Dominated Conv. Thm

Monotone conv thm is too specific and difficult to apply in concrete applications because the main assumption

$$0 \leq f_n \leq f_{n+1} \quad \text{a.e}$$

implies

1) all f_n must be ≥ 0
2) f_n is increasing in n.

However, by monot conv it follows the most powerful limit thm we have in int. th.

Thm: $(f_n) \subset L^1(D)$ be s.t.

1. $f_n \to f$ a.e. $\left(\lim_{n \to +\infty} f_n(x) \overset{=: f(x)}{} \text{ exists, a.e. } x \in D \right)$

2. $\exists g \in L^1(D) : |f_n| \leq g$ a.e.
$(|f_n(x)| \leq g(x) \quad \text{a.e. } x \in D, \ \forall n \in \mathbb{N})$
(g is called <u>integrable dominant for (f_n)</u>)

Then, $f \in L^1(D)$ and

$$\lim_n \int_D f_n = \int_D f \quad \left(= \int_D \lim_n f_n \right)$$

Proof: f is meas because it is pointwise limit of $f_n \in L$. Furthermore

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of $f_n \in L$. Furthermore

$$\int_D |f| < +\infty.$$

Indeed, by 2.

$$0 \leq |f_n(x)| \leq g(x) \quad \text{a.e. } x \in D$$
$$\downarrow$$
$$|f(x)| \leq g(x) \quad \text{a.e. } x \in D$$
$$\Downarrow$$
$$\int_D |f| \leq \int_D g < +\infty \quad (\text{because } g \in L^1)$$

Now the core of the proof is to prove that

$$\int_D f_n \longrightarrow \int_D f$$
$$\Leftrightarrow \quad \int_D f_n - \int_D f \longrightarrow 0$$
$$\Leftrightarrow \quad \int_D (f_n - f) \longrightarrow 0$$
$$\Leftarrow \quad \int_D |f_n - f| \longrightarrow 0$$

(Why? $0 \leq \left| \int_D (f_n - f) \right| \overset{\Delta}{\leq} \int_D |f_n - f| \longrightarrow 0$

Goal: $\boxed{\int_D \underbrace{|f_n - f|}_{\geq 0} \longrightarrow 0}$

$|f_n - f| \geq 0$
$|f_n - f| \longrightarrow 0$

$\Gamma \; \varepsilon_n := |f_n - f| \overset{\geq 0}{\longrightarrow} 0$

not true in gen.

$\varepsilon_n \overset{?}{\geq} \varepsilon_{n+1}$

$\int \varepsilon_1 < +\infty$

$$\lim_n \int_D \varepsilon_n \overset{?}{=} \int_D (\lim_n \varepsilon_n) = \int_D 0 = 0$$

$|f_n - f| \to 0$ $\quad \lim_n \int_D \varepsilon_n \overset{\cdot}{=} \int_D (\lim \varepsilon_n) = \int_D 0 = 0$

$\lim_n \varepsilon_n = 0$

$\int |f_1 - f|$

true because $f_1 \in L^1$, $f \in L^1$ $\Rightarrow f_1 - f \in L^1$

$|f_1 - f|$

$|f_2 - f|$

$|f_3 - f|$

$\vdots$

$|f_n - f|$

$|f_{n+1} - f|$

$|f_{n+2} - f|$

$\vdots$

$$\boxed{\varepsilon_n(x) := \sup_{k \geq n} |f_k(x) - f(x)|} \geq |f_n(x) - f(x)|$$

$$\mathbin{\rotatebox[origin=c]{90}{$\geq$}}$$

$$\varepsilon_{n+1}(x) = \sup_{k \geq n+1} |f_k(x) - f(x)|$$

$$\boxed{\varepsilon_n \geq \varepsilon_{n+1} \geq 0 \text{ a.e.}}$$

$$\int_D \varepsilon_1 < +\infty$$

$$\varepsilon_1 = \sup_{k \in \mathbb{N}} |f_k - f|$$

$$|f_k - f| \leq \underbrace{|f_k|}_{\leq g} + \underbrace{|f|}_{\leq g} \overset{2.}{\leq} 2g \in L^1 \quad \forall k$$

$$\Rightarrow \varepsilon_1 = \sup_k |f_k - f| \leq 2g \in L^1$$

$$\Rightarrow \varepsilon_1 \in L^1$$

$\Rightarrow$ (monot conv for decreasing seqs)

$$\lim_n \int_D \varepsilon_n = \int_D \lim \varepsilon_n$$

Now:

