi\xi)^n \hat{f}(\xi) = \widehat{\underbrace{(i2\pi\#)^n f}}(\xi) \to 0$$

provided $(i2\pi\#)^n f \in L^1$

$\partial_x [\, (i2\pi x)^n f(x) \,] \in L^1$ true because $f \in \mathcal{S}$

Finally:

- FT is injective (gen fact)

$$\hat{f} = \hat{g} \iff \widehat{f-g} = 0$$

$f - g \in L^1$ + $\widehat{f-g} \in L^1$

$$\overset{\text{INV F}}{\Longrightarrow} \quad \hat{\widehat{f-g}} = f - g(-\#)$$

$$0 = \widehat{0} \implies f \equiv g. \qquad \square$$

- FT is surjective

$g \in \mathcal{S} \; : \; \exists f \in \mathcal{S} \; : \; \hat{f} = g \,?$

Yes, because $\underset{L^1}{\underset{\cap}{g \in \mathcal{S}}}, \; \underset{L^1}{\underset{\cap}{\hat{g} \in \mathcal{S}}} \underset{\text{INV F}}{\implies} \hat{\hat{g}} = g(-\#)$

$$\iff g(-\#) = \hat{f} \qquad f = \hat{g}$$

$$g(\#) = \hat{f}(-\#)$$

$$= \widehat{f(-\#)}(x) \qquad \square$$

Prop: (duality Lemma)

$f, g \in L^1$

$$\int_{\mathbb{R}} f \hat{g} = \int_{\mathbb{R}} \hat{f} g.$$

Proof:

$$\int_{\mathbb{R}} f(x)\hat{g}(x) = \int_{\mathbb{R}} \left(f(x) \int_{\mathbb{R}} g(y) e^{-i2\pi xy} dy \right) dx$$

$$\hat{g}(x) = \int_{\mathbb{R}} g(y) e^{-i2\pi x \cdot y} dy$$

$$= \int_{\mathbb{R}} g(y) \underbrace{\boxed{\int_{\mathbb{R}} f(x) e^{-i2\pi x y} \, dx}}_{= \hat{f}(y)} dy$$

$$= \int_{\mathbb{R}} g(y) \hat{f}(y) \, dy. \qquad \blacksquare$$

Rmk: Introducing the real prod of $L^2_{\mathbb{R}}$

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \varphi(x) \psi(x) \, dx$$

$$\int_{\mathbb{R}} \varphi(x) \overline{\psi(x)} \, dx \qquad L^2_{\mathbb{C}}$$

then duality says

$$\langle f, \hat{g} \rangle = \langle \hat{f}, g \rangle$$

or, denoting by $FT(f) = \hat{f}$

$$\langle f, FT(g) \rangle = \langle FT(f), g \rangle$$

This is wrong because $\langle \, , \, \rangle_{L^2_{\mathbb{R}}}$ is not the nat. scalar prod.

Prop: FT is an isometry respect to $L^2_{\mathbb{C}}$ hermitian prod.

that is:

$$\langle \hat{f}, \hat{g} \rangle_{L^2_{\mathbb{C}}} = \langle f, g \rangle_{L^2_{\mathbb{C}}} \qquad \forall f, g \in \mathcal{S}(\mathbb{R})$$

In particular

$$\| \hat{f} \|_{L^2_{\mathbb{C}}} = \| f \|_{L^2_{\mathbb{C}}} \qquad \forall f \in \mathcal{S}(\mathbb{R})$$

Proof:

$$\langle \hat{f}, \hat{g} \rangle_{L^2_{\mathbb{C}}} = \int_{\mathbb{R}} \hat{f}(\xi) \, \overline{\hat{g}}(\xi) \, d\xi$$

$$\overset{\text{duality}}{=} \int_{\mathbb{R}} f(x) \, \widehat{\overline{\hat{g}}}(x) \, dx \overset{(*)}{=}$$

$$\overline{\hat{g}}(\xi) = \overline{\int_{\mathbb{R}} g(y) \, e^{-i2\pi\xi y} \, dy}$$

$$= \int_{\mathbb{R}} \overline{g(y)} \; \overline{e^{-i2\pi\xi y}} \, dy$$

$$\overline{e^{-i2\pi\xi y}} = e^{+i2\pi\xi y}$$

$\parallel$

$$e^{+i2\pi\xi y}$$

$$= \int_{\mathbb{R}} \overline{g(y)} \; e^{i2\pi\xi y} \, dy$$

$$= \hat{\bar{g}}(-\xi)$$

$$\hat{\hat{\bar{g}}}(x) = \widehat{\overline{\hat{g}}(\#)}(x) = \widehat{\hat{\bar{g}}(-\#)}(x)$$

$$= \widehat{\hat{\bar{g}}}(-x)$$

$$= \hat{\hat{\bar{g}}}(-x)$$

$$\overset{\text{INV F.}}{=} \bar{g}(-(-x)) = \bar{g}(x)$$

$$\overset{(*)}{=} \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, dx = \langle f, g \rangle_{L^2_{\mathbb{C}}}.$$

