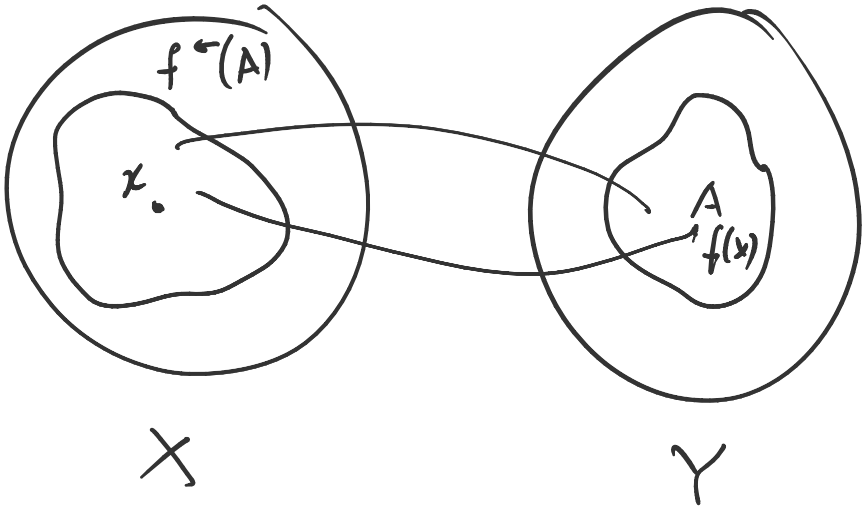
lunedì 7 ottobre 2019 12:44

$f^{\leftarrow}(A)$ $\qquad f: X \to Y$




$$f^{\leftarrow}(A) := \{x \in X : f(x) \in A\}$$

Ex 1.9.7 $f$ is meas $\Leftrightarrow$

$\to$ i) $\{f \geq a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}$

ii) $\{f \in I\} \in \mathcal{M}_d \quad \forall I \subset \mathbb{R}$ interval.

Sol: Recall that

$f \in L(\mathbb{R}^d) \Leftrightarrow$ (Def) $f^{\leftarrow}(]a,+\infty[) = \{x \in \mathbb{R}^d : f(x) > a\}$ ($\{f>a\}$) $\in \mathcal{M}_d \quad \forall a \in \mathbb{R}$.

i) Hp: Def

Th: i)

$\{f \geq a\} \quad \supset \quad \{f > a\}$

$\subset$

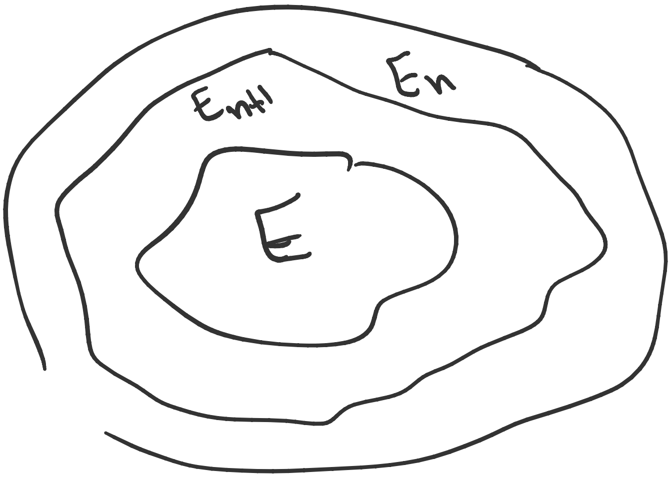
$\{f \geq a\}$ $\subset$ $\{f > \quad\}$

$\{f > a-\varepsilon\}$ $\forall \varepsilon > 0$

$\{f \geq a\} \subset E_n = \{f > a - 1/n\}$

$E$

$E_{n+1} = \{f > a - \frac{1}{n+1}\}$




$m_d$ by Hyp

(*) $E \overset{?}{=} \bigcap_n E_n$

(If this is true, then $E$ is meas because

$E^c = \bigcup_n E_n^c \in m_d$ (each $E_n^c \in m_d$)

$\Downarrow$

$(E^c)^c = E \in m_d$ )

So it remains to check (*).

