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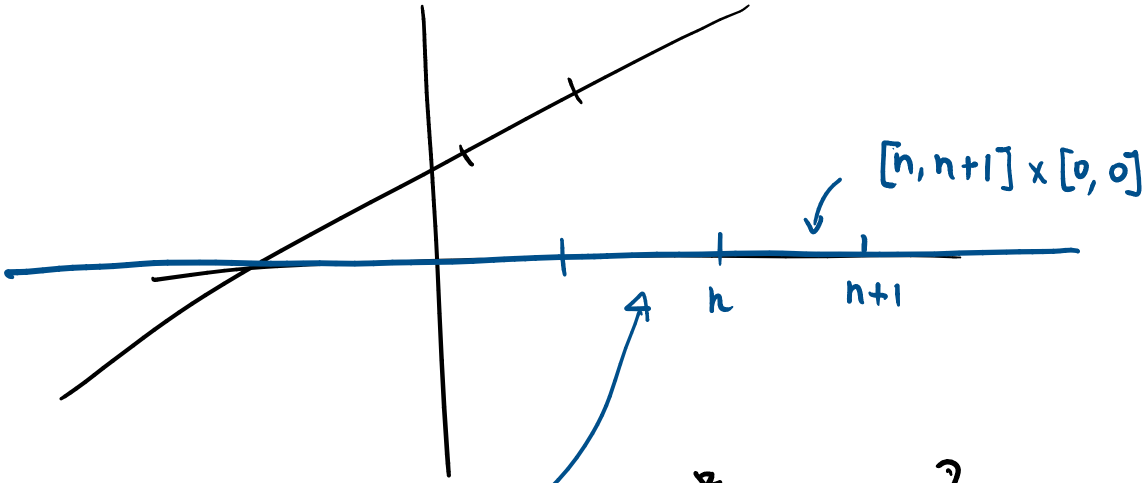
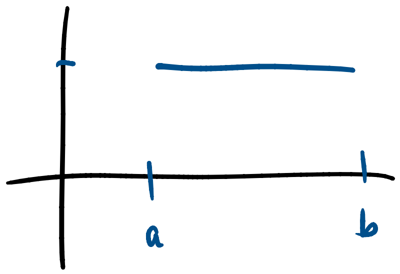
Pb $\lambda_2^*\left(\{(x,y) : y = ax+b\}\right) = 0$.



$\{(x,0) : x \in \mathbb{R}\}$ $\qquad \lambda_2^* = 0$?



$\lambda_2^*\left([a,b]\times[c,c]\right) = 0$

$$\lambda_2^*\left([n,n+1]\times[0,0]\right) = 0$$

$$\Rightarrow 0 \le \lambda_2^*(\mathbb{R}\times\{0\}) = \lambda_2^*\left(\bigcup_{n=-\infty}^{+\infty}[n,n+1]\times[0,0]\right)$$

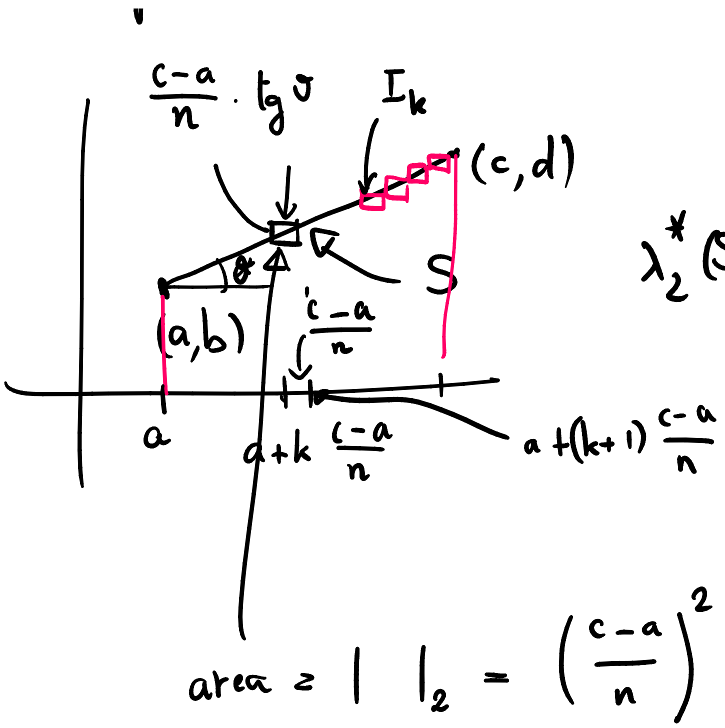
$$\overset{\text{sub add}}{\le} \sum_n \lambda_2^*\left([n,n+1]\times[0,0]\right)$$

