Ex 1.9.21

i) $$\lim_{n \to +\infty} \int_n^{+\infty} \frac{e^{-n(x-n)}}{1+x^2}\, dx.$$

Sol: $\lim_n \int_D f_n \, dx$

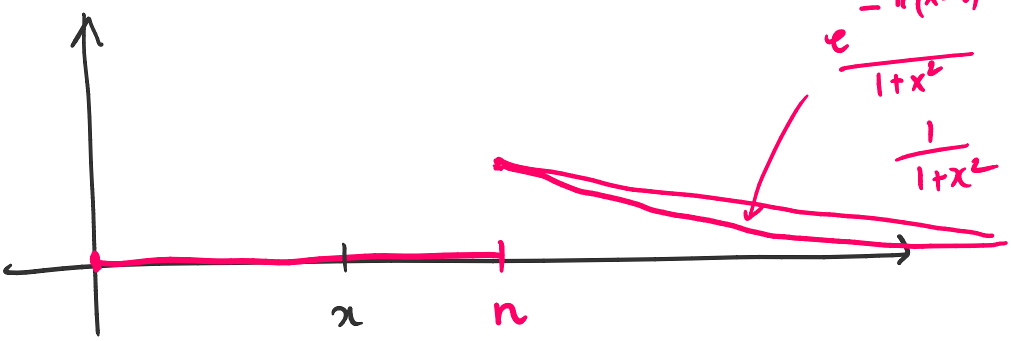
Rmk $$\int_n^{+\infty} \text{—} = \int_0^{+\infty} \text{—}\ \mathbb{1}_{[n,+\infty[}$$

$$= \int_0^{+\infty} \underbrace{\frac{e^{-n(x-n)}}{1+x^2} \mathbb{1}_{[n,+\infty[}}_{f_n(x)} \, dx$$

Let's try to apply the dominated conv.

We have

i) $\lim_{n \to +\infty} f_n(x)$



$f_n(x) = 0 \qquad \forall n \geq [x]+1$

$\Downarrow$

$$\Downarrow$$

$$\lim_n f_n(x) = 0 \qquad \forall x \in [0,+\infty[$$

$$\overset{\|}{f(x)}$$

ii) $|f_n(x)| \underset{f_n \geq 0}{=} f_n(x) \leq \frac{1}{1+x^2} = g(x) \quad \forall x \in [0,+\infty[\quad \forall n \in \mathbb{N}$

$$\left(f_n(x) = \frac{e^{-n(x-n)}}{1+x^2} \mathbb{1}_{[n,+\infty[} \right.$$

$$= \begin{cases} 0 \leq \frac{1}{1+x^2} & x < n \\ \frac{e^{\boxed{-n(x-n)}}}{1+x^2} \leq \frac{1}{1+x^2} & x \geq n \end{cases}$$

$e^{\ominus} \leq 1$

and because $g \in L^1([0,+\infty[) \Rightarrow g$ is an int dom for (f_n)

$\Rightarrow$ by dom conv

$$\lim_n \int_n^{+\infty} \text{———} = \lim_{n \to +\infty} \int_0^{+\infty} f_n(x)\,dx$$

$$= \int_0^{+\infty} \underset{\substack{\| \\ 0}}{\lim_n f_n(x)}\,dx = 0. \qquad \blacksquare$$

ii) $\lim \int_0^{+\infty} n\left(1+\frac{x}{n}\right)^{-n} \sin\frac{x}{n}\,dx$

ii) $\lim_n \int_0^{+\infty} \underbrace{n\left(1+\frac{x}{n}\right)^{-n} \sin \frac{x}{n}}_{f_n(x)} dx$

Is $f_n \in L^1([0,+\infty[)$ $n \geq 1$

$\left(n = 1 \quad \frac{1}{1+x} \sin x \notin L^1 \right)$

$n \geq 2 \quad f_n(x) = n\left(1+\frac{x}{n}\right)^{-n} \sin \frac{x}{n},$

$|f_n(x)| \leq n\left(1+\frac{x}{n}\right)^{-n} \cdot 1 \qquad x \in [0,+\infty[$

$n = 2 \qquad 2 \cdot \frac{1}{\left(1+\frac{x}{2}\right)^2} \sim_{+\infty} 2 \cdot \frac{1}{\frac{x^2}{4}} = \frac{C}{x^2}$ int at $+\infty$

$n \geq 2 \qquad \sim_{+\infty} \frac{C}{x^n}$ int at $+\infty$.

