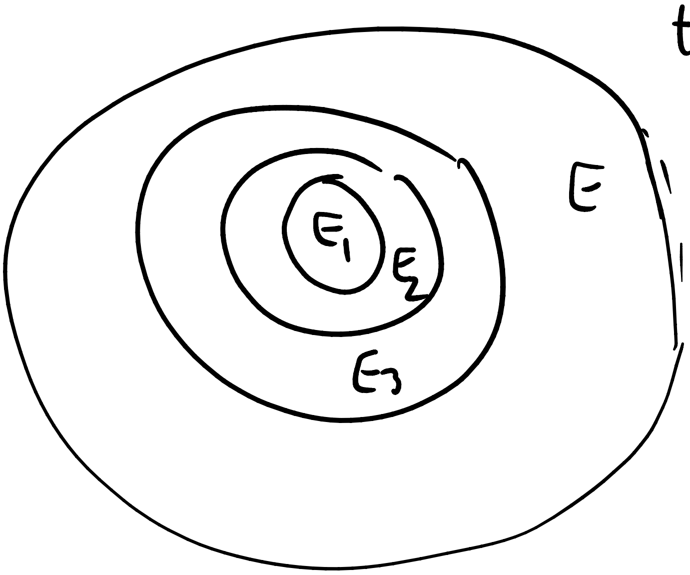
Continuity from above

Recall that if $(E_n) \subset \mathcal{M}_d$, $E_n \nearrow$ $\left(E_n \subset E_{n+1} \ \forall n\right)$



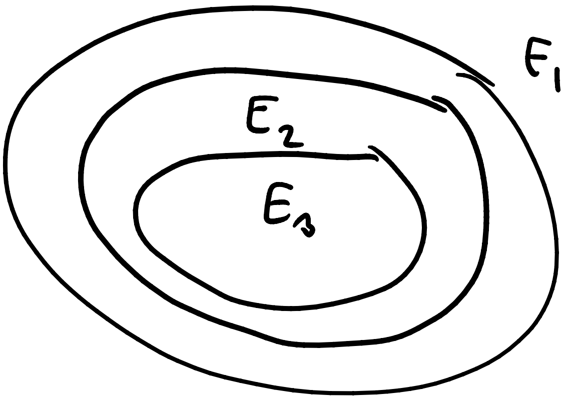
then if

$$E = \bigcup_n E_n \quad (\in \mathcal{M}_d)$$

$$\lambda_d(E) = \lim_{n \to \infty} \lambda_d(E_n)$$

(cont from below)

A similar prop. could be the following:



Consider $(E_n) \subset \mathcal{M}_d$,

$$E_n \searrow \iff E_n \supset E_{n+1} \ \forall n.$$

If we define

$$E = \bigcap_n E_n$$

we wonder if

$$\lambda_d(E) = \lim_n \lambda_d(E_n).$$

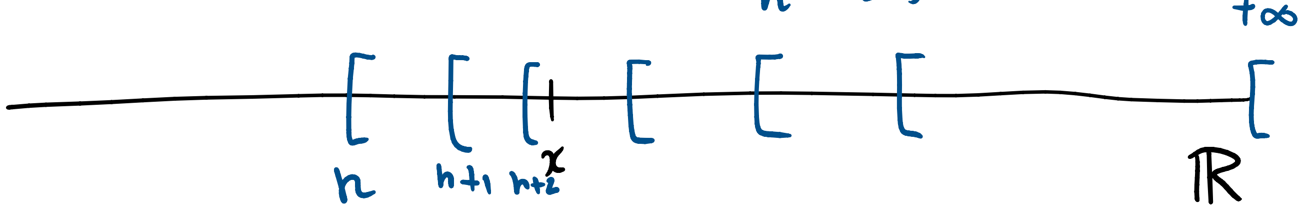
This is false in general.

