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Computing Integrals

I) Connections with Riemann and Generalized Integrals

Thm: Riemann integrable fcts are Leb. int

(if $f \in \mathcal{R}([a,b]) \Rightarrow f \in L^1([a,b])$)

$$\int_a^b f(x)\,dx = \int_{[a,b]} f$$

(Riemann int) (Lebesgue)

In part, $f \in \mathcal{R}([a,b]) \Leftrightarrow f$ is a.e. cont

$(f \in \mathcal{C}([a,b] \setminus N) : \lambda_1(N) = 0)$

Ex. So for inst. $f = \mathbb{1}_{[0,1] \setminus \mathbb{Q}}$ is not R. int

because $S = \{x \in [a,b] : f \text{ is not cont at } x\}$

$= [a,b]$

$\lim_{x \to x_0} \mathbb{1}_{[0,1] \setminus \mathbb{Q}} \nexists \Rightarrow f \notin \mathcal{R}([0,1])$. $\boxtimes$

For generalized integrals things are a bit different.

Let's consider the case of

$+\infty$?

$$\int_a^{+\infty} f(x)\,dx \overset{?}{=} \int_{[a,+\infty[} f \,.$$

Recall that if $f \in R([a,r])$ $\forall r \geq a$

we can compute $\int_a^r f(x)\,dx$.

We say that f is int on $[a,+\infty[$ in gen. sense

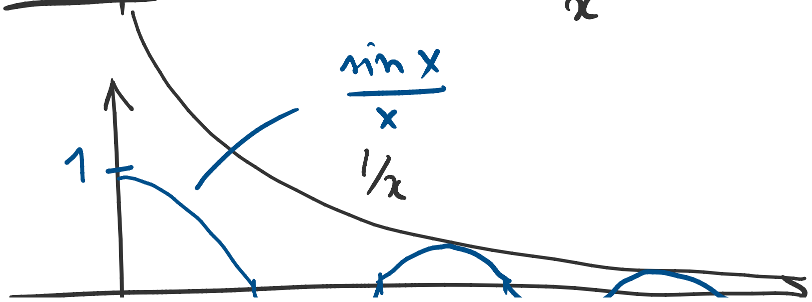
if $\exists \int_a^{+\infty} f(x)\,dx := \lim_{r\to+\infty} \int_a^r f(x)\,dx \in \mathbb{R}$

Pb: Does this imply that $f \in L^1([a,+\infty[)$ and

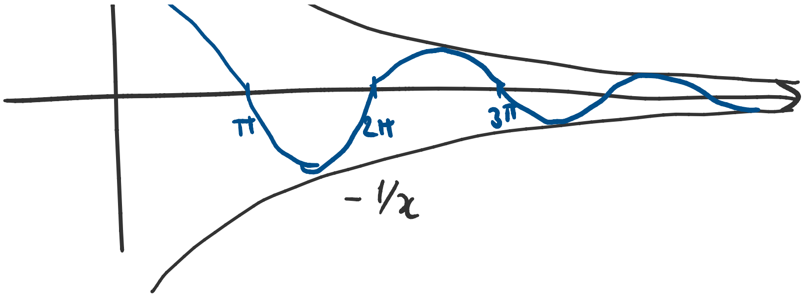
$$\underset{\text{(gen. int)}}{\int_a^{+\infty} f(x)\,dx} = \underset{\text{(Leb.)}}{\int_{[a,+\infty[} f} \quad ?$$

NO

Example $f(x) = \dfrac{\sin x}{x} \in \mathscr{C}([0,+\infty[)$ $\left(= 1 \text{ for } x = 0 \right)$



It may be checked that.