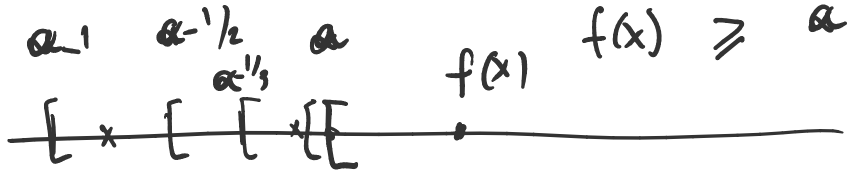
$E \subset \bigcap_n E_n$ is evident because $E_n \supset E$ $\forall n$

$E \overset{?}{\supset} \bigcap_n E_n \ni x \Rightarrow x \in E_n = \{f > a - 1/n\}$

$\Rightarrow f(x) > a - 1/n \quad \forall n$

$\Downarrow$ $n \to +\infty$

$a-1 \quad a-1/2 \quad a \qquad f(x) \geq a$




$f(x) \geq \alpha$

$\Rightarrow \quad x \in E = \{f \geq \alpha\}$

$\Rightarrow \quad E = \bigcap_n E_n .$

Vice versa

Hp: $\{f \geq a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}$

Th: $\{f > a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}$

Work out details in a similar way as the first part

ii) $\{f \in I\} \in \mathcal{M}_d \quad \forall I$ int.

$\Updownarrow$ $\quad\quad \Downarrow \quad \Uparrow$

(Def) $f \in L(\mathbb{R}^d)$

$\{f > a\} \in \mathcal{M}_d \quad \forall a$ $\quad$ trivial

$\{f \in ]a,+\infty[\} = \{f > a\}$

$\Uparrow$ : $\{f \in [a,b]\} \overset{?}{\in} \mathcal{M}_d$

$\{a \leq f \leq b\} = \{f \geq a\} \cap \{f \leq b\} \in \mathcal{M}_d$

by i) $\cap$ $\mathcal{M}_d$ $\quad \mathbb{R}^d \setminus \{f > b\}$

$= \{f > b\}^c$

by ... $m_d$ ... $\mathbb{R} \setminus \{f > b\}$

$$= \underbrace{\{f > b\}}_{\in m_d}{}^c \in m_d$$

$\{f \in ]a,b]\}$ $\{f \in ]a,b[\}$

$\{f \in [a,b[\}$ $\{f \in [a,+\infty[\}$ . . . all cases

finish.

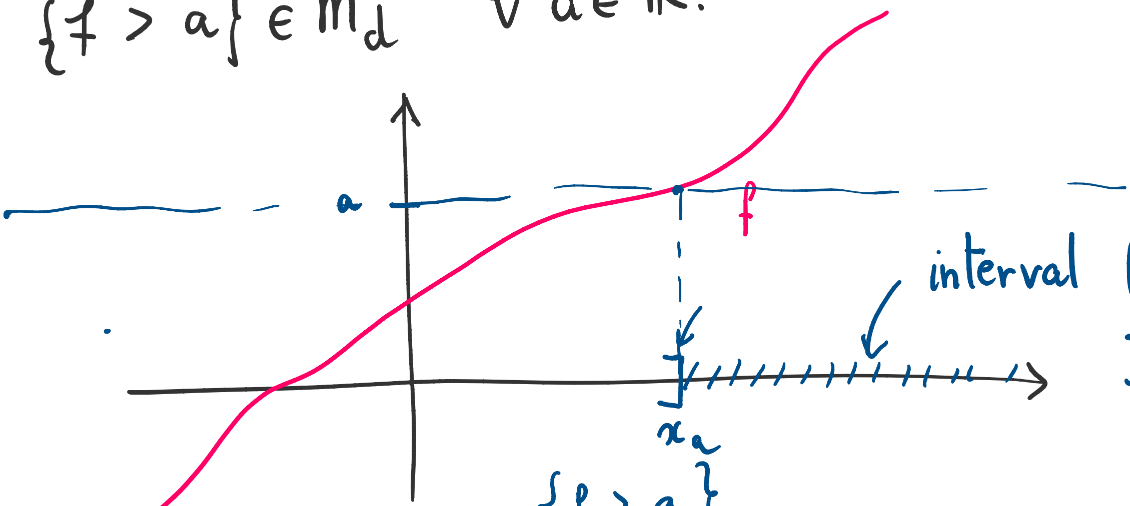
**Ex 1.9.9** If $f$ is monotone $f:\mathbb{R}\to\mathbb{R}$
( $f \nearrow$ : $f(x) \le f(y)$ for $x \le y$)
$\searrow$ $\ge$

$\Downarrow$

$f \in L(\mathbb{R})$.

**Sol:** We have to check if

$\{f > a\} \in m_d$ $\forall a \in \mathbb{R}$.