Example $(d=1)$

$E_n =]n, +\infty[$

Example $(d = 1)$

$$E_n = [n, +\infty[$$



In this case

$$E = \bigcap_n E_n = \bigcap_{n=1}^{\infty} [n, +\infty[\; = \emptyset$$

$$\Rightarrow \quad \lambda_1(E) = \lambda_1(\emptyset) = 0$$

$$\lim_n \lambda_1(E_n) = \lim_n \underbrace{\lambda_1([n, +\infty[)}_{= +\infty} = +\infty.$$

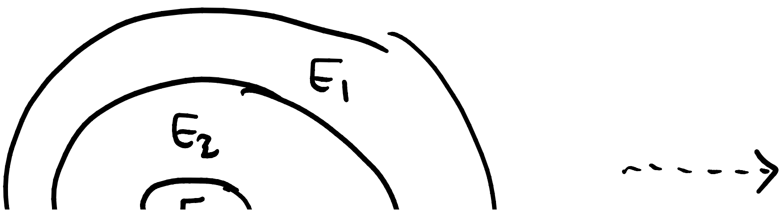
However, the conv from above turns out to be true if we add some assumption:

Prop: Let $(E_n) \subset \mathcal{M}_d$ s.t. $E_n \searrow$ $\left(E_n \supset E_{n+1} \; \forall n \right)$

If $\lambda_d(E_1) < +\infty \Rightarrow$

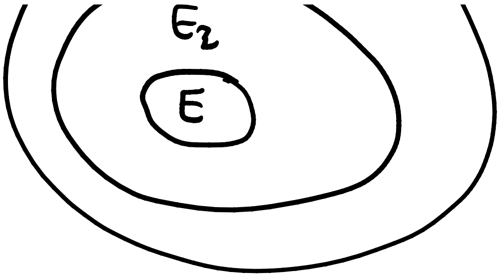
$$\lambda_d(E) = \lim_n \lambda_d(E_n) \qquad E = \bigcap_n E_n .$$

Proof: We reduce the concl to the cont from below:

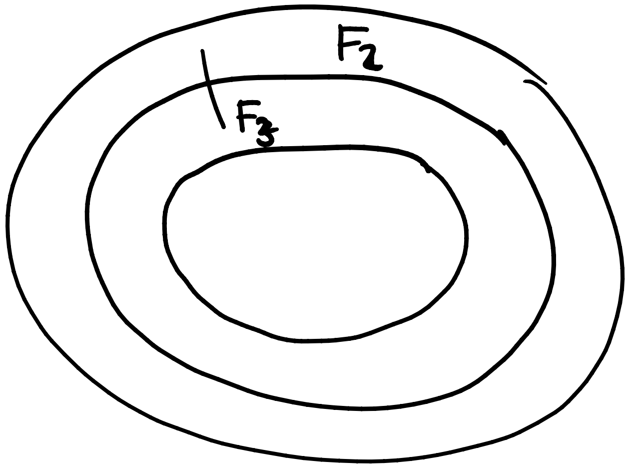


$$F_n = E_1 \setminus E_n$$

(complementary of E_n resp to E_1)



(complementary ... E_n resp to E_1)



$F_1 = E_1 \setminus E_1 = \emptyset$

$F_2 = E_1 \setminus E_2$

$F_3 = E_1 \setminus E_3$

$$F_n \nearrow F = \bigcup F_n = \bigcup_n E_1 \setminus E_n = \bigcup_n E_1 \cap E_n^c$$