To discuss the limit, let's see if dom. conv conds are fulfilled here:

i) $\lim_n n \left(1+\frac{x}{n}\right)^{-n} \sin \frac{x}{n}$ $\quad n \to +\infty,\ \left(1+\frac{x}{n}\right)^{-n} \to e^{-x},\ \sin\frac{x}{n} \to 0$

$\| $

$\frac{1}{\left(1+\frac{x}{n}\right)^n}$ $\quad x$

$$\frac{1}{\left(1+\frac{x}{n}\right)^n} \to e^x$$

$$= \lim_n \underset{\downarrow \atop e^{-x}}{\left(e^{-x}\right)_n} \cdot x \frac{\sin\frac{x}{n}}{\frac{x}{n}} \nearrow 1 \quad = x e^{-x} =: f(x)$$

$\forall x \geq 0$

ii) Let's det an int dom:

$$|f_n(x)| = \underset{\uparrow}{n}\left(\left(1+\frac{x}{n}\right)^{-n}\right)\left|\sin\frac{x}{n}\right|$$

$$\left(1+\frac{x}{n}\right)^n \longrightarrow e^x \qquad n \to +\infty$$

$$\left(1+\frac{x}{n}\right)^n \nearrow e^x \qquad n \to +\infty \quad x>0$$

$$\frac{1}{\left(1+\frac{x}{n}\right)^n} \searrow e^{-x}$$

$$(1+x)^{-1} = \frac{1}{1+x} \swarrow \qquad \notin L^1$$

$$\geq$$

$$\left(1+\frac{x}{2}\right)^{-2} = \frac{1}{\left(1+\frac{x}{2}\right)^2} \qquad \in L^1$$

$$\geq$$

$$\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{\left(1+\frac{x}{3}\right)^3} \qquad \in L^1$$

$$\left(1+\frac{x}{3}\right)^{-3} = \frac{1}{\left(1+\frac{x}{3}\right)^3} \in L^1$$

$$\geq$$

$$\left(1+\frac{x}{4}\right)^{-4} = \frac{1}{\left(1+\frac{x}{4}\right)^4} \in L^1$$

$$n\left|\sin\frac{x}{n}\right| \leq n\left|\frac{x}{n}\right| \underset{x\geq 0}{=} \cancel{n}\,\frac{x}{\cancel{n}}$$

$$|\sin t| \leq |t|$$

So

$$|f_n(x)| \leq x\cdot\left(1+\frac{x}{n}\right)^{-n} \leq \frac{x}{1+x} \notin L^1([0,+\infty[)$$

$$\leq x\cdot\frac{1}{\left(1+\frac{x}{2}\right)^2} \sim_{+\infty} \frac{c}{x} \notin L^1$$

$$\leq x\cdot\frac{1}{\left(1+\frac{x}{3}\right)^3} =: g(x) \sim_{+\infty} \frac{c}{x^2} \in L^1$$

$$\forall n \geq 3$$

$\Rightarrow$ by dom conv

$$\lim\int^{+\infty} f_n = \int^{+\infty}\lim f_n = \int^{+\infty} x e^{-x}\,dx$$

$$\lim_n \int_0^{+\infty} f_n = \int_0 \lim_n f_n = \int_0 x \underbrace{e^{-x}}_{(-e^{-x})'} dx$$

$$= -x e^{-x} \Big|_0^{+\infty} - \int_0^{+\infty} 1 \cdot (-e^{-x})\, dx$$

$$= - \int_0^{+\infty} (e^{-x})'\, dx$$

$$= - e^{-x} \Big|_0^{+\infty} = 1 \qquad \blacksquare$$

Do 1.9.22 (similar).

Cont/Deriv dep by parameters.

An important pb is the following:

$$I(\lambda) := \int_D f(x, \lambda)\, dx$$

where $f : \underset{\mathbb{R}^d}{\overset{\cap}{D}} \times \underset{\mathbb{R}^k}{\overset{\cap}{\Lambda}} \longrightarrow \mathbb{R}$

If $\Lambda \subset \mathbb{R}^k$ is a "continuum" we wonder under which conds I depends cont. by λ regularly

Continuity: $\lim_{\lambda \to \lambda_0} I(\lambda) = I(\lambda_0)$

$$\lim_{\lambda \to \lambda_0} I(\lambda) = I(\lambda_0)$$
$$\parallel \qquad \Uparrow$$
$$\lim_{\lambda \to \lambda_0} \int_D f(x, \lambda)\, dx$$