$$= 0$$

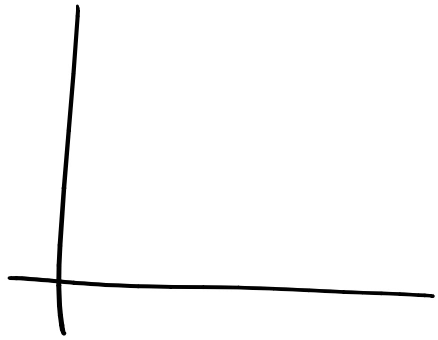
$$\Rightarrow \lambda_2^*(\mathbb{R}\times\{0\}) = 0$$

$\Rightarrow$ by invariance $\lambda_2^*(y = ax+b) = 0$ $\boxed{\text{(b)}}$

. $\underline{c-a}$. $t_a \circ$ I_b



$\lambda_2^*(S)=0$



$$\lambda_2^*(S) \leq \sum_{k=1}^{n} |I_k|_2 = \sum_{k=1}^{n} \left(\frac{c-a}{n}\right)^2 tg\,\vartheta = \cancel{n}\cdot\frac{(c-a)^2}{n^{\cancel{2}}}\,tg\,\vartheta$$

$$= \frac{(c-a)^2\,tg\,\vartheta}{n}$$

$$\Rightarrow \quad \lambda_2^*(S) \leq \frac{K}{n} \quad \forall n \geq 1$$

$$\Rightarrow 0 \leq \lambda_2^*(S) \leq 0 .$$

$\blacksquare$

Ex 1.9.4 $\quad S \subset \mathbb{R}^2 \quad S = \{(x,y)\in\mathbb{R}^2 : \exists\, m,n\in\mathbb{N} \mid mx+ny=0,\ m\cdot n \neq 0\}$

$\lambda_2(S)=0$.