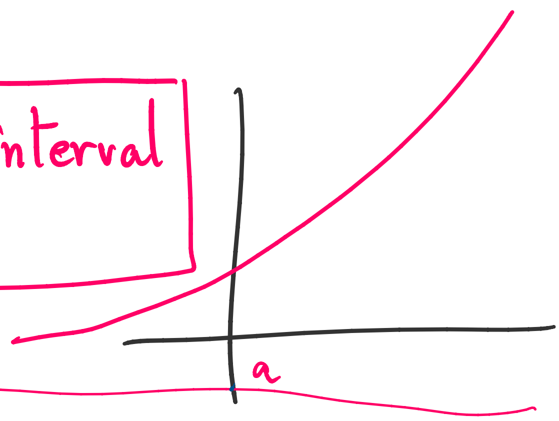



interval $[x_a, +\infty[$

$]x_a, +\infty[$

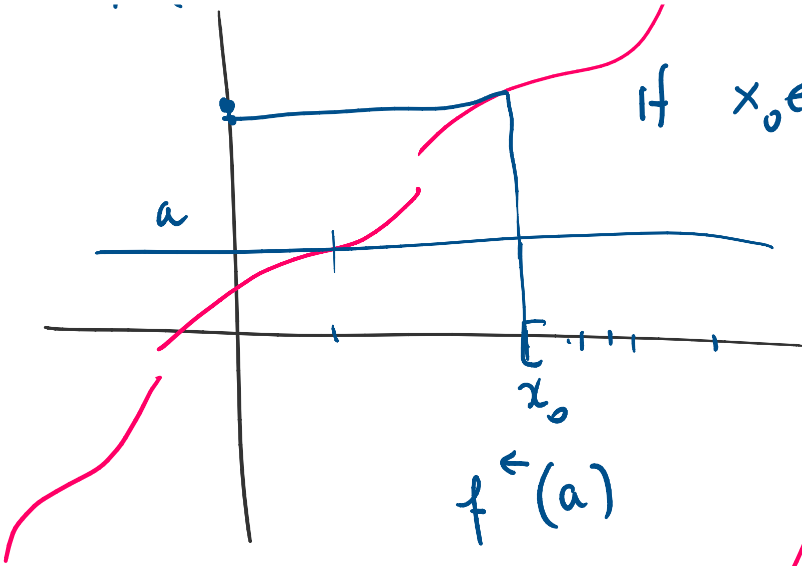
$\{f > a\}$

Guess: $\{f > a\}$ is an interval $\forall a$



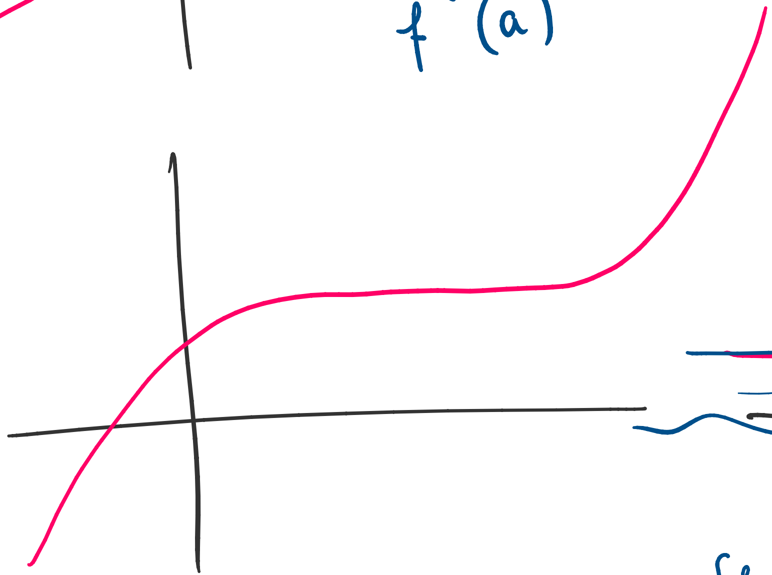

$f^{\leftarrow}(a)$

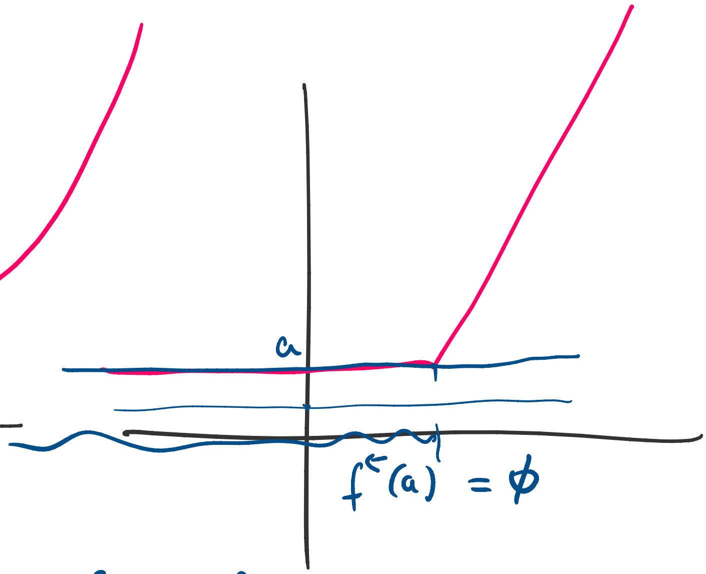
if $x \in \{f > a\} \Rightarrow [x_a, +\infty[ \subset \{f > a\}$