Deriv. $\exists\, I'(\lambda)$? and how do we compute $I'(\lambda)$?

$$\partial_\lambda = \frac{d}{d\lambda}$$

$$\partial_\lambda \int_D f(x,\lambda)\, dx \overset{?}{=} \int_D \partial_\lambda f(x,\lambda)\, dx$$

Continuity:

Let $f : \underset{\mathbb{R}^d}{\overset{\cap}{D}} \times \underset{\mathbb{R}^k}{\overset{\cap}{\Lambda}} \longrightarrow \mathbb{R}$ be s.t.

$$f(\#, \lambda) \in L^1(D) \quad \forall \lambda \in \Lambda.$$

(is the funct $x \longmapsto f(x,\lambda)$ λ fixed)

Under this ass.

$$I(\lambda) := \int_D f(x,\lambda)\, dx \quad \text{is well def} \quad \forall \lambda \in \Lambda$$

$I : \Lambda \subset \mathbb{R}^k \longrightarrow \mathbb{R}$. We assume that

Λ be an open set in $\mathbb{R}^k$

$\lambda_0 \in \Lambda$, it makes sense to compute

$\lim\limits_{\lambda \to \lambda_0} I(\lambda)$. <u>Pb</u>: $= I(\lambda_0)$?

<u>Thm</u>: Let $f : D \times \Lambda \longrightarrow \mathbb{R}$ be s.t

1. $f(\#, \lambda) \in L^1(D) \quad \forall \lambda \in \Lambda$.
2. $f(x, \#) \in \mathcal{C}(\Lambda)$ a.e. $x \in D$
3. $|f(x,\lambda)| \leq g(x)$ a.e. $x \in D$ $\forall \lambda \in \Lambda$, $g \in L^1(D)$

Then

$$\lim_{\lambda \to \lambda_0} \int_D f(x,\lambda)\, dx = \int_D f(x, \lambda_0)\, dx$$

<u>Proof</u>: Recall that

$$\lim_{\lambda \to \lambda_0} I(\lambda) = I(\lambda_0) \iff \boxed{\forall (\lambda_n) \subset \Lambda \quad \lambda_n \to \lambda_0 \Rightarrow I(\lambda_n) \to I(\lambda_0).}$$

Set $(\lambda_n) \subset \Lambda$, $\lambda_n \to \lambda_0$.

$$I(\lambda_n) = \int_D \underbrace{f(x, \lambda_n)}_{f_n(x)} dx$$

$$\lim_{n \to +\infty} I(\lambda_n) = \lim_{n \to +\infty} \int_D f_n(x) dx$$

Now

by 2. $\lim_{n \to +\infty} f_n(x) = \lim_n f(x, \underset{\lambda_0}{\underset{\downarrow}{\lambda_n}}) \overset{2.}{=} f(x, \lambda_0)$

a.e. $x \in D$

by 3. $|f_n(x)| \left(= |f(x, \lambda_n)| \right) \leq g(x)$ $\forall n$ a.e $x \in D$

$\Rightarrow$ by dom conv

$$\Rightarrow \lim_n \int_D f_n(x) dx = \int_D \lim_n f_n(x) dx = \int_D f(x, \lambda_0) dx$$

$$\lim_n \int_D f_n(x)\,dx = \lim_n I(\lambda_n), \qquad \int_D f(x,\lambda_0)\,dx = I(\lambda_0)$$

$\boxtimes$

Now, let's cons the Pb of deriv of $I(\lambda)$

$$\frac{\partial}{\partial \lambda} I(\lambda) = \frac{\partial}{\partial \lambda} \int f(x, \lambda) dx$$

$$\exists\ \partial_\lambda I(\lambda) = \partial_\lambda \int_D f(x,\lambda)\,dx$$

$$\stackrel{?}{=} \int_D \partial_\lambda f(x,\lambda)\,dx$$

Thm: Let $f : D \times \Lambda \subset \mathbb{R}^d \times \mathbb{R}^k \longrightarrow \mathbb{R}$ be st.