$\boxtimes$

<u>Thm</u> (Fourier - Plancherel)

$$\text{FT}: \quad \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}) \qquad \text{bijection}$$

$$\begin{array}{ccc} \cap & & \cap \\ L^2_{\mathbb{C}}(\mathbb{R}) & & L^2_{\mathbb{C}}(\mathbb{R}) \end{array}$$

on which we have the nat. struct of Hilb sp with herm. prod

$$\langle f, g \rangle_{L^2_{\mathbb{C}}} = \int_{\mathbb{R}} f \bar{g}$$

$$\| \hat{f} \|_2 = \| f \|_2 \quad \forall f \in \mathcal{S}$$

This produces an extension of FT. to $L^2_{\mathbb{C}}$:

$$f \in L^2_{\mathbb{C}} \qquad \exists (f_n) \subset \mathcal{S}(\mathbb{R}) \qquad \begin{tikzcd} f_n \arrow[r, "L^2"] \arrow[d, "\text{FT}"'] & f \\ \hat{f}_n \arrow[r, "L^2"] & \text{something} \end{tikzcd}$$

Why? $(\hat{f}_n)$ is a C. seq

$$\| \hat{f}_n - \hat{f}_m \|_2 = \| \widehat{f_n - f_m} \|_2$$

is C. $\overset{\text{isom}}{=} \| f_n - f_m \|_2$ is C. because (f_n) converges

$$\hat{f} := \lim_n \hat{f}_n$$

$$\hat{f} := \lim_n \hat{f_n}$$

$$\hat{f}(\xi) = \lim_{R \to +\infty} \int_{-R}^{R} f(x)\, e^{-i2\pi\xi x}\, dx$$

Rmk: Of course, if $f \in L^1 \cap L^2$

$$\hat{f} \text{ (new FT on } L^2) \equiv \hat{f} \text{ (old FT on } L^1)$$

so for inst

$$\widehat{\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}} = e^{-2\pi^2\sigma^2\xi^2}$$

φ

$$\widehat{e^{-a|x|}}(\xi) = \frac{2a}{a^2 + 4\pi^2\xi^2}$$

$\in L^1 \cap L^2$

There're other FT that cannot be computed in L^1 but makes sense in L^2.

Remind that

Remind that

$$\underset{\substack{\uparrow \\ L^1}}{\widehat{\mathrm{rect}_a}}(\xi) = \frac{\sin(\pi a \xi)}{\pi \xi} \notin L^1$$

so in part. $\frac{\sin x}{x} \notin L^1$ so $\widehat{\frac{\sin \#}{\#}}$ cannot be computed with L^1 FT.

However $\frac{\sin x}{x} \in L^2 \setminus L^1$

$$\int_{\mathbb{R}} \Big| \underbrace{\frac{\sin x}{x}}_{\substack{x \to 0 \\ 1}} \Big|^2 dx = \int_{-\infty}^{-1} + \underbrace{\int_{-1}^{1}}_{\substack{\uparrow \\ \mathbb{R}}} + \int_{1}^{+\infty}$$

because f cont at $x = 0$

$$\int^{+\infty} \Big| \frac{\sin x}{x} \Big|^2 \leq \int^{+\infty} \frac{1}{x^2} < +\infty$$