If $x_0 \in \{f>a\} \Rightarrow [x_0, +\infty[ \subset \{f>a\}$

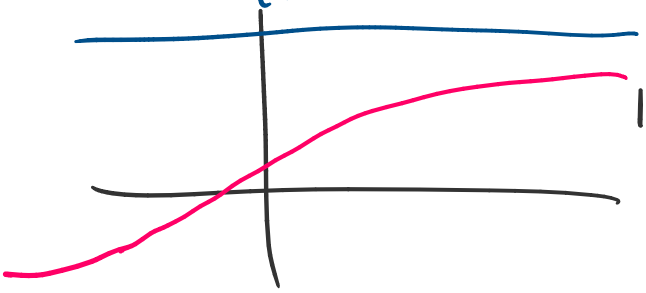
$\{f>a\}$






$\{f>a\} = \phi \in \mathcal{M}_d$

$\{f>a\} = \mathbb{R}$



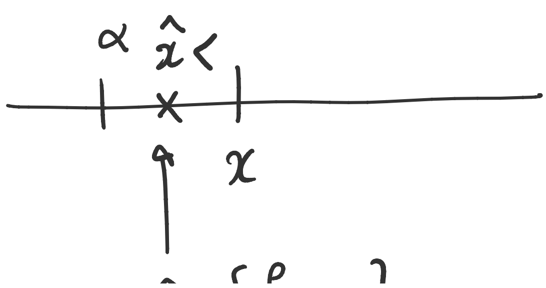
If $\{f>a\} = \phi \in \mathcal{M}_d$

$\{f>a\} \neq \phi$

we claim this set is an int of type $]\,??\,, +\infty[$ or $[$

$\underbrace{\{x \in \mathbb{R} : f>a\}}_{\alpha := \operatorname{Inf}} \subset \mathbb{R}$

$\{f>a\} \supseteq ]\alpha, +\infty[$



$\{f>a\} \overset{?}{\supseteq} \, ]\alpha,+\infty[$

$x>\alpha$

$\exists\, \hat{x} \in \{f>a\}$

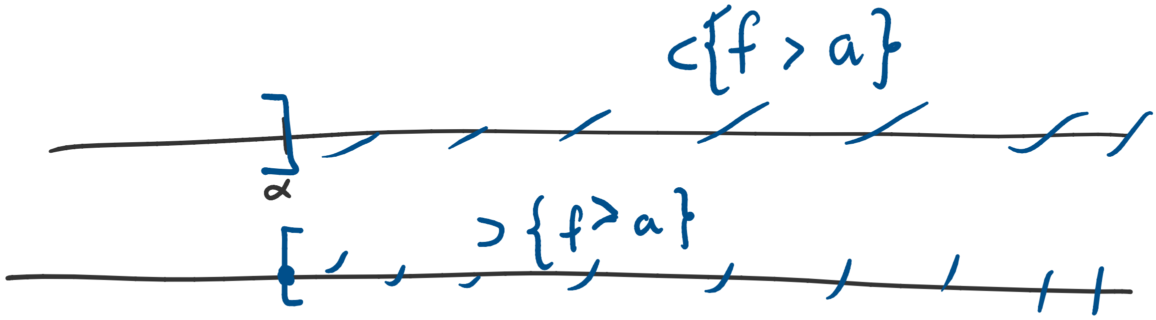
$a < f(\hat{x}) \leq f(x)$

$\Downarrow$

$x \in \{f>a\}$

$\overset{?}{\subset}$ (try)

$\{f>a\} \subset [\alpha,+\infty[$




$]\alpha,+\infty[ \; \subset \; \boxed{\{f>a\} \overset{?}{\subset} [\alpha,+\infty[}$

$\left( \Rightarrow \{f>a\} \setminus \, ]\alpha,+\infty[ \; \subset \{\alpha\} \; \lambda_1 \text{ meas } 0 \text{ set} \Rightarrow \{f>a\} \text{ meas} \right)$