1. $f(\#,\lambda) \in L^1(D)$ $\forall \lambda \in \Lambda$ ($I(\lambda)$ makes sense $\forall \lambda \in \Lambda$)

2. $\partial_\lambda f(x,\lambda)$ $\forall \lambda \in \Lambda$, a.e. $x \in D$

3. $\exists g \in L^1(D)$: $|\partial_\lambda f(x,\lambda)| \leq g(x)$ $\forall \lambda \in \Lambda$ a.e. $x \in D$

Then

$$\exists\ \partial_\lambda \int_D f(x,\lambda)\,dx = \int_D \partial_\lambda f(x,\lambda)\,dx.$$

$\ulcorner$ $$\partial_\lambda I(\lambda) = \lim_{h \to 0} \frac{I(\lambda+h) - I(\lambda)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\int_D f(x,\lambda+h) - \int_D f(x,\lambda) \right)$$

$$= \lim_{h \to 0} \int_D \frac{f(x,\lambda+h) - f(x,\lambda)}{h} \, dx$$

$$= \int_D \partial_\lambda f(x,\lambda) \, dx$$

$\lrcorner$

Example $\sigma \neq 0, \sigma \in \mathbb{R}$

$$I(\lambda) := \int_{\mathbb{R}} e^{-\frac{x^2}{2\sigma^2}} e^{-i2\pi\lambda x} \, dx$$

(Fourier transform of gaussian distribution)

Let's compute $I(\lambda)$.

Sol: $I(\lambda) = \int_{\mathbb{R}} f(x,\lambda) \, dx \qquad f(x,\lambda) = e^{-\frac{x^2}{2\sigma^2}} e^{-i2\pi\lambda x}$.

$\mathbb{R} \leftarrow D \qquad \Lambda = \mathbb{R}$

Let's first check that I is well def'd $\forall \lambda \in \mathbb{R}$

$\Leftrightarrow \quad f(\#,\lambda) \in L^1(\mathbb{R})$.

We check

$$\int_{\mathbb{R}} |f(x,\lambda)| \, dx < +\infty \qquad \forall \lambda \in \mathbb{R}$$

$$|f(x,\lambda)| = \left| e^{-x^2/2\sigma^2} e^{-i2\pi\lambda x} \right| = e^{-x^2/2\sigma^2} \cdot 1$$

$$|f(x,\lambda)| = \left| \underbrace{e^{-x^2/2\sigma^2}}\ \underbrace{e^{-i2\pi\lambda x}}_{|\quad|=1} \right| = e^{-x^2/2\sigma^2}\, 1$$

therefore $\displaystyle\int_{\mathbb{R}} |f(x,\lambda)|\, dx = \int_{\mathbb{R}} e^{-x^2/2\sigma^2} dx$

$$\underset{y = x/\sigma}{=} \int_{\mathbb{R}} e^{-y^2/2}\, \sigma\, dy$$

$$= \sigma\sqrt{2\pi}.$$

Let's see if we can apply the cont thm to deduce $I \in \mathcal{C}(\mathbb{R})$

1. true (checked above)

2. $f(x, \#) \in \mathcal{C}(\mathbb{R})$ a.e. $x \in \mathbb{R}$

True because $f(x,\lambda) = e^{-x^2/2\sigma^2}\, \underbrace{e^{-i2\pi\lambda x}}_{\in \mathcal{C}_\lambda}$

3. $|f(x,\lambda)| \leq g(x)$ $g \in L^1$ $\forall \lambda$, a.e. x

$|f(x,\lambda)| = e^{-x^2/2\sigma^2} =: g(x) \in L^1(\mathbb{R})$ $\forall \lambda \in \mathbb{R}$.

$\Rightarrow$ $I(\lambda)$ is cont funct of $\lambda \in \mathbb{R}$.

Let's pass to the deriv. thm.

1. already checked

2. $\exists\ \partial_\lambda f(x,\lambda) \quad \forall \lambda,$ a.e. x.

$$\partial_\lambda f(x,\lambda) = \partial_\lambda \left(e^{-x^2/2\sigma^2} e^{-i2\pi\lambda x} \right)$$

$$= e^{-x^2/2\sigma^2} \cdot (-i2\pi x)\, e^{-i2\pi\lambda x}$$

$\forall \lambda \in \mathbb{R} \quad \forall x \in \mathbb{R}$

3. $|\partial_\lambda f(x,\lambda)| \leq g(x) \qquad \forall \lambda \in \Lambda = \mathbb{R},$ a.e. $x \in D = \mathbb{R}$

$$|\partial_\lambda f(x,\lambda)| = 2\pi |x|\, e^{-x^2/2\sigma^2} =: g(x) \in L^1(\mathbb{R})$$