$$\Rightarrow \frac{\sin x}{x} \in L^2$$

$$\Rightarrow \exists \; \widehat{\frac{\sin \#}{\#}} = ?$$

Recall

$$\widehat{\mathrm{rect}_a}(\xi) = \frac{\sin(\pi a \xi)}{\pi \xi}$$

$$\widehat{\mathrm{rect}_a}(\xi) = \frac{\sin(\pi a \xi)}{\pi \xi}$$

$a = 1/\pi$

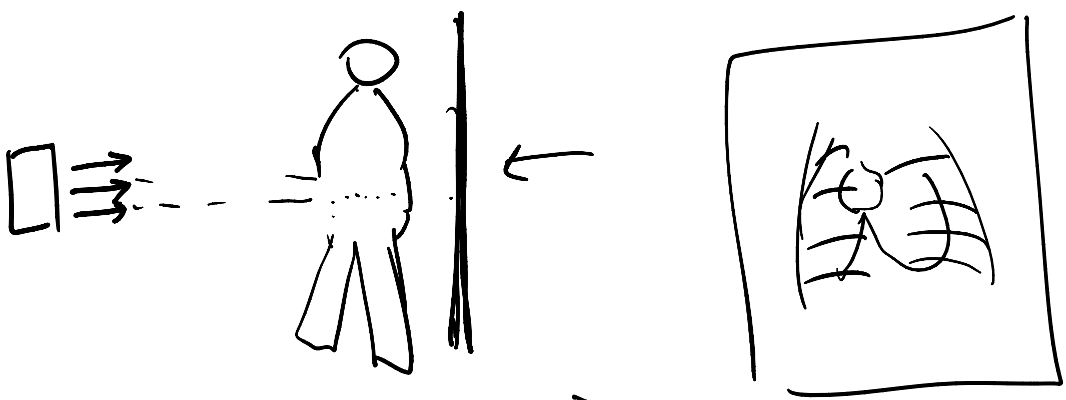
$$\underbrace{\widehat{\pi\,\mathrm{rect}_{1/\pi}}(\xi)}_{\in L^2} = \frac{\sin \xi}{\xi} \in L^2$$

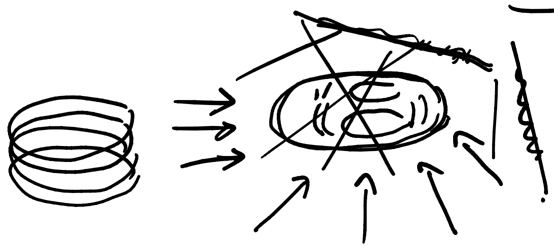
$$\Big\Downarrow \text{INV F}$$

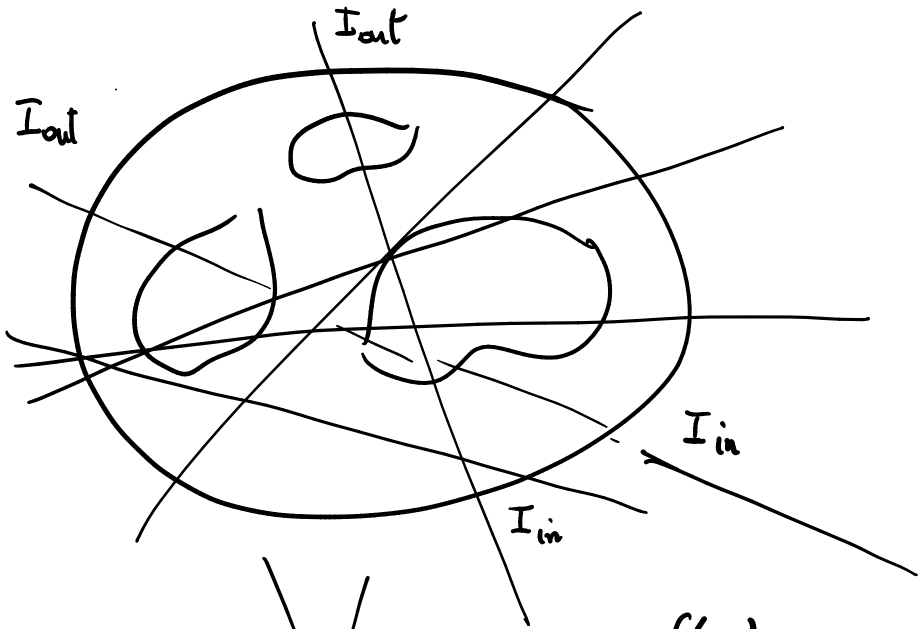
$$\pi\,\mathrm{rect}_{1/\pi}(-x) = \widehat{\widehat{\pi\,\mathrm{rect}_{1/\pi}}}(x) = \widehat{\frac{\sin \#}{\#}}(x)$$

$$\Rightarrow \boxed{\widehat{\frac{\sin \#}{\#}}(\xi) = \pi\,\mathbb{1}_{[-1/\pi,\,1/\pi]}(\xi)}$$