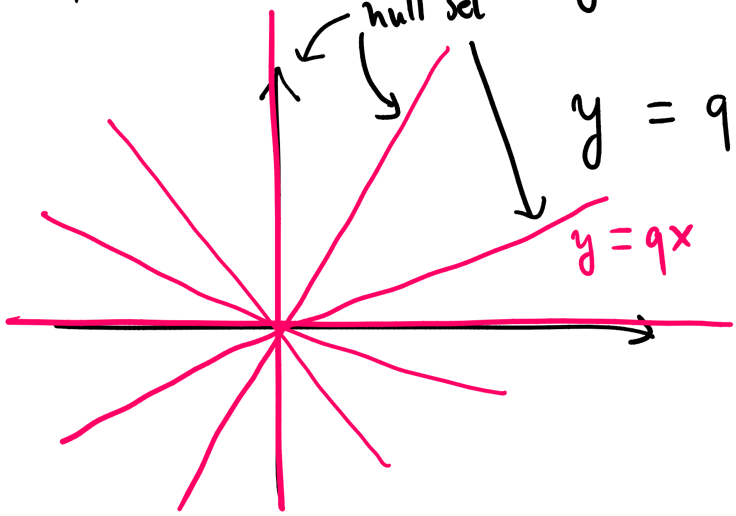
Notice that $\quad mx+ny=0$

if $\quad n=0 \Rightarrow mx=0 \xRightarrow[m\neq 0]{} x=0$

if $\quad n\neq 0 \Rightarrow y=-\frac{m}{n}x$

if $n \neq 0 \Rightarrow y = -\frac{m}{n} x$

$y = qx \qquad q \in \mathbb{Q}.$



Now we proved above $\lambda_2(y=qx) = 0$

$\lambda_2(x=0) = 0$

and $S = \bigcup_{q \in \mathbb{Q}} \{y = qx\} \cup \{x = 0\}$

countable union of meas 0 sets

⌈If $(N_k)_{k \in \mathbb{N}}$ is a family of meas 0 sets

$\bigcup_k N_k$ is a meas 0 set

because $\lambda_d\left(\bigcup_k N_k\right) \underset{\text{subadd}}{\leq} \sum_k \overbrace{\lambda_d(N_k)}^{0} = 0$ ⌋

$\Rightarrow \lambda_2(S) = 0$ ☒

1.9.3 (*) $\lambda_1(N) = 0 \Longrightarrow \lambda_1(N^2) = 0$

where $N^2 = \{x^2 : x \in N\}$.

Consider case when $N \subset [0,1]$ ($\Rightarrow N^2 \subset [0,1]$).

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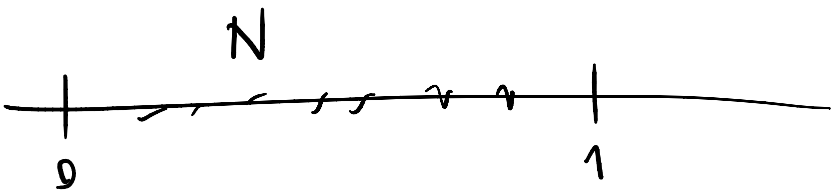
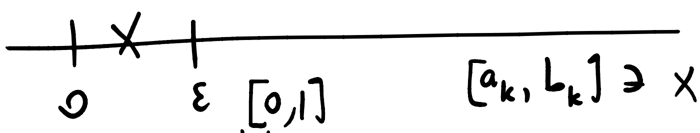


$$\lambda_1(N) = \inf\left\{ \sum |I_k|_1 : \bigcup I_k \supset N \right\} = 0$$



$a_k \leq x \leq b_k$

$[0,1]$

$[a_k, b_k] \ni x$

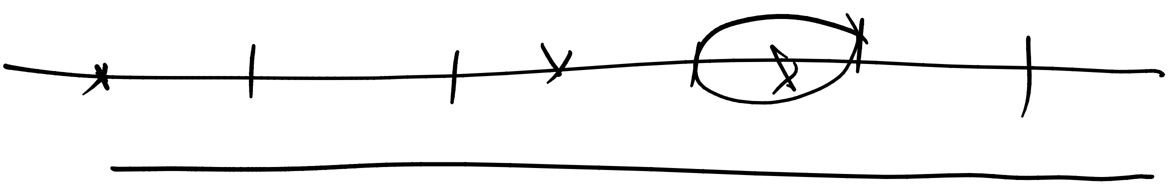
$$\forall \varepsilon > 0 \quad \exists (I_k) : \bigcup I_k \supset N, \quad \sum |I_k|_1 \leq \varepsilon.$$

$[a_k^2, b_k^2] \ni x^2 \qquad a_k^2 \leq x^2 \leq b_k^2$

$\lambda_1(N^2) \leq$ small number

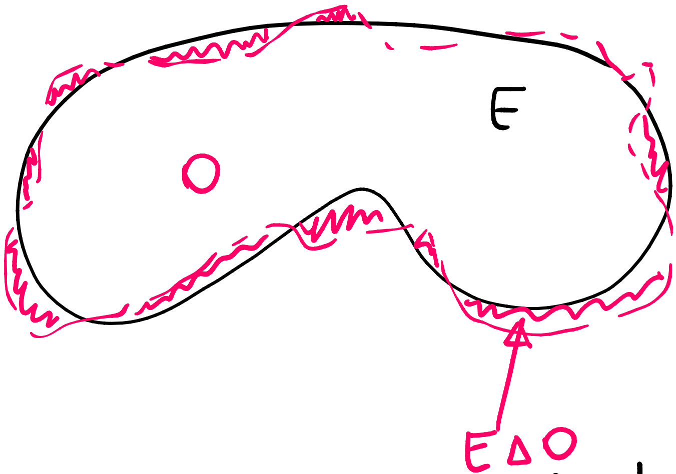
<u>Second step:</u> remove restriction $N \subset [0,1]$