$\hookrightarrow$ we have to prove that if $f(x)>a$

$\Downarrow$

$x \geq \alpha$

$= \quad \|$

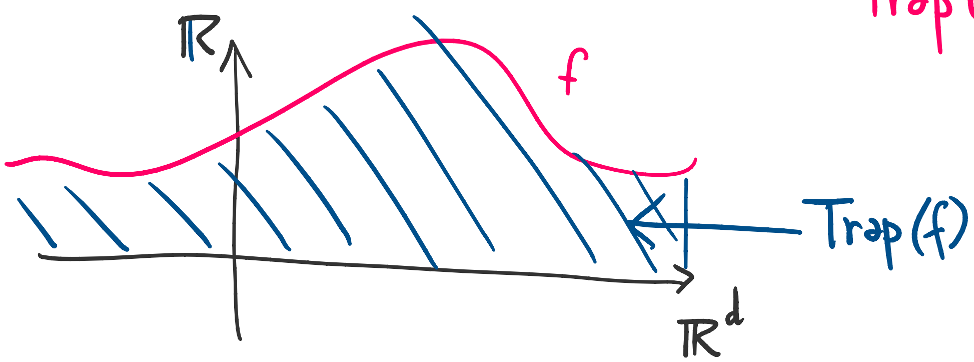
$\operatorname{Inf}\{f>a\}$

☒

**Do 1.9.10, 1.9.13**

**Thm:** $f : \mathbb{R}^d \longrightarrow [0, +\infty]$

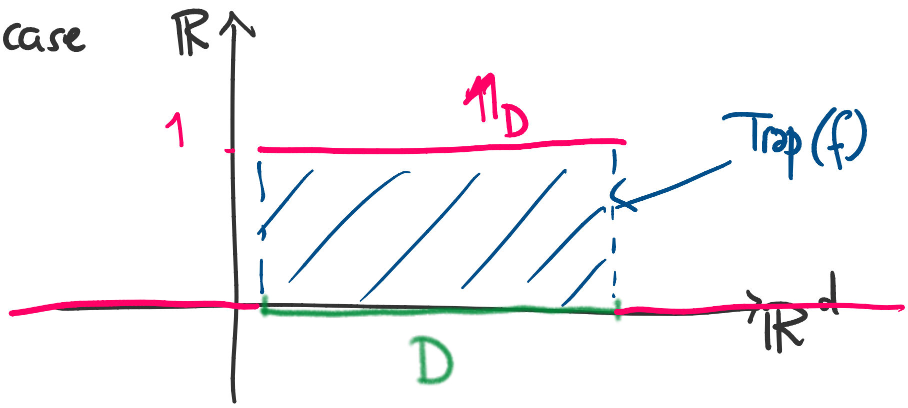
$$f \in L(\mathbb{R}^d) \iff \underbrace{\{ (x,y) \in \overbrace{\mathbb{R}^d \times \mathbb{R}}^{\mathbb{R}^{d+1}} : \ 0 \leq y < f(x) \}}_{\text{Trap}(f)} \in \mathcal{M}_{d+1}$$




**Example:** $f = 1\!\!1_D = \begin{cases} 1 & x \in D \\ 0 & x \notin D \end{cases}$.

We know that $1\!\!1_D \in L(\mathbb{R}^d) \iff D \in \mathcal{M}_d$

In this case



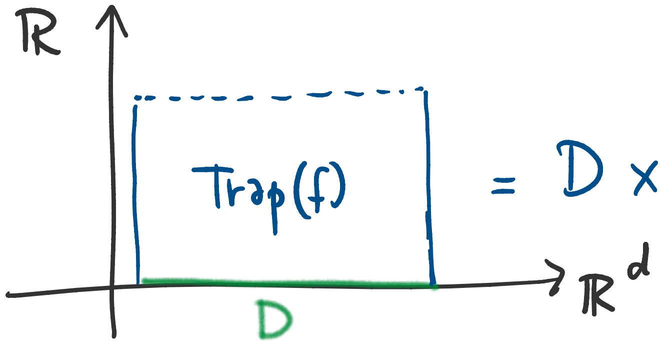

$$\text{Trap}(f) = \{ (x,y) \in \mathbb{R}^d \times \mathbb{R} : \ 0 \leq y < 1\!\!1_D(x) \}$$

$$= \{ (x,y) : \ x \in D, \ 0 \leq y < \overset{1}{\overset{\shortparallel}{1\!\!1_D(x)}} \}$$

$$\cup \underbrace{\{ (x,y) : \ x \notin D, \ 0 \leq y < \underset{\underset{0}{\shortparallel}}{1\!\!1_D(x)} \}}_{\emptyset}$$