First $\quad \varepsilon_n(x) \longrightarrow 0 \quad n \to +\infty \quad$ a.e. x

$$\varepsilon_n(x) = \sup_{k \geq n} |f_k(x) - f(x)|$$

Indeed by 1. $\quad f_n(x) \to f(x)$

$$\forall \delta > 0 \quad \exists N : \quad |f_k(x) - f(x)| \leq \delta \quad \forall k \geq N$$

$$\varepsilon_N(x) = \sup_{k \geq N} |f_k(x) - f(x)| \leq \delta$$

$$\Rightarrow \forall \delta > 0 \quad \exists N : \quad \varepsilon_{N+1} \leq \varepsilon_N(x) \leq \delta$$

$$0 \leq \quad - - - - \leq \varepsilon_{N+3} \leq \varepsilon_{N+2} \leq$$

$$\Rightarrow \forall \delta > 0 \quad \exists N : \quad 0 \leq \varepsilon_n(x) \leq \delta \quad \forall n \geq N$$

$$\Leftrightarrow \varepsilon_n(x) \longrightarrow 0 \quad n \to +\infty.$$

$$\Longrightarrow \int_D \lim \varepsilon_n = \int_D 0 = 0$$

$$\Longrightarrow \lim \int_D \varepsilon_n = 0$$

So

$$\int_D \sup_{k \geq n} |f_k(x) - f(x)| \longrightarrow 0$$

$$\geq \qquad \geq$$

$$\int_D |f_n(x) - f(x)| \geq 0$$

$$\Longrightarrow \int_D |f_n - f| \longrightarrow 0 \quad \blacksquare$$

Example 1.7.6.

Compute $\lim_{n\to+\infty} \int_0^{+\infty} n^2\left(1-\cos\frac{x}{n}\right) e^{-\frac{n}{n+1}x}\, dx.$

Pb: $\lim_n \int_D f_n \overset{?}{=} \int_D \lim_n f_n$

1. $f_n \to f$ a.e.
2. $|f_n| \leqslant g \in L^1 \quad \forall n \geqslant n_0$

Here $f_n(x) := n^2\left(1-\cos\frac{x}{n}\right) e^{-\frac{n}{n+1}x}$

on $D = [0,+\infty[$. Clearly $f_n \in \mathscr{C}([0,+\infty[)$

$\Rightarrow f_n \in L([0,+\infty[) \quad \forall n \in \mathbb{N}$.

We have

1. $\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} n^2\left(1-\cos\frac{x}{n}\right)e^{-\frac{n}{n+1}x}$

$n^2 \to +\infty$, $1-\cos\frac{x}{n} \to 0$, $\cos\frac{x}{n} \to 1$, $\frac{x}{n} \to 0$, $\frac{n}{n+1} \to 1$, $e^{-\frac{n}{n+1}x} \to e^{-x}$

$$n^2\left(1-\cos\frac{x}{n}\right) = n^2\left(\cancel{1} - \left(\cancel{1} - \frac{x^2}{2n^2} + o\left(\frac{x^2}{n^2}\right)\right)\right)$$

$1-\cos t \qquad t\to 0$

$\cos t = 1 - \frac{t^2}{2} + o(t^2)$ | $\frac{1-\cos t}{t^2} \to \frac{1}{2}$

$$= n^2\left(\frac{x^2}{2n^2} + o\left(\frac{1}{n^2}\right)\right)$$

$$= \frac{x^2}{2} + n^2\, o\left(\frac{1}{n^2}\right) \longrightarrow \frac{x^2}{2}$$

$$\overset{\shortparallel}{o(1)} \to 0$$

$$\Rightarrow \lim_{n\to+\infty} f_n(x) = \frac{x^2}{2} e^{-x} \quad \forall x \in [0,+\infty[$$

$$\shortparallel$$

$$f(x)$$

$\frac{n}{n+1}$

$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

$$2. \quad |f_n(x)| = \Big(n^2 \underbrace{\left|1 - \cos\frac{x}{n}\right|}_{\leq 2} \Big) \cancel{e^{-\frac{n}{n+1}x}} \cancel{\geq e^{-x}} \leq 1$$

$$\leq |1| + \left|\cos\frac{x}{n}\right| \leq 1+1 = 2$$

$$\leq 2n^2 \notin L^1([0,+\infty[)$$

$$\left(n \geq 1 \qquad e^{-\frac{n}{n+1}x} \leq e^{-\frac{x}{2}} \quad n \geq 1 \right.$$