$\left(N \subset [a,b], \quad N \subset \mathbb{R} \right)$



We introduced the Leb class

$$\mathcal{M}_d = \{ E \subset \mathbb{R}^d : \exists\, O \text{ open } \lambda_d^*(E \Delta O) = 0 \}$$

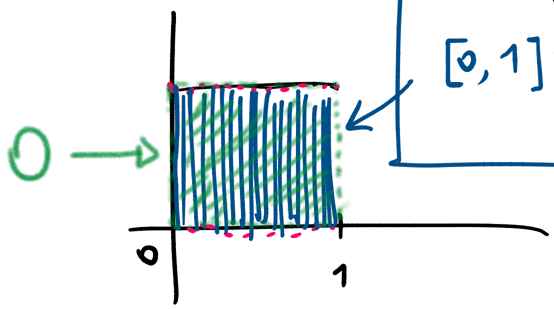


$E \Delta O$ is null set

Example: For Dirichlet funct we wish to compute area between

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \cap [0,1] \\ 1 & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

and x - axis.



$$\boxed{[0,1] \times [0,1] \setminus \bigcup_{q \in \mathbb{Q} \cap [0,1]} \{q\} \times [0,1]} = E$$

Pb: Is E a Leb meas set ($E \in \mathcal{M}_2$)?

To prove that $E \in \mathcal{M}_2$ we need to prove $\exists\, O \subset \mathbb{R}^2$ open s.t. $\lambda_2(E \Delta O) = 0$.

Let

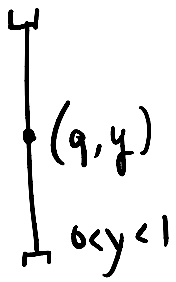
$$O = \,]0,1[\times]0,1[\quad \text{open.}$$

$$E \,\Delta\, O =$$

$$E \setminus O = \overset{\lambda_2=0}{\{0\}\times[0,1]} \cup \overset{\lambda_2=0}{\{1\}\times[0,1]}$$

$$\overset{\lambda_2=0}{[0,1]\setminus\mathbb{Q}\times\{0\}} \cup$$

$$\underset{\lambda_2=0}{[0,1]\setminus\mathbb{Q}\times\{1\}}$$



$$O \setminus E = \bigcup_{q\in\mathbb{Q}\cap[0,1]} \{q\}\times]0,1[$$

$$\lambda_2 = 0$$

$$\Rightarrow E\Delta O = \text{null set.}$$

$$\Rightarrow E \in \mathcal{M}_2 \quad \blacksquare$$

Rmks.

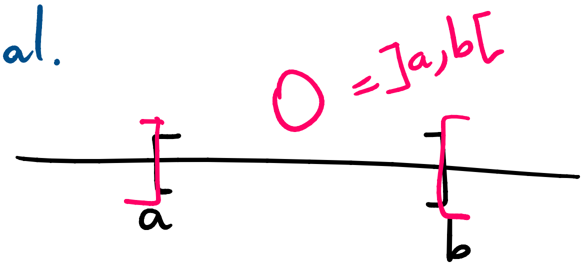
1. $O \in \mathcal{M}_d \quad \forall\, O$ open.

(If E is open then $O = E$ is open st.

$$\lambda_d^*(E\,\Delta\,O) = 0 \text{)}$$

2. $I \in \mathcal{M}_d \quad \forall\, I$ interval. $\quad O = \,]a,b[$

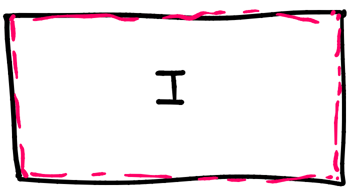
2. $I \in \mathcal{M}_d$ $\forall I$ interval.