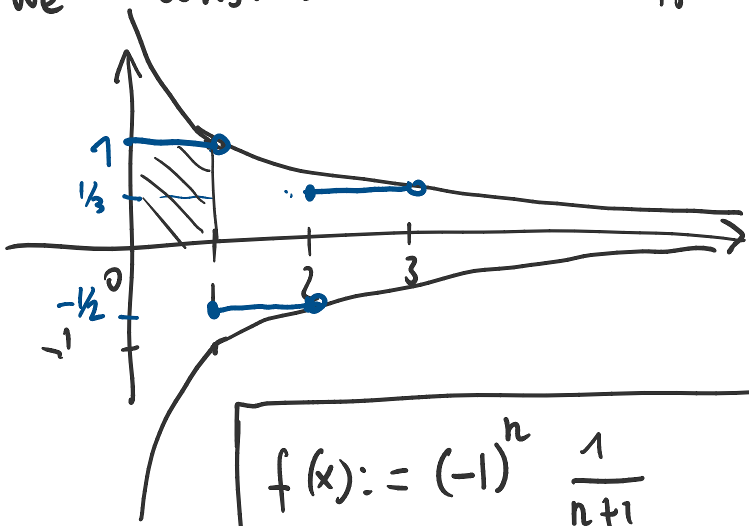
that

1. $\exists \int_0^{+\infty} \frac{\sin x}{x}\, dx$ (gen sense)

but

2. $f \notin L^1([0,+\infty[)$

Instead to work all details with this example we construct a different but similar one.



$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \mathbb{1}_{[n,n+1[}$$

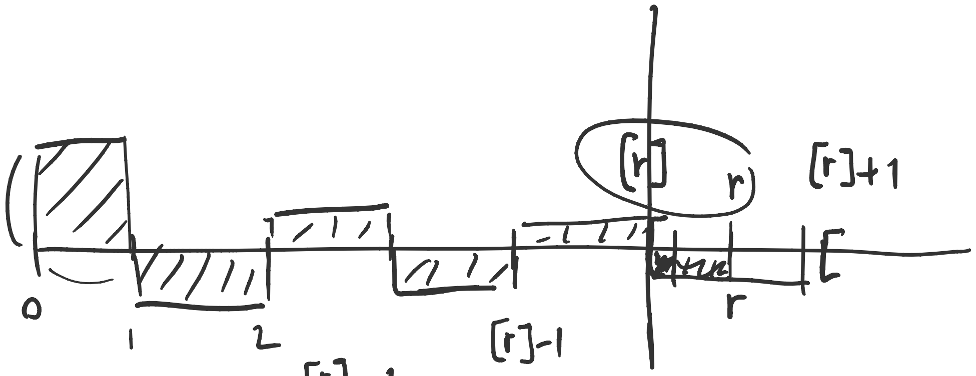
$$\boxed{f(x) := (-1)^n \frac{1}{n+1} \qquad n \le x < n+1, \quad n \in \mathbb{N}}$$

Now for this f we have

1. $\exists \int_0^{+\infty} f(x)dx$

2. $f \notin L^1([0,+\infty[)$

1. $$\int_0^{+\infty} f(x)\, dx = \lim_{r \to +\infty} \int_0^r f(x)\, dx$$



$$\int_0^r f(x)\,dx = \sum_{n=0}^{[r]-1} (-1)^n\, 1\cdot \frac{1}{n+1} + (-1)^{[r]} \overset{0\le\ <1}{(r-[r])}\, \frac{1}{[r]+1}$$

$$\downarrow$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

$$\|$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \in \mathbb{R} \quad \text{(Leibniz test)}$$

$$(-1)^{[r]} \qquad \frac{1}{[r]+1} \to 0$$

$$\pm 1 \qquad 0$$

$$\to 0$$

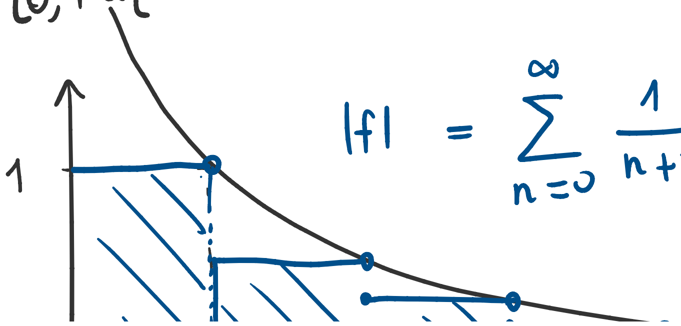
$$\Rightarrow \int_0^r f(x)\,dx \xrightarrow[r\to+\infty]{} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \in \mathbb{R}$$