$$= \{ (x,y) : \ x \in D \ \ 0 \leq y < 1 \}$$

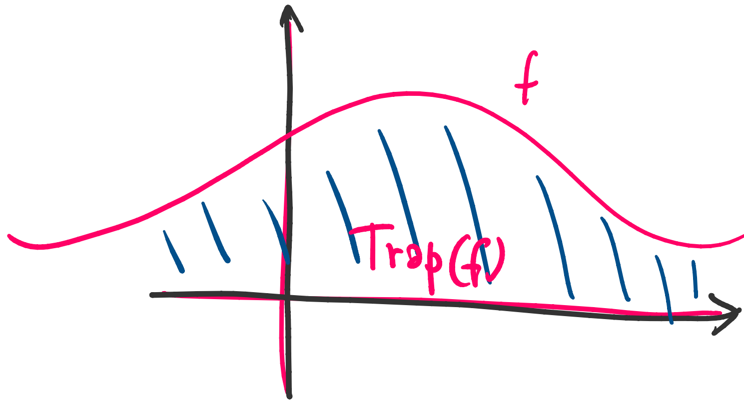
$$= \{(x,y) : x \in D,\ 0 \leq y < 1\}$$




$= D \times [0,1[.$

**Def:** (Integral) Let $f : \mathbb{R}^d \to [0,+\infty]$, $f \in L(\mathbb{R}^d)$

We define




$$\int_{\mathbb{R}^d} f \left( \equiv \int_{\mathbb{R}^d} f(x)\,dx \right)$$

$$:= \lambda_{d+1}\left(\mathrm{Trap}(f)\right)$$

**Ex.** If $f(x)$ is Dirichlet fct $f(x) = \mathbb{1}_{\mathbb{Q}^c \cap [0,1]}$

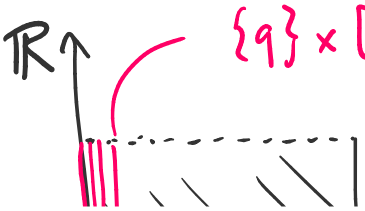
$[0,1] \setminus \mathbb{Q}$

$f \geq 0$ and meas,

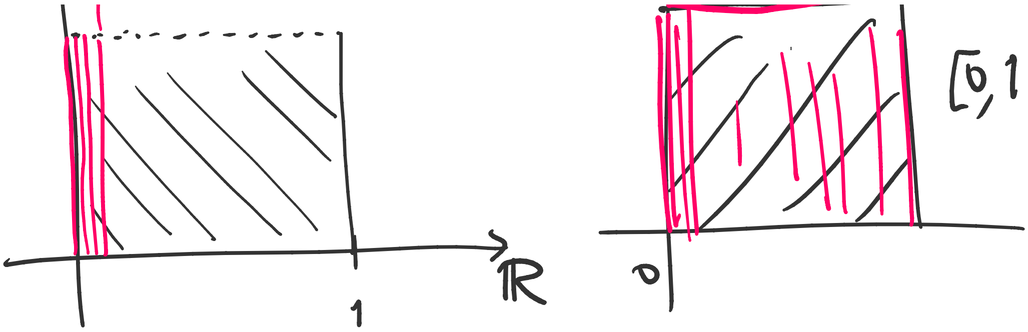
$$\int_{\mathbb{R}} f = \lambda_2\left(\mathrm{Trap}(f)\right)$$

In this case $\mathrm{Trap}(f) = \left([0,1] \setminus \mathbb{Q}\right) \times [0,1[$