$d=1$ $I = [a,b]$ 

$O =]a,b[$

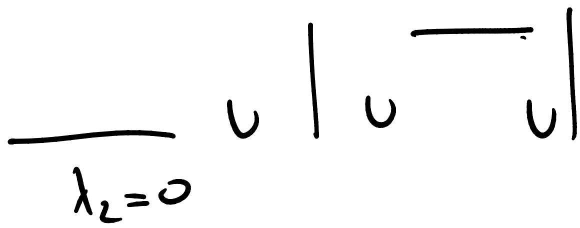
$I \Delta O = \{a\} \cup \{b\}$

$\Rightarrow \lambda_1(I \Delta O) = 0$

$d=2$ 

$O = \mathrm{Int}(I) =]a_1,b_1[\times]a_2,b_2[$

$I = [a_1,b_1] \times [a_2,b_2]$

$O \Delta I$ = boundary = 

$\Rightarrow \lambda_2(O \Delta I) = 0.$

3. Meas 0 sets are in $\mathcal{M}_d$.

N, $\lambda_d(N) = 0$

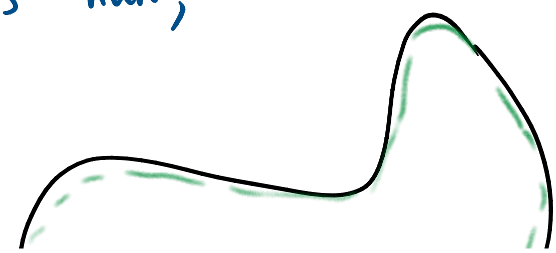
Find O open in such a way

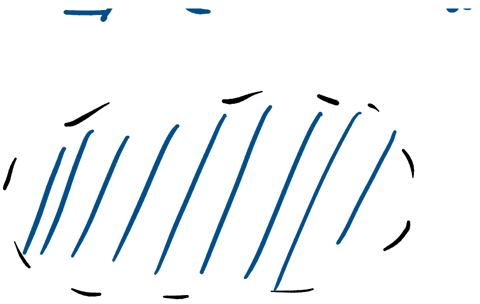
$$\lambda_d(N \Delta O) = 0$$

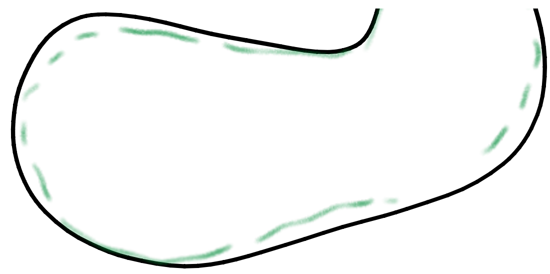
$O = \emptyset$ $N \Delta \emptyset = N$

4. If $E \in \mathcal{M}_d$ and N is null,

$\Rightarrow E \cup N \in \mathcal{M}_d$.





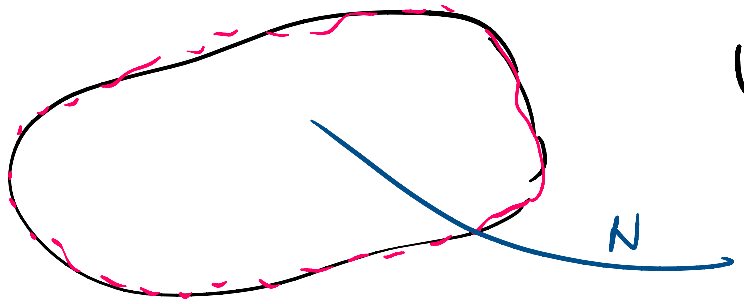


Try to prove as exercise

Hypo: $E \in \mathcal{M}_d \Rightarrow \exists\, O$ open s.t $\lambda_d(E \Delta O) = 0$

$N \in \mathcal{M}_d \quad \lambda_d(N) = 0$

$E \cup N$: pb find $\tilde{O}$: $\lambda_d((E\cup N)\Delta \tilde{O}) = 0$



Use $\tilde{O} = O$.

