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Ex 1.9.24

$$I(x) := \int_0^{+\infty} \frac{e^{-xt} - e^{-t}}{t}\, dt$$

Show $I(x)$ is well defd $\forall x > 0$

$\exists\, I'(x)$ and compute it

Deduce $I(x)$

Sol: $f(t,x) := \dfrac{e^{-xt} - e^{-t}}{t} \in \mathcal{C}(]0,+\infty[)$

Let $x > 0$ be fixed and discuss $f(\#, x) \in L^1(]0,+\infty[)$

Notice that, as $t \to 0$, recalling

$$e^y = 1 + y + o(y) \qquad y \to 0$$

$$\Rightarrow \quad f(t,x) = \frac{\cancel{1} - xt + o(\cancel{x}t) - (\cancel{1} - t + o(t))}{t}$$

$$= (-x+1) + \frac{o(t)}{t} \nearrow 0 \to -x+1 \in \mathbb{R}$$

in part $f(t,x)$ could be extended at $t=0$

as a cont func $\Rightarrow \exists \int_0 f(t,x)\, dt$.

As $t \to +\infty$

As $t \to +\infty$

$$|f(t,x)| = \left| \frac{\overset{\leq 1}{e^{-xt}} - \overset{\leq 1}{e^{-t}}}{t} \right| \leq \frac{1}{t} \notin L^1(]0,+\infty[)$$

$$\leq \frac{e^{-xt} + e^{-t}}{t} \underset{t \geq 1}{\leq} e^{-\hat{x}t} + e^{-t} \in L^1(]0,+\infty[)$$

$\Rightarrow \exists \int^{+\infty} |f(t,x)|\, dt \quad \forall x > 0.$ $[1,+\infty[$

$\Rightarrow \exists \int_0^{+\infty} |f(t,x)|\, dt \quad \forall x > 0.$

II) Existence of

$$\partial_x I(x) = \partial_x \int_0^{+\infty} f(t,x)\, dt$$

$$\overset{?}{=} \int_0^{+\infty} \partial_x f(t,x)\, dt.$$

To show that this is possible we apply the derivability thm, provided the following conds hold true:

i) $f(\#, x) \in L^1(]0,+\infty[) \quad \forall x > 0$

(already checked)

$[0,+\infty[$

ii) $\exists\, \partial_x f(t,x) \qquad \forall x>0, \quad$ a.e. $t \in]0,+\infty[$

$$\partial_x \frac{e^{-xt}-e^{-t}}{t} = \frac{1}{t}\,\partial_x\left(e^{-xt} - e^{-t}\right)$$

$$\partial_x e^{-xt} = (-t)\,e^{-xt}$$

$$= \frac{1}{\cancel{t}}\left(-\cancel{t}\, e^{-xt}\right) = -\,e^{-xt}.$$

$\forall x>0 \qquad \forall t \in]0,+\infty[.$

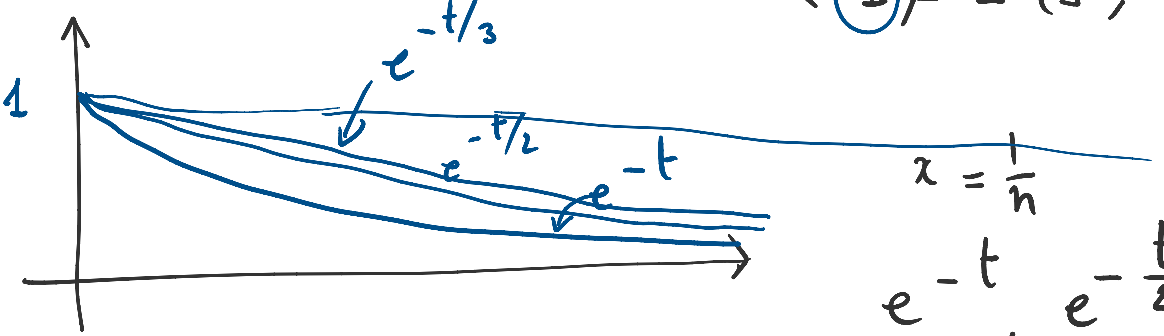
iii) $\exists\, g \in L^1 \qquad g = g(t) \quad$ s.t.