2. Why $f \notin L^1([0,+\infty[)$?

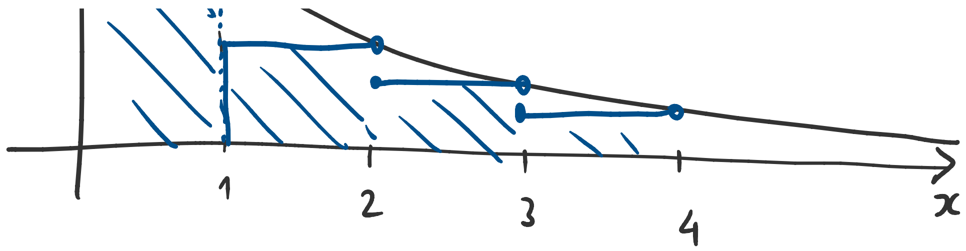
(f is clearly meas because f a.e. cont)

and to be in L^1 we need

$$\int_{[0,+\infty[} |f| < +\infty$$



$$|f| = \sum_{n=0}^{\infty} \frac{1}{n+1} \mathbb{1}_{[n,n+1[}$$



$$\int_{[0,+\infty[} |f| = \lambda_2\left(\mathrm{Trap}(f)\right) \underset{\text{count add}}{=} \sum_{n=0}^{\infty} \lambda_2\left([n,n+1[\times [0, \tfrac{1}{n+1}]\right)$$

$$\mathrm{Trap}(f) = \bigcup_{n=0}^{\infty} [n,n+1[\times [0, \tfrac{1}{n+1}]$$

disjoint

$$= \sum_{n=0}^{\infty} 1 \cdot \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Conclusion: integrable in gen sense $\not\Rightarrow$ Leb int.

There's another concept of integrability for gen. int that turns out to be stricter than Leb int:

Def: $f \in \mathcal{R}([a,r])$ $\forall r > a$ be s.t.

$$\exists \int_a^{+\infty} |f(x)|\, dx \quad \text{in gen sense}$$

$$\left(\exists \lim_{r \to +\infty} \int_a^r |f(x)|\, dx \in \mathbb{R} \right) \quad \left(\begin{array}{l} \text{In part} \\ \exists \int_a^{+\infty} f(x)\, dx \end{array} \right)$$

$\Rightarrow f \in L^1([a,+\infty[)$. and

$$\int_a^{+\infty} f(x)\, dx = \int_{[a,+\infty[} f$$

The same conclusion holds for all other possible

The same conclusion holds for all other possible cases as for

$$\int_{-\infty}^{b}, \quad \int_{-\infty}^{+\infty}, \quad \int_{a}^{b} \quad f \text{ unbded in } a \text{ or } b \text{ or both.}$$

$$\left(\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{r \to 0+} \int_r^1 \frac{1}{\sqrt{x}}\, dx \right.$$

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Folder: Analytical Methods

Limit Thms

Pb: $(f_n) \subset L^1(D)$ $\quad \int_D f_n \longrightarrow \int_D f$

$\boxed{f_n \to f}$

in some sense

$$\Updownarrow$$

$$\lim_n \int_D f_n = \int_D \lim_n f_n$$

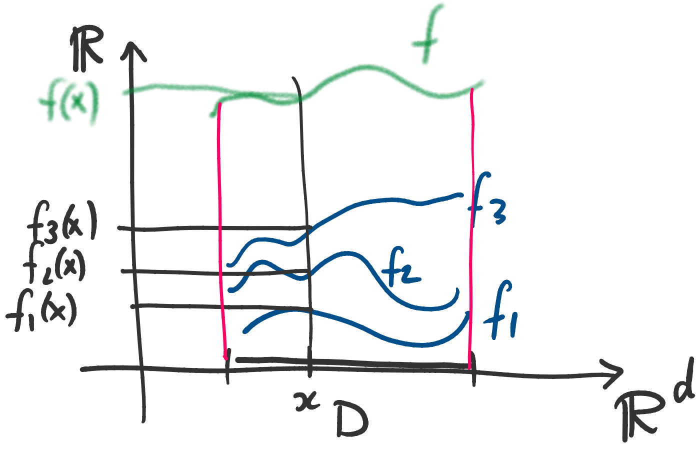
Thm: (monotone conv)