$\frac{n}{n+1} \geq \frac{1}{2}$

$$|f_n(x)| \leq 2n^2 e^{-x/2} \quad n \geq 1$$

$?$

$\in L^1([0,+\infty[)$

$$n^2\left(1 - \cos\frac{x}{n}\right) \qquad \left(= \left|n^2\left(1 - \cos\frac{x}{n}\right)\right| \right.$$

$$= n^2 \frac{1 - \cos\left(\frac{x}{n}\right)}{t^2/2} \cdot t^2/2$$

t

$$= \cancel{n^2} \boxed{\frac{1 - \cos\frac{x}{n}}{\left(\frac{x^2}{2n^2}\right)}} \cdot \frac{x^2}{2\cancel{n^2}} \overset{?}{\leq} C \frac{x^2}{2}$$

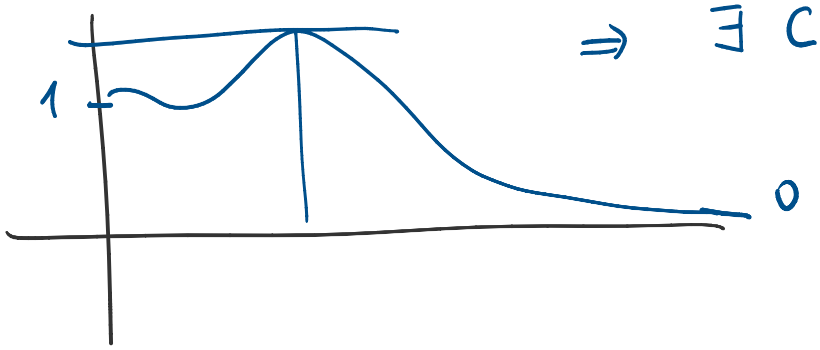
$$\varphi(t) = \frac{1 - \cos t}{\dots} \leq C$$

$$\varphi(t) = \frac{1-\cos t}{t^2/2} \overset{\cdot}{\leqslant} C$$

$[0,+\infty[\ni t = \frac{x}{n}$ $\qquad x \in [0,+\infty[$

$\varphi \in \mathcal{C}([0,+\infty[)$ $\qquad \left(\varphi(0) = \lim_{t\to 0} \varphi(t) = 1 \right)$

$\varphi(+\infty) = \lim_{t\to+\infty} \varphi(t) = 0$



$\Rightarrow \exists\, C : \quad \varphi(t) \leqslant C$

$t \in [0,+\infty[$.

$$\Rightarrow \quad n^2\left(1 - \cos\frac{x}{n}\right) \leqslant C x^2 \quad \forall x \in [0,+\infty[\quad \forall n$$

$$e^{-\frac{n}{n+1}x} \leqslant e^{-x/2} \quad \forall n \geqslant 1 \quad \forall x \in [0,+\infty[$$

$$|f_n(x)| \underset{f_n \geqslant 0}{=} f_n(x) \leqslant C x^2 e^{-x/2} = g(x)$$

$g \in L^1([0,+\infty[)$

$\Rightarrow$ int. dominant for (f_n)

$\Rightarrow$ by dom conv, we have

$$\lim_{n\to+\infty} \int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty} \lim_{n\to+\infty} f_n(x)\,dx$$

$$= \underbrace{\int_0^{+\infty} \frac{x^2}{2} e^{-x}}_{\text{abs int in gen sense}} dx$$

$$= \lim_{r\to+\infty} \frac{1}{2}\int_0^r x^2 \underbrace{e^{-x}}_{= (-e^{-x})'} dx$$

$$= \lim_{r\to+\infty} \frac{1}{2}\left[-e^{-x}x^2 \Big|_0^r - \int_0^r 2x \cdot (-e^{-x})\, dx \right]$$

$$\left[-e^{-r}r^2 - 2\int_0^r x \cdot (e^{-x})'\, dx \right]$$

$$= \lim_{r\to+\infty} \frac{1}{2}\left[\underbrace{-e^{-r}r^2}_{\to 0} - 2\left[xe^{-x}\Big|_0^r + \int_0^r \underbrace{-e^{-x}}_{= (e^{-x})'} dx \right]\right]$$