Less trivial facts.

- closed sets (C closed $\Leftrightarrow$ C^c is open)

  So in part $\{x \in \mathbb{R}^d : f(x) \geq 0\}$, $f \in \mathcal{C}(\mathbb{R}^d)$

  is closed $\Rightarrow$ it is measurable.

Thm: The outer measure λ_d^* restricted to $\mathcal{M}_d$,

$$\lambda_d^* : \mathcal{M}_d \longrightarrow [0, +\infty]$$

is a measure in the sense that:

i) The family $\mathcal{M}_d$ is a <u>σ-algebra of sets</u> that is:

- $\emptyset, \mathbb{R}^d \in \mathcal{M}_d$
- if $E \in \mathcal{M}_d \Rightarrow E^c \in \mathcal{M}_d$
- if $(E_n)_{n \in \mathbb{N}} \subset \mathcal{M}_d \Rightarrow \bigcup_n E_n \in \mathcal{M}_d$.

ii) • $\lambda_d^*(\emptyset) = 0$

• $\lambda_d^*\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda_d^*(E_n)$ if $(E_n) \subset \mathcal{M}_d$ and $E_n \cap E_m = \emptyset$.

$\lambda_d^* \equiv \lambda_d$ is called Lebesgue measure and it fulfills the invariance props

$$\lambda_d(TE + v) = |\det T| \, \lambda_d(E) \quad \forall E \in \mathcal{M}_d, \ \forall T \text{ inv. matr.}, \ \forall v \in \mathbb{R}^d.$$

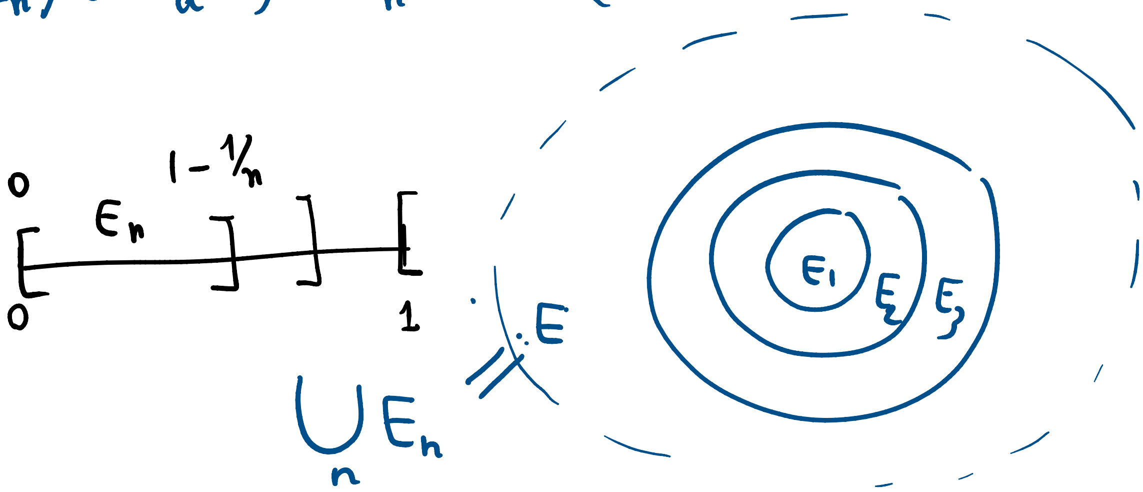
$$\lambda_d(I) = |I|_d$$

<u>Prop</u> (Continuity of Leb meas)

Cont from below:

$(E_n) \subset \mathcal{M}_d$, $E_n \nearrow$ $\left(E_n \subset E_{n+1} \ \forall n\right)$