$(f_n) \subset L(D)$, s.t. $\boxed{0 \leq f_n \leq f_{n+1} \quad \forall n}$

$$\left(f_n(x) \leq f_{n+1}(x) \quad \forall x \in D \right)$$

Then if

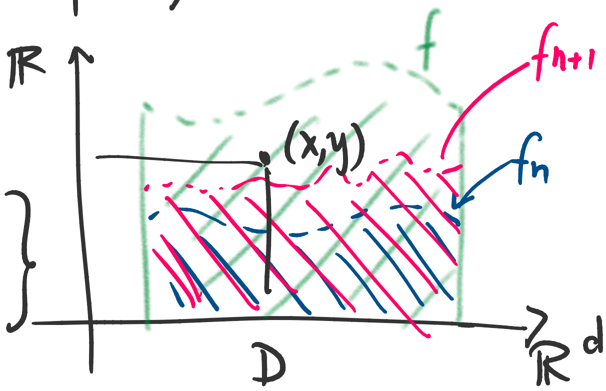
$$f(x) := \lim_{n \to +\infty} f_n(x) \qquad x \in D$$



$$\Rightarrow \quad \int_D f = \lim \int_D f_n$$

<u>Proof:</u> $\int_D f \underset{\text{def}}{=} \lambda_{d+1}(\mathrm{Trap}(f))$

$$\mathrm{Trap}(f) = \left\{ (x,y) : x \in D, \; 0 \leq y < f(x) \right\}$$



$$\int_D f_n = \lambda_{d+1}(\mathrm{Trap}(f_n))$$

Notice that because $f_n \leq f_{n+1} \quad \forall x$

$$E_n = \mathrm{Trap}(f_n) \subset \mathrm{Trap}(f_{n+1}) \qquad \forall n$$

$\Rightarrow E_n \nearrow$

$$\Rightarrow \quad \lim_n \int_D f_n = \lim \lambda_{d+1}(\mathrm{Trap}(f_n))$$

$$= \lim \lambda_{d+1}(E_n)$$

$= \lambda_{d+1}(E)$ where $E = \bigcup E_n$

$\Uparrow$ by cont of λ_{d+1}

Is $\boxed{E = \text{Trap } f}$? If yes

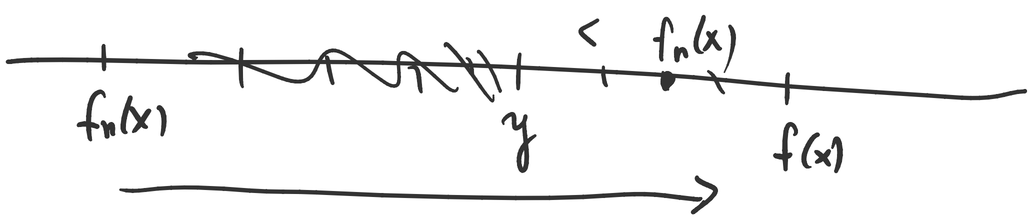
$= \lambda_{d+1}(\text{Trap}(f)) \underset{\text{def Leb int}}{=} \int_D f.$

$$E = \bigcup_n \text{Trap}(f_n) \overset{\supset}{\underset{\subset}{?}} \text{Trap}(f)$$

$(x,y) \in E \Rightarrow (x,y) \in \text{Trap}(f_n)$ for some n $\Rightarrow$ $x \in D$, $0 \le y \cancel{< f_n(x)} \le f(x)$

$\Downarrow$

$(x,y) \in \text{Trap}(f) \Leftarrow$ $x \in D$, $0 \le y < f(x)$



$\blacksquare$

Corollary $(f_n) \subset L(D)$: $0 \le f_n \le f_{n+1}$ a.e.

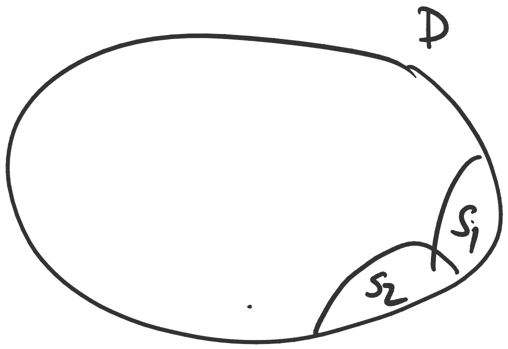
$(\forall n\ \exists S_n\ \lambda(S_n) = 0 : f_n \le f_{n+1}\ \forall x \in D \setminus S_n)$

Then.