$$= E_1 \cap \bigcup_n E_n^c$$

$$= E_1 \cap \left(\bigcap_n E_n\right)^c$$

$$= E_1 \setminus E$$

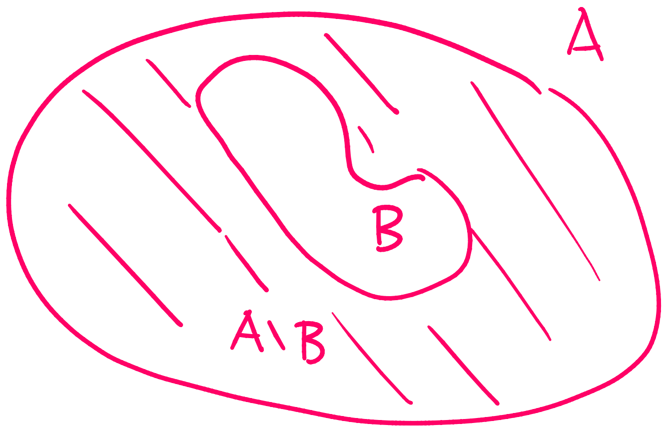
By cont from below

$$\lambda_d(F) = \lim_n \lambda_d(F_n)$$

$$(*) \quad \lambda_d(E_1 \setminus E) = \lim_n \lambda_d(E_1 \setminus E_n)$$

$\lambda_d(A \setminus B)$

$B \subset A$



$A = (A \setminus B) \cup B$ (disj union)

$\lambda_d(A) = \lambda_d(A \setminus B) + \lambda_d(B)$

If $\lambda_d(A) < +\infty \Rightarrow \lambda_d(A \setminus B), \lambda_d(B) < +\infty$

If $\lambda_d(A) < +\infty \Rightarrow \lambda_d(A\setminus B), \lambda_d(B) < +\infty$

$$\boxed{\lambda_d(A\setminus B) = \lambda_d(A) - \lambda_d(B)}$$

Returning to the proof, because $\lambda_d(E_1) < +\infty$

$$\Rightarrow \cancel{\lambda_d(E_1)} - \lambda_d(E) = \lim_n \big(\lambda_d(E_1) - \lambda_d(E_n)\big)$$

$$\qquad // \qquad = \cancel{\lambda_d(E_1)} - \lim_n \lambda_d(E_n)$$

$$\Rightarrow \lambda_d(E) = \lim_n \lambda_d(E_n) \quad \blacksquare$$

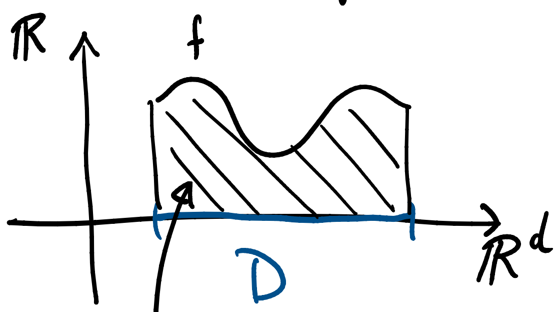
The main application of measure is

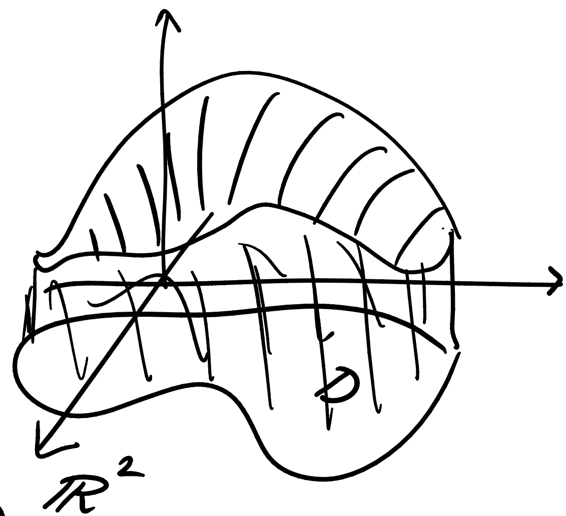
- to compute areas / volumes / measure of sets

  Pb: How do we proceed in practical cases?

- to develop a powerful concept of integral.

Consider $f : D \subset \mathbb{R}^d \longrightarrow [0, +\infty[$





$$\left\{ (x,y) \in \overbrace{\mathbb{R}^d \times \mathbb{R}}^{\mathbb{R}^{d+1}} : \begin{array}{l} x \in D \\ 0 \le y \le f(x) \end{array} \right\} =: \mathrm{Trap}(f).$$

Natural Idea:

$$\int_D f := \text{meas of } \mathrm{Trap}(f)$$
$$= \underset{\substack{\uparrow \\ m_{d+1}\ ?}}{\lambda_{d+1}} (\mathrm{Trap}(f))$$

To have this well defd we need

$$Trap(f) \in \mathcal{M}_{d+1} \quad \text{(be meas.)}$$

and this demands some "technical" cond on f.