$|\partial_x f(t,x)| \leq g(t) \quad$ a.e. $t \in]0,+\infty[, \ \forall x>0.$

By ii) $\quad \partial_x f(t,x) = -\,e^{-xt}$

$$\Rightarrow \quad |\partial_x f(t,x)| = \boxed{e^{-xt} \overset{?}{\leq} g(t)} \qquad \begin{matrix}\forall x>0\\ t>0\end{matrix}$$

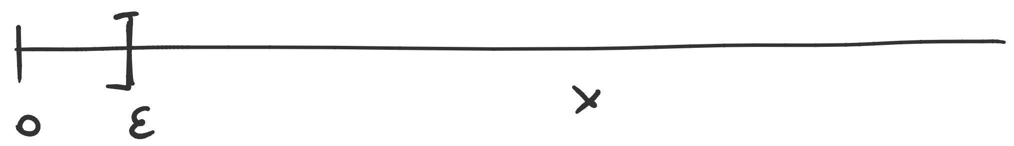
$$\leq 1 \notin L^1(]0,+\infty[)$$



$x = \frac{1}{n}$

$e^{-t}, \quad e^{-\frac{t}{2}}, \quad e^{-t/3}$

Consider, instead of $x>0$, $\underline{x>\varepsilon}>0$



and apply the deriv. thm on $\Lambda =]\varepsilon, +\infty[$

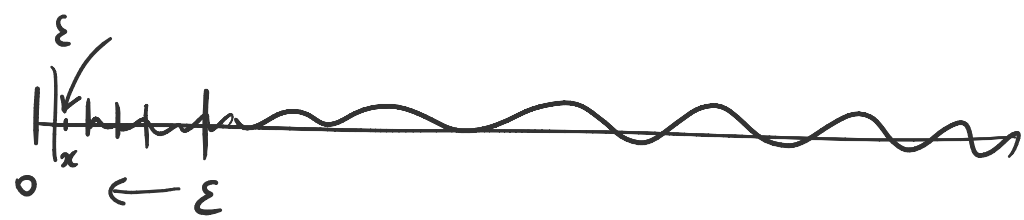
i) $f(t,x) \in L^1(]0,+\infty[) \quad \forall x>\varepsilon$

ii) $\exists \partial_x f(t,x) = -e^{-xt} \quad \forall x>\varepsilon, \quad$ a.e. $t \in]0,+\infty[$

iii) $|\partial_x f(t,x)| = e^{-\hat{x}^{\varepsilon} t} \leq e^{-\varepsilon t} =: g(t) \in L^1(]0,+\infty[)$

a.e. $t \in]0,+\infty[\quad \forall \boxed{x>\varepsilon}$

$$\Rightarrow \forall x>\varepsilon \quad \exists I'(x) = \int_0^{+\infty} \partial_x f(t,x)\,dt$$



and because $\varepsilon>0$ is free $\Rightarrow$

$$\exists I'(x) = \int_0^{+\infty} \partial_x f(t,x)\,dt \quad \forall x>0.$$

$$= \frac{1}{x}\int_0^{+\infty} -x e^{-xt}\,dt$$

||

$$\partial_t\left(e^{-xt}\right)$$

$$= \frac{1}{x} \underbrace{e^{-xt}\Big|_0^{+\infty}}_{=-1} = -\frac{1}{x}$$

$$\Rightarrow \quad I'(x) = -1/x \qquad \forall\, x>0$$

III) $I(x) = -\log x + c \qquad \forall\, x>0$

$$\left(c = ? \right.$$

$$= \int_0^{+\infty} \frac{e^{-xt} - e^{-t}}{t}\, dt$$

$$I(1) = \int_0^{+\infty} \frac{e^{-t} - e^{-t}}{t}\, dt = 0$$

$$\| $$

$$-\underbrace{\log 1}_{=0} + c \quad \Rightarrow \quad c = 0 \quad \Rightarrow \quad I(x) = -\log x \quad \blacksquare$$

<u>1.9.25</u> $\quad I(\lambda) = \int_0^1 \frac{x^\lambda - 1}{\log x}\, dx$

I is well defd $\forall\, \lambda \geq 0$

$\exists\, \partial_\lambda I \quad \lambda > 0 \quad (\lambda \geq 0)$

Det I.