$\{q\} \times [0,1[ \quad q \in \mathbb{Q}$




$[0,1]^2 = I$




$[0,1]^2 = I$

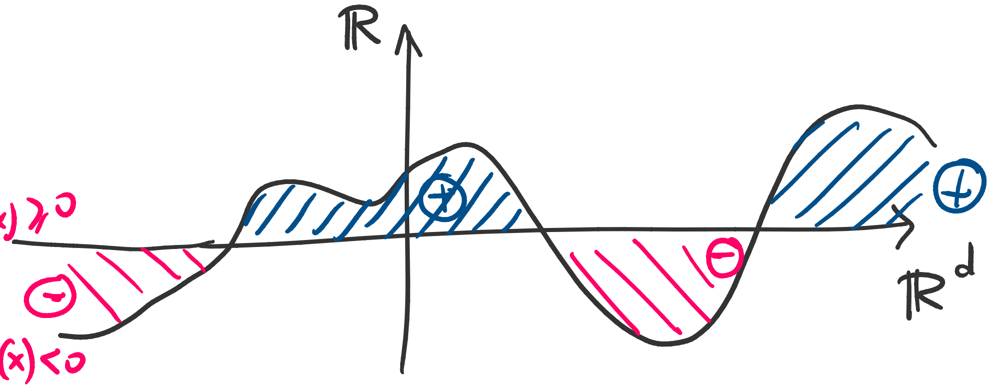
$I \setminus Trap(f)$ is a meas 0 set

$\Rightarrow \lambda_2(Trap(f)) = \lambda_2(I) = 1$

$\Rightarrow \int_{\mathbb{R}} 1\!\!1_D = 1$ in Leb. theory. ∎

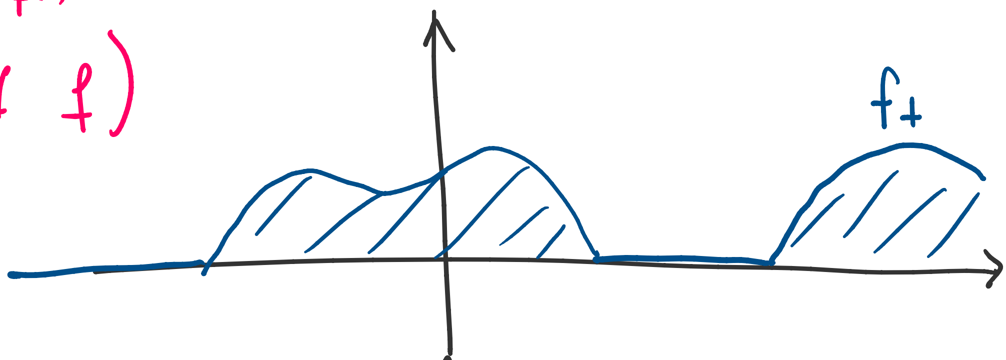
<u>Def</u>: Let $f : \mathbb{R}^2 \longrightarrow [\infty, +\infty]$, $f \in L(\mathbb{R}^d)$

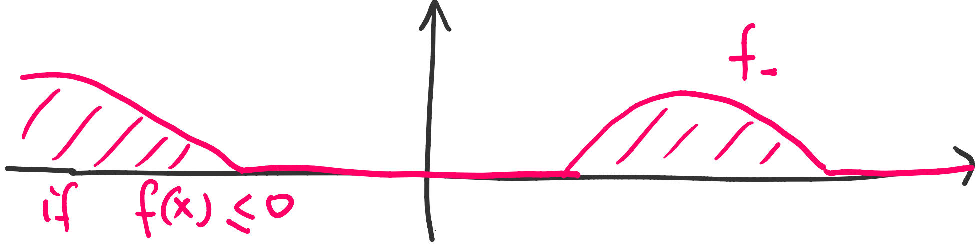
Def




$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

(positive part of f)




$$f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$$

(negative part of f.)

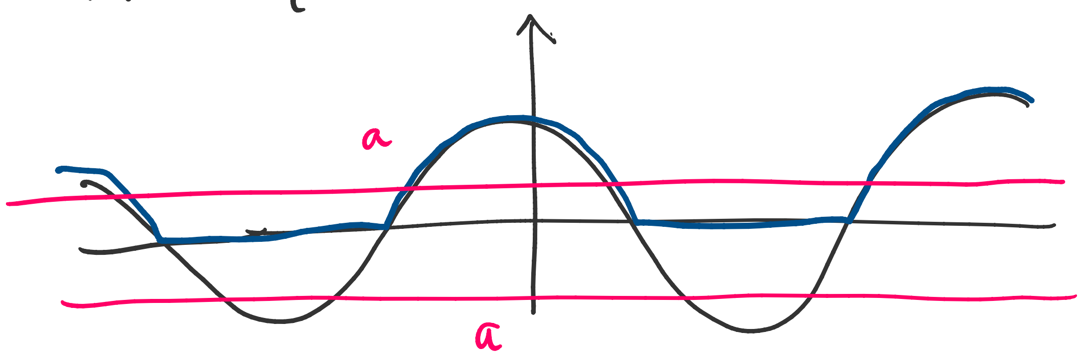
(negative part of $f$.)