Then

$$\int_D \lim_n f_n = \lim_n \int_D f_n .$$

__Proof:__



$f_1 \leq f_2$ $D \setminus S_1$

$f_2 \leq f_3$ $D \setminus S_2$

$\vdots$

$$f_n \leq f_{n+1} \quad \text{on} \quad \overbrace{D \setminus \bigcup_{n=1}^{\infty} S_n}^{\tilde{D} :=}$$

on $\tilde{D}$ we apply the prev thm and we get

$$\int_{\tilde{D}} \lim f_n = \lim \int_{\tilde{D}} f_n$$

and because $N := \bigcup S_n$ $\quad \lambda(N) = 0$

$$\left(\leq \sum \lambda(S_n) = 0 \right)$$

$$\Rightarrow \int_D = \underbrace{\int_{D \setminus N}}_{\tilde{D}} + \underbrace{\int_N}_{=\,0} = \int_{\tilde{D}}$$

∎

__Corollary:__ $(f_n) \subset L(D)$, $f_n \geq 0$ a.e.

Then

$$\int_D \left(\sum_n f_n\right) = \sum_n \int_D f_n$$

<u>Proof:</u> We apply monot conv to g_n where

$$\left(\sum_{n=0}^{\infty} f_n = \lim_{n\to+\infty} \underbrace{\sum_{k=0}^{n} f_k}_{g_n} \right.$$

$$g_{n+1} \geq g_n \quad \text{a.e.}$$

$$f_1+f_2+\dots+f_n+f_{n+1} \searrow f_1+f_2+\dots f_n$$

$$f_{n+1} \geq 0$$

monot conv

$$\Rightarrow \int_D \lim g_n = \lim_n \int_D g_n$$

$$\int_D \sum_n f_n = \lim_n \int_D \sum_{k=0}^{n} f_k$$

$$= \lim_n \sum_{k=0}^{n} \int_D f_k$$

$$= \sum_{k=0}^{\infty} \int_D f_k$$

∎

<u>Rmk:</u> What happens if $(f_n) \subset L(D)$ s.t

Rmk: What happens if $(f_n) \subset L(D)$ ...

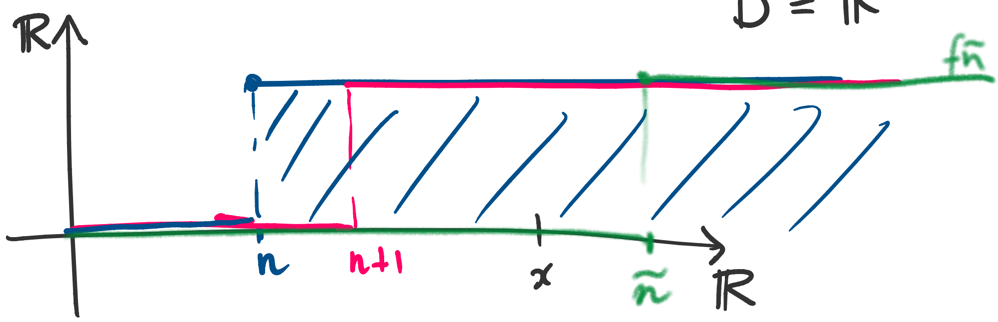
$$f_n \geq f_{n+1} \geq 0 \quad \forall n \quad \text{a.e.} \quad ?$$

Is it true that $\int_D \lim f_n \overset{?}{=} \lim \int_D f_n$?

In general, this is false

Example: $f_n = 1\!\!1_{[n,+\infty[}$ $\qquad f_n \geq f_{n+1}$ a.e.

$D = \mathbb{R}$



$$\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} 1\!\!1_{[n,+\infty[} = +\infty \longrightarrow +\infty$$

$$\int_{\mathbb{R}} \lim_n f_n = \int_{\mathbb{R}} 0 = 0$$

$$\lim_{n\to+\infty} f_n(x) = 0 \qquad f_n(x) = 0 \quad \forall n \geq N_x$$

So

$$+\infty = \lim \int_{\mathbb{R}} f_n \neq \int_{\mathbb{R}} \lim f_n = 0. \qquad \blacksquare$$

Exercise $(f_n) \subset L(D)$, $f_n \geq f_{n+1} \geq 0 \quad \forall n$ a.e.

<u>Exercise</u> $(f_n) \subset L(D)$, $f_n \geq f_{n+1} \geq 0$ $\forall n$

s.t $\int_D f_1 < +\infty$. Then

$$\lim_n \int_D f_n = \int_D \lim f_n .$$