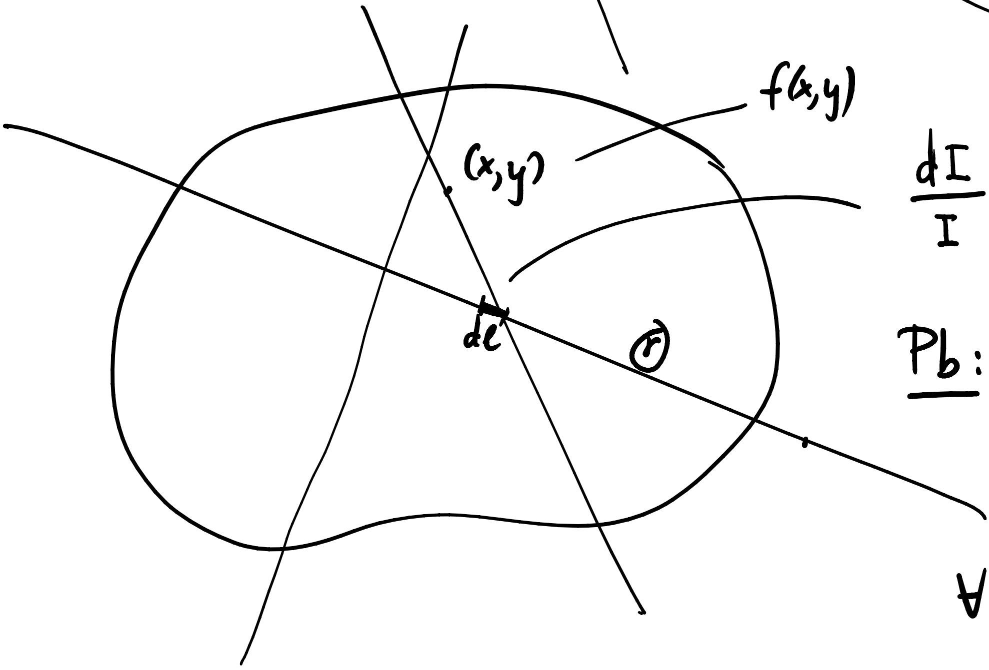
Tomography







2 - DIM



$$\frac{dI}{I} = -f\,d\ell$$

Pb: We meas

$I_{in}(r)$, $I_{out}(r)$

$\forall r$.

$I_{in}(r)$, $I_{out}(r)$ known

Det f

$$\int_r \frac{dI}{I} = \int_r -f\,d\ell$$

$$\|$$

$$d \log I$$

$$\|$$

$$\log I_{out} - \log I_{in}$$

$$\parallel$$

$$-\log\left(\frac{I_{out}}{I_{in}}\right)(r) = \cancel{\otimes}\int_r f\, d\ell$$

Because we assume to know $I_{in}(r), I_{out}(r)$ $\forall r$ (beam) Pb becomes:

Def $f = f(x,y)$: $\displaystyle\int_r f\, d\ell = \underline{F(r)}$ known.

$$\int_r f\, d\ell$$

$$\parallel$$

$$\boxed{\int_{(x,y)\,:\,(a,b)\cdot(x,y)=c} f\, d\ell}$$

$$\underbrace{ax+by}_{}=c$$

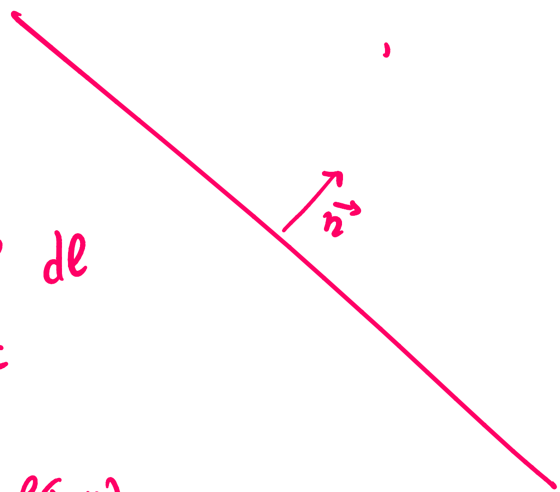
$(a,b) \neq 0_2$

$\|(a,b)\| = 1$ $\quad \lambda a x + \lambda b y = \lambda c$ $\quad r$

(Radón Transform of f)

$$Rf(\vec{n}, c) := \int_{\vec{n}\cdot(x,y)=c} f\, d\ell$$

$\|\vec{n}\| = 1$



$$f = f(x,y)$$

We need to invert this formula to compute f in terms of Rf

There's a connection between FT and RT.