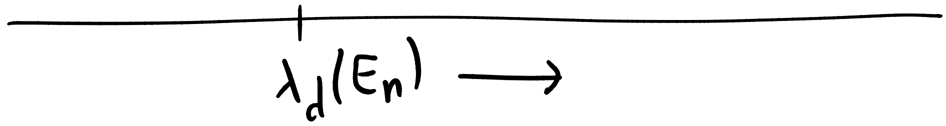
$(E_n) \subset \mathcal{M}_d$, $E_n \nearrow$ $(E_n \subset E_{n+1})$



$$\Rightarrow \boxed{\lambda_d(E) = \lim_{n \to +\infty} \lambda_d(E_n).}$$

<u>Proof</u>: Because $E_n \nearrow$ $\quad E_n \subset E_{n+1}$

$$\lambda_d(E_n) \leqslant \lambda_d(E_{n+1})$$



$\Rightarrow \exists \, \ell = \lim \lambda_d(E_n)$

Goal: $\lambda_d(E) = \ell$

$E = \bigcup E_n$

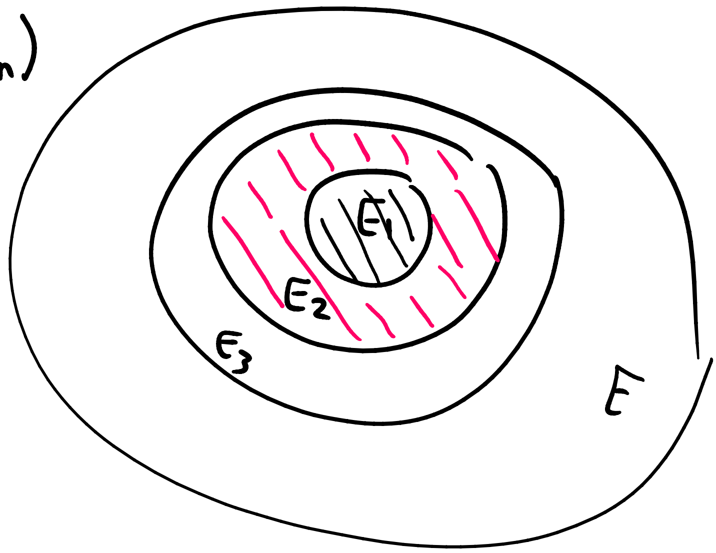
$\setminus E_0 = \phi!$

$= E_1 \cup E_2 \setminus E_1 \cup$

$E_3 \setminus E_2$

$\cup$

∞ ...



$$= \bigcup_{n=1}^{\infty} E_n \setminus E_{n-1} \qquad \text{disj union}$$

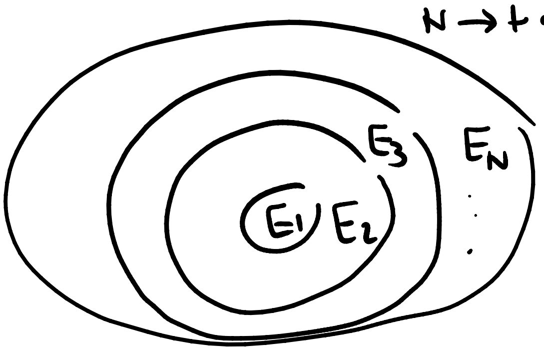
$$\Rightarrow \lambda_d(E) = \lambda_d\Big(\bigcup_n E_n \setminus E_{n-1}\Big)$$

$$\overset{\text{count add}}{=} \sum_{n=1}^{\infty} \lambda_d(E_n \setminus E_{n-1})$$

$$\sum_{n=1}^{\infty} a_n = \lim_{N \to +\infty} \sum_{n=1}^{N} a_n$$

$$= \lim_{N \to +\infty} \sum_{n=1}^{N} \lambda_d(E_n \setminus E_{n-1})$$

$$= \lim_{N \to +\infty} \lambda_d\Big(\underbrace{\bigcup_{n=1}^{N} E_n \setminus E_{n-1}}_{= E_N}\Big)$$



$$= \lim_{N \to +\infty} \lambda_d(E_N).$$

$\boxtimes$