$$= \lim_{r\to+\infty} \frac{1}{2}\left[-2\left[\underbrace{re^{-r}}_{\to 0} + \underbrace{e^{-x}\Big|_0^r}_{= e^{-r}-1} \right]\right]$$

$$= -1 \cdot (e^{-r} - 1) = 1 .$$ $\square$

Ex 1.9.21

Do i), ii)

iii) $\displaystyle \lim_{n\to+\infty} \int_0^{+\infty} \frac{n}{x(1+x^2)} \sin\frac{x}{n}\, dx$

Sol: Here $f_n(x) = \dfrac{n}{x(1+x^2)} \sin\dfrac{x}{n} \in \mathcal{C}(]0,+\infty[)$

(a.e. cont on $[0,+\infty[$ $\Rightarrow$ $f_n \in L([0,+\infty[))$

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Let's see if we can apply the dom conv to say that

$$\lim_n \int_0^{+\infty} f_n = \int_0^{+\infty} \lim_n f_n .$$

First:

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{n}{x(1+x^2)} \sin\frac{x}{n}$$

$$= \frac{1}{1+x^2} \lim_{n\to+\infty} \frac{\sin x/n}{x/n} \quad (x/n \to 0)$$

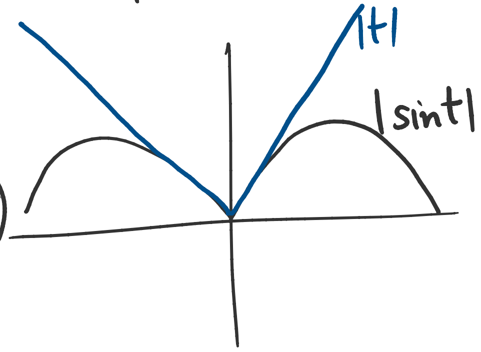
$$= \frac{1}{1+x^2} \cdot 1 \qquad \forall x>0$$

$$=: f(x)$$

Second: We look for an integrable dominant that is a $g \in L^1([0,+\infty[)$ s.t.

$$|f_n(x)| \leq g(x) \qquad \text{a.e. } x\in[0,+\infty[$$
$$\forall n \geq N.$$

$$|f_n(x)| = \frac{n}{x(1+x^2)} \left|\sin\frac{x}{n}\right| \qquad \underbrace{|\sin t|}_{\leq 1} \leq |t|$$



$$\underset{|\sin t|\leq 1}{\overset{x>0}{\leq}} \frac{n}{x(1+x^2)} \notin L^1([0,+\infty[)$$

$$\int^{+\infty} \frac{1}{x(1+x^2)}$$

$$\int_0^{+\infty} \frac{1}{x(1+x^2)}$$

$\sim_0 \ 1/x$ non int. at 0

$\sim_{+\infty} \ \dfrac{1}{x^3}$

$\exists \int^{+\infty} \frac{1}{x^\alpha} \iff \alpha > 1$

$\exists \int_0 \frac{1}{x^\alpha} \iff \alpha < 1$

$\nexists \int_0^{+\infty} \frac{1}{x^\alpha}$

$$|f_n(x)| \underset{|\sin t| \leq |t|}{\leq} \frac{\cancel{n}}{\cancel{x}(1+x^2)} \cancel{\left|\frac{x}{n}\right|} \qquad x>0$$

$$= \frac{1}{1+x^2} =: g(x) \in L^1([0,+\infty[)$$

$\Rightarrow$ we can apply dom conv and get

$$\lim_{n\to+\infty} \int_0^{+\infty} f_n = \int_0^{+\infty} \frac{1}{1+x^2}\,dx = \operatorname{arctg} x \Big|_0^{+\infty}$$

$$= \frac{\pi}{2}. \quad \blacksquare$$

Do 1.9.22

Corollary: $(f_n) \in L^1(D)$. Suppose that $\exists g \in L^1(D)$ s.t.

$$\sum |f_n| \leq g$$

Then

$$\int_D \sum_n f_n = \sum_n \int_D f_n$$

... applying dom conv to

Proof: Try to do applying dom conv to

$$S_n = \sum_{k=0}^{n} f_k .$$ ☒