Measurable Functs

<u>Def</u>: We say that $f: D \subset \mathbb{R}^d \longrightarrow [-\infty, +\infty]$ is <u>measurable</u> if

$$\{x \in D : f(x) > a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}.$$

<u>Examples</u>:

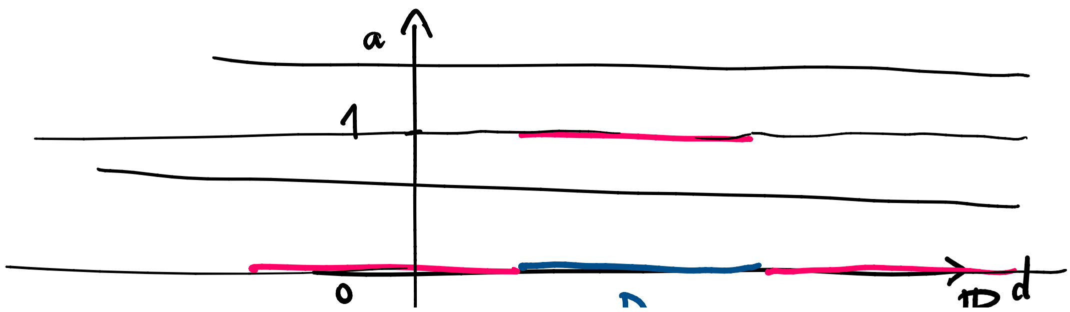
1. (unit functs / ~~(characteristic)~~ / indicator functs)

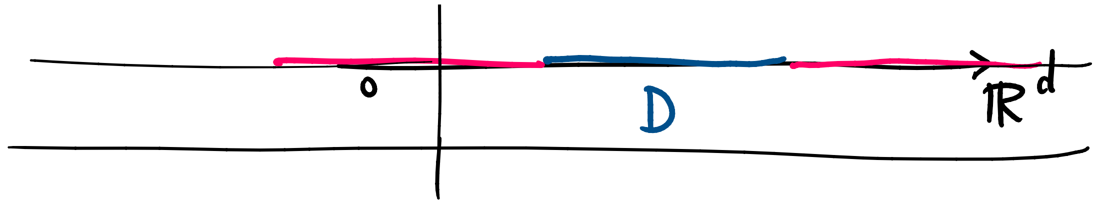
$$\mathbb{1}_D(x) = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases}$$

$\mathbb{1}_D$ is meas. $\iff$ $D \in \mathcal{M}_d$

Indeed:

$$\{x \in \mathbb{R}^d : \mathbb{1}_D > a\}$$





$$= \begin{cases} \emptyset & \text{if } a \geq 1 \\ D & \text{if } 0 \leq a < 1 \\ \mathbb{R}^d & \text{if } a < 0 \end{cases}$$

Therefore

$$\{x \in \mathbb{R}^d : \mathbb{1}_D > a\} \in m_d \iff \emptyset, \mathbb{R}^d, D \in m_d$$

$$\iff D \in m_d. \qquad \blacksquare$$

2. (functs with finite rank)

$$f : \mathbb{R}^d \longrightarrow \{c_1, c_2, \dots, c_N\} \subset [-\infty, +\infty]$$

(with $c_i \neq c_j$).

If $D_j = \{x \in \mathbb{R}^d : f(x) = c_j\}$, $\underset{i \neq j}{D_i \cap D_j = \emptyset}$

$$\sum_{j=1}^{N} c_j \mathbb{1}_{D_j} = f. \qquad \text{(simple funct.)}$$

Simple functs are meas $\iff D_j \in m_d \ \forall j = 1 \dots N$

(do as exercise)

Prop: Cont functs are meas.