$\Rightarrow$

$$\exists\ I'(\lambda) = \int_{\mathbb{R}} \partial_\lambda f(x,\lambda)\, dx$$

$$= \int_{\mathbb{R}} e^{-x^2/2\sigma^2} (-i2\pi x)\, e^{-i2\pi\lambda x}\, dx$$

$$= -i2\pi(-\sigma^2) \int -\frac{x}{\sigma^2}\, e^{-x^2/2\sigma^2}\, e^{-i2\pi\lambda x}\, dx$$

$$= -i2\pi(-\sigma^2)\int_{\mathbb{R}} \underbrace{-\frac{x}{\sigma^2}\, e^{-x^2/2\sigma^2}}_{} \; e^{-i2\pi\lambda x}\, dx$$

$$\|$$

$$\partial_x\left(e^{-x^2/2\sigma^2}\right) = e^{-x^2/2\sigma^2}\cdot\left(\frac{-\cancel{2}x}{\cancel{2}\sigma^2}\right)$$

$$= +i2\pi\sigma^2\Big[\; \overbrace{e^{-x^2/2\sigma^2}\, e^{-i2\pi\lambda x}\Big|_{-\infty}^{+\infty}}^{= 0}$$

$$- \int_{\mathbb{R}} e^{-x^2/2\sigma^2}\; \partial_x\left(e^{-i2\pi\lambda x}\right) dx \Big]$$

$$= i2\pi\sigma^2\Big[-\int_{\mathbb{R}} e^{-x^2/2\sigma^2}\;(-i2\pi\lambda)\; e^{-i2\pi\lambda x}\, dx\Big]$$

$$= +(-1)\,4\pi^2\sigma^2\lambda\; \underbrace{\int_{\mathbb{R}} e^{-x^2/2\sigma^2}\; e^{-i2\pi\lambda x}\, dx}_{I(\lambda)}$$

$$\Rightarrow \quad \boxed{I'(\lambda) = \underbrace{-\,4\pi^2\sigma^2}_{}\,\lambda\, I(\lambda)}$$

$$I' = k\lambda I$$

$$\Rightarrow \log I(\lambda) = k\,\frac{\lambda^2}{2} + C$$

$$(\log I)' = \left(k\,\frac{\lambda^2}{2}\right)'$$

$$\Rightarrow I(\lambda) = C\, e^{-4\pi^2\sigma^2\frac{\lambda^2}{2}}$$

$$\Rightarrow I(\lambda) = C\, e^{-4\pi^2\sigma^2 \frac{\lambda^2}{2}}$$

$$I(\lambda) = C\, e^{-2\pi^2\sigma^2\lambda^2}$$

Finally $I(0) = \int_{\mathbb{R}} e^{-x^2/2\sigma^2} \underbrace{e^0}_{=1} dx = \sigma\sqrt{2\pi}$

$$I(0) = C$$

$$\Rightarrow \boxed{I(\lambda) = \sigma\sqrt{2\pi}\, e^{-2\pi^2\sigma^2\lambda^2}} \qquad \blacksquare$$

Do 1.9.23 → 26.

<u>1.9.23</u> $I(x) := \int_0^{+\infty} e^{-y} \frac{\sin(xy)}{y}\, dy$

Show that I is well def $\forall x \in \mathbb{R}$, compute I' deduce I.

<u>Sol:</u> $f(x,y) = e^{-y} \frac{\sin(xy)}{y}$

We have first to check $f(x,\#) \in L^1(\mathbb{R})$ $\forall x \in \mathbb{R}$.

$\sin t \sim_0 t$

$f(x,y) \sim_{y\to 0} e^{-y}\frac{x\cancel{y}}{\cancel{y}} = x e^{-y} \in \mathbb{R}$ no pb at $y=0$

$|f(x,y)| \leq e^{-y}$ ... $\forall x \in \mathbb{R}$

$$|f(x,y)| \leq \frac{e^{-y}}{y} \quad \text{int at } +\infty . \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow f(x,\#) \in L^1 \quad \forall x \in \mathbb{R}.$$

$$\partial_x f(x,y) = \frac{e^{-y}}{\cancel{y}} \cos(xy)\,\cancel{y} = e^{-y}\cos(xy)$$

$$\forall x \in \mathbb{R} \quad y \in]0,+\infty[$$

$$|\partial_x f(x,y)| = |e^{-y}\cos(xy)| \leq e^{-y} =: g(y) \in L^1([0,+\infty[)$$

$$\Rightarrow I'(x) = \int_0^{+\infty} e^{-y}\cos(xy)\,dy = \dots$$

$\boxtimes$