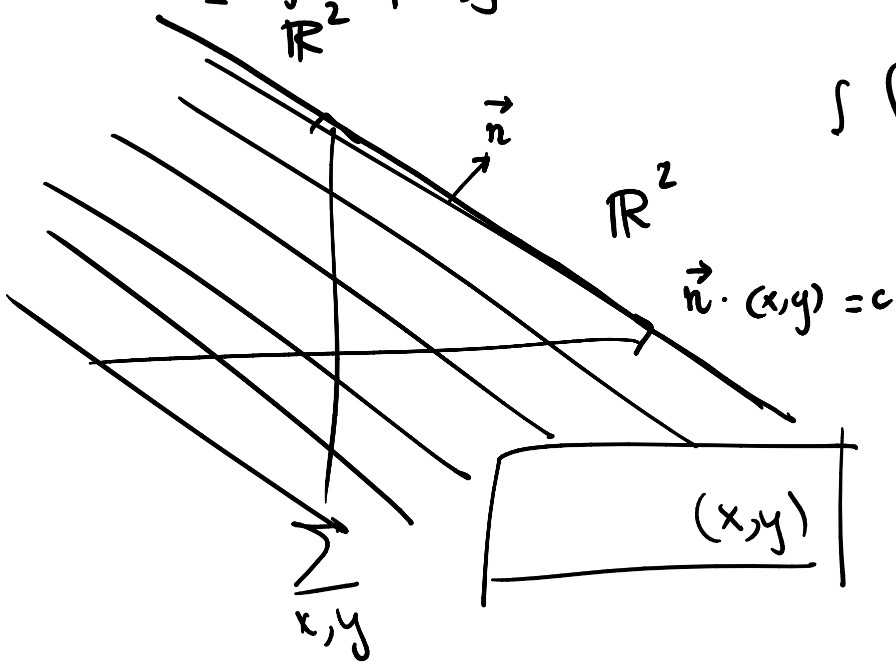
$$\hat{f}(\vec{\xi}) = \int_{\mathbb{R}^2} f(x,y)\, e^{-i2\pi \vec{\xi}\cdot(x,y)} \, dx\,dy$$

$\vec{\xi} \in \mathbb{R}^2$

$\vec{\xi} = \vec{n} \cdot \|\vec{\xi}\| \qquad \vec{n} = \dfrac{\vec{\xi}}{\|\vec{\xi}\|}$

$$= \int_{\mathbb{R}^2} f(x,y)\, e^{-i2\pi \|\vec{\xi}\|\, \vec{n}\cdot(x,y)} \, dx\,dy$$

$\int \left(\int \, dx \right) dy$



$dx\,dy$

$$= \int_{-\infty}^{+\infty} \left(\int_{\vec{n}\cdot(x,y)=c} f(x,y)\, e^{-i2\pi \|\vec{\xi}\|\, \vec{n}\cdot(x,y)} \, dl \right) dc$$

$$\int \quad f(x,y) \left(e^{-i2\pi \|\vec{\xi}\| c} \right) dl$$

$$\int_{\vec{n}\cdot(x,y)=c} f(x,y)\, e^{-i2\pi\|\xi\|c}\, dl$$

$$= \int_{-\infty}^{+\infty} \left(\int_{\vec{n}\cdot(x,y)=c} f \; dl \right) e^{-i2\pi\|\vec{\xi}\|c}\, dc$$

$$\int_{-\infty}^{+\infty} Rf(\vec{n}, c)\; e^{-i2\pi\|\vec{\xi}\|c}\, dc$$

$$= \widehat{Rf(\vec{n}, \#)}\,(\|\vec{\xi}\|)$$

$$\boxed{\hat{f}(\vec{\xi}) = \widehat{Rf(\vec{n}(\vec{\xi}), \#)}\,(\|\vec{\xi}\|)}$$

$\Downarrow$ INV F

$$f(-x,-y) = \widehat{\widehat{Rf(\vec{n}(\heartsuit), \#)}(\heartsuit)}\,(x,y)$$

$\Downarrow$

$$f(x,y) = \widehat{\widehat{Rf(\vec{n}(\heartsuit), \#)}(\heartsuit)}\,(-x,-y)$$