Proof: $f \in \mathcal{C}$ $\quad \{x \in \mathbb{R}^d : f(x) > a\}$

$$= f^{\leftarrow}(\underbrace{]a, +\infty[}_{\text{open}}) \text{ is open} \Rightarrow \text{it is meas.}$$

Rmk: $\cancel{(\Rightarrow)}$ f is cont $\Leftrightarrow$ $f^{\leftarrow}(A)$ is open $\forall A$ open.

$\boxtimes$

Prop: Suppose that $f, g : D \to [-\infty, +\infty]$ s.t.

$$f(x) = g(x) \quad \forall x \in D \setminus N \quad \lambda_d(N) = 0$$

Then f is meas $\Leftrightarrow$ g is meas.

Rmk: So for inst we may change a meas. funct on a meas. 0 set, and we have still a meas funct.

Ex: $f : [0,1] \longrightarrow \mathbb{R}$ $\quad f \equiv 1$, $f \in \mathcal{C}([0,1])$

$\Downarrow$

f meas.

$N = \mathbb{Q} \cap [0,1]$ $\quad \lambda_1(N) = 0$ and consider

$$g : [0,1] \longrightarrow \mathbb{R}$$

$$g : [0,1] \longrightarrow \mathbb{R}$$

$$g(x) = \begin{cases} f(x) \equiv 1 & x \in [0,1] \setminus N \\ \text{your choice} & x \in N \end{cases}$$

$\Rightarrow$ g is meas.

So for inst the Dirichlet funct is meas.

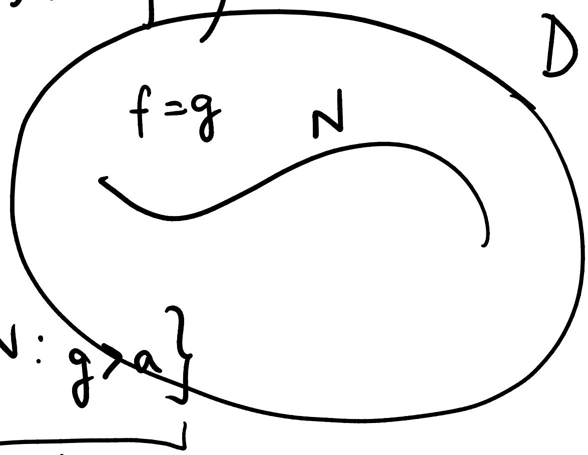
(by the way Dirichlet funct is $\mathbb{1}_{[0,1]\setminus \mathbb{Q}}$)

<u>Proof</u>: Hypo: 1. f is meas $\{f>a\} \in \mathcal{M}_d$ $\forall a$.

2. $f = g$ on $D \setminus N$.

Th: $\{g>a\} \in \mathcal{M}_d$ $\forall a$.

$(\{f>a\} = \{x \in \mathbb{R}^d : f(x) > a\})$



$\{g>a\} =$

$= \{x \in D \setminus N : g > a\} \cup \underbrace{\{x \in N : g > a\}}_{\tilde{N}}$

$= \underbrace{\{x \in D \setminus N : f > a\}}\cup \tilde{N}$ $\quad \tilde{N}$

$\tilde{\tilde{N}}$

$\underbrace{\{f>a\}}_{A} = \underbrace{\{x \in D \setminus N : f > a\}} \cup \{x \in N : f > a\}$

$= \overbrace{\{f>a\}}^{A} \setminus \tilde{\tilde{N}} \cup \tilde{N}$

$$\{g>a\} = \overbrace{\{f>a\}}^{A} \setminus \tilde{\tilde{N}} \cup \tilde{N}$$

$\{f>a\} \in \mathcal{M}_d$, $\setminus \tilde{\tilde{N}} = \cap\, \tilde{\tilde{N}}^c$

(Two poss arg: $A \in \mathcal{M}_d$

$\tilde{\tilde{N}} \in \mathcal{M}_d \Rightarrow \tilde{\tilde{N}}_d^c \in \mathcal{M}_d$