**Rmk:** If $f \in L(\mathbb{R}^d) \Rightarrow f_+, f_- \in L(\mathbb{R}^d)$

Indeed

Hypo: $\{f > a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}$

Th: $\{f_+ > a\} \in \mathcal{M}_d \quad \forall a \in \mathbb{R}$




If $a < 0$: $\{f_+ > a\} = \mathbb{R}^d \in \mathcal{M}_d$

$a \geq 0$: $\{f_+ > a\} = \{f > a\} \in \mathcal{M}_d$ ☒

**Def:** Let $f \in L(\mathbb{R}^d)$. We say that $f$ is *integrable* if

$$(0 \leq) \int_{\mathbb{R}^d} f_+ , \int_{\mathbb{R}^d} f_- < +\infty$$

and in this case we pose

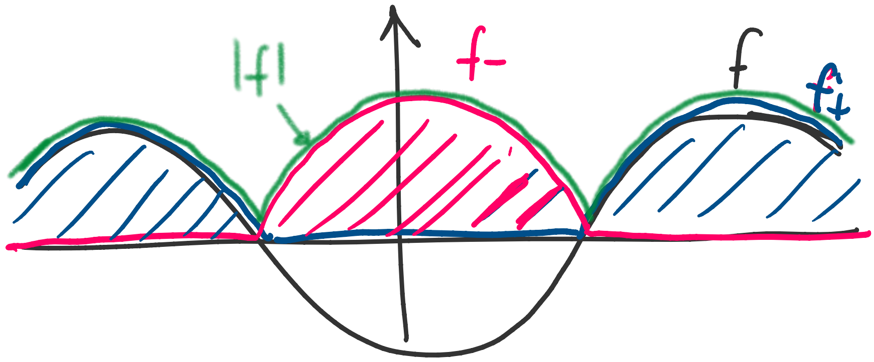
$$\int_{\mathbb{R}^d} f := \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-$$

$$(\lambda_{d+1}(\mathrm{Trap}(f_+)) - (\lambda_{d+1}(\mathrm{Trap}(f_-))$$

**Rmk:**

**Rmk:**

$$\int_{\mathbb{R}^d} f_+ + \int_{\mathbb{R}^d} f_- = \int_{\mathbb{R}^d} |f|$$




so in a more elegant way, the cond of integrability may be written as

$$\int_{\mathbb{R}^d} |f| < +\infty .$$

The class of all the (Lebesgue) integrable functs is denoted by $L^1(\mathbb{R}^d)$

$$L^1(\mathbb{R}^d) = \left\{ f \in L(\mathbb{R}^d) : \int_{\mathbb{R}^d} |f|^{1} < +\infty \right\}$$

**Def:** (Int for $\mathbb{C}$-val functs)

$f : \mathbb{R}^d \to \mathbb{C}$, $f = u + iv$ $\quad u = \mathrm{Re} f$

$$\left( f(x) = u(x) + iv(x) \quad \begin{array}{l} u(x) = \mathrm{Re} f(x) \\ v(x) = \mathrm{Im} f(x) \end{array} \right)$$

We say that $f \in L(\mathbb{R}^d)$ if $\mathrm{Re} f, \mathrm{Im} f \in L(\mathbb{R}^d)$

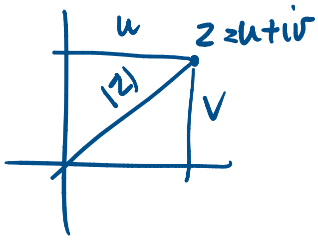
call integral of $f$ the quantity

Moreover, we call <u>integral of $f$</u> the quantity

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} u + i \int_{\mathbb{R}^d} v$$

provided $u, v \in L^1(\mathbb{R}^d) \Leftrightarrow \int |u|, \int |v| < +\infty$

$$\overset{?}{\Leftrightarrow} \int |f| < +\infty$$




( Yes: $\boxed{|f| = \sqrt{u^2 + v^2}}$

If $\int |u|, \int |v| < +\infty \Rightarrow \int |f| = \int |u + iv| \leq \int |u| + \int |v| < +\infty$ (where $|u+iv| \leq |u| + |v|$)

$$0 \leq \varphi \leq \psi \Longrightarrow \int \varphi \leq \int \psi$$

$$\lambda_{d+1}(\operatorname{Trap} \varphi) \leq \lambda_{d+1}(\operatorname{Trap} \psi)$$

$\subset \operatorname{Trap} \psi$

Vice versa if $\int |f| < +\infty \Rightarrow |u| \leq |f|$, $|v| \leq |f|$

$$\Rightarrow \int |u| \leq \int |f|, \quad \int |v| \leq \int |f| \quad < +\infty$$

This int fulfills all mat conds we may expect as, for inst

- $f, g \in L^1, \Rightarrow \alpha f + \beta g \in L^1 \quad \forall \alpha, \beta \in \mathbb{R}$

• $f, g$ ...

and $\displaystyle\int_{\mathbb{R}^d} (\alpha f + \beta g) = \alpha \int_{\mathbb{R}^d} f + \beta \int_{\mathbb{R}^d} g$

• $\boxed{f \leq g}$ a.e., $f, g \in L^1 \Rightarrow \displaystyle\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g$

$(f(x) \leq g(x) \ \forall x \in \mathbb{R}^d \setminus N, \ \lambda_d(N) = 0)$

• $f \in L^1$ $\displaystyle\left| \int_{\mathbb{R}^d} f \right| \leq \int_{\mathbb{R}} |f|$

<u>Def:</u> $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{C}$. We say that $f \in L^1(D)$

if $\displaystyle\int_D |f| \equiv \int_{\mathbb{R}^d} |f| \mathbb{1}_D < +\infty$

and we pose

$$\int_D f = \int_{\mathbb{R}^d} f \mathbb{1}_D .$$