I) $f(x,\lambda) = \dfrac{x^\lambda - 1}{\log x} \in \mathcal{C}(]0, 1[)$

$\log x$

When $x \to 0+$ $\qquad \lambda \geq 0$

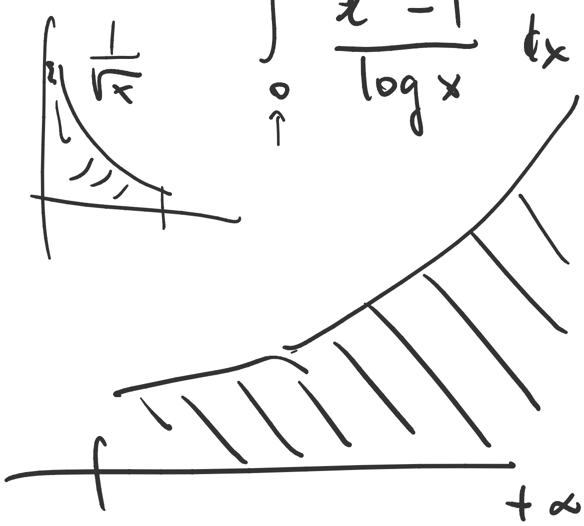
$$f(x,\lambda) \approx \frac{-1}{\log x} \longrightarrow 0$$

$\lambda < 0$

$$\frac{\infty}{\infty}$$

$$\lim_{x\to 0+} \frac{x^{\lambda}-1}{\log x} \underset{\substack{y=\log x\\ x=e^y}}{=} \lim_{y\to -\infty} \frac{e^{\lambda y}-1}{y}$$

$$\to -\infty$$



$$\int_0 \frac{x^{\lambda}-1}{\log x}\,dx \underset{y=\log x}{=} \int_{-\infty} \frac{e^{\lambda y}-1}{y}\,dy$$

$$\underset{z=-y}{=} -\int^{+\infty} \frac{e^{(-\lambda)z}-1}{z}\,dz$$

$+\infty$

$+\infty$

$$= -\infty.$$

So $\exists \int_0 f(x,\lambda)\,dx \quad \forall \lambda \geq 0$

At $x = 1$: $\quad f(x,\lambda) = \dfrac{x^{\lambda}-1}{\log x}$

$$= \frac{e^{(\lambda \log x)} - 1}{\log(1 + \underbrace{x-1}_{\downarrow 0})} \sim x - 1$$

$\log(1+t) \sim t$, $t \to 0$

$$\frac{\log(1+t)}{t} \to 1$$

$$e^t = 1 + t + o(t) \qquad e^{\lambda \log x} = x^\lambda$$

$$= \frac{\overbrace{\cancel{1} + \lambda \cancel{\log x} + o(\log x)} - \cancel{1}}{\cancel{\log x}}$$

$$= \lambda + \frac{o(\log x)}{\log x} \to \lambda$$

$$\Rightarrow \frac{x^\lambda - 1}{\log x} \to \lambda \qquad x \to 1 \qquad \forall \lambda \in \mathbb{R}$$

$$\exists \int^1 \frac{x^\lambda - 1}{\log x}\, dx \qquad \forall \lambda \in \mathbb{R}.$$

$$\Rightarrow \exists \underbrace{\int_0^1 \frac{x^\lambda - 1}{\log x}\, dx}_{I(\lambda)\,!!} \iff \lambda \geq 0.$$

II) To compute $\partial_\lambda I$ we apply the deriv thm:

i) $f(t, \lambda) \in L^1([0,1])$ $\forall \lambda \geq 0$ (checked)

i) $f(\#, \lambda) \in L^1([0,1]) \quad \forall \lambda \geq 0$ (checked)

ii) $$\partial_\lambda f(x,\lambda) = \partial_\lambda \frac{x^\lambda - 1}{\log x}$$

$$= \frac{1}{\log x} \partial_\lambda (\underbrace{x^\lambda}_{= e^{\lambda \log x}})$$

$$= \frac{1}{\cancel{\log x}} (\cancel{\log x})\, e^{\lambda \log x} = x^\lambda .$$

$$\forall x \in]0,1[, \quad \forall \