$\Downarrow ?$

$A \cap \tilde{\tilde{N}}^c \in \mathcal{M}_d$?

$$= \big(\underset{\text{meas}}{A^c} \cup \underset{\text{meas}}{\tilde{\tilde{N}}}\big)^c$$

$$\Rightarrow \underbrace{A \setminus \tilde{\tilde{N}}}_{\in \mathcal{M}_d} \cup \underbrace{\tilde{N}}_{\in \mathcal{M}_d} \in \mathcal{M}_d \quad)$$

Alternative: $A \setminus \tilde{\tilde{N}}$ differs from A by $\tilde{\tilde{N}}$ which is a meas 0 set $\Rightarrow$ $A \setminus \tilde{\tilde{N}}$ is meas being A meas

next

$$\underbrace{A \setminus \tilde{\tilde{N}}}_{\in \mathcal{M}_d} \cup \underbrace{\tilde{N}}_{\text{meas}=0} \Rightarrow \in \mathcal{M}_d \quad)$$

$\Rightarrow \{g>a\}$ is meas $\forall a$ ∎

Do 1.9.5/6/7/8/9/10
1.9.11 (*) 1.9.15 (*)
1.9.12/13/14

Prop:
1. f, g meas $\Rightarrow$ $f \pm g$, $f \cdot g$ are meas.
2. f, g " $\Rightarrow$ f/g is meas on $\{g \neq 0\}$.

Def: We say that a certain property holds almost everywhere (a.e.) if it is fulfilled on every pt. except for a meas. 0 set.

Examples:
A funct f is continuous almost everywhere means

$$f \in \mathcal{C}(\mathbb{R}^d \setminus N), \quad \lambda_d(N) = 0.$$

For instance, one of the previous props could be written in the form:

If $f = g$ a.e. on D, then f is meas $\Leftrightarrow$ g is meas

$$\left(f(x) = g(x) \ \forall x \in D \setminus N, \ \lambda_d(N) = 0 \right)$$

$\left(f(x)=g(x) \quad \forall x \in D \setminus N, \ \lambda_d(N)=0 \right)$

Notation: We denote the class of all meas functs on D with $L(D)$ ← Lebesgue

$$\boxed{f=g \text{ a.e. on } D \Rightarrow \quad f \in L(D) \Leftrightarrow g \in L(D)}$$

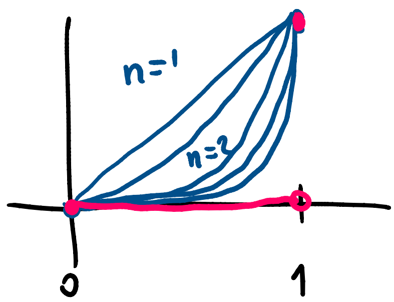
Thm: $(f_n) \subset L(D)$ s.t. $\boxed{f_n \longrightarrow f \text{ a.e. on } D}$

$\left(f_n(x) \longrightarrow f(x) \quad n \to +\infty, \quad \forall x \in D \setminus N, \ \lambda_d(N)=0 \right)$

$\Rightarrow \quad f \in L(D)$

In words: pointwise limits of meas functs is a meas funct. This doesn't happen if we replace measurability with continuity:

Ex: $f_n(x) = x^n \qquad x \in [0,1]$



$$f_n(x) \xrightarrow[n \to +\infty]{} \begin{cases} 1 & x=1 \\ 0 & 0 \leq x < 1 \end{cases} =: f(x)$$

$$\lim_{n \to +\infty} x^n = \begin{cases} \lim_n 1^n = 1 & x = 1 \\ \lim x^n & 0 \leq x < 1 \end{cases}$$

$$= \begin{cases} \lim x^n & 0 \leq x < 1 \end{cases}$$

Notice that even if $f_n \in \mathcal{C}^\infty$, $f \notin \mathcal{C}$

$$\Downarrow$$

$$f_n \in L \quad \text{and} \quad f \in L$$

$$f = \mathbb{1}_{\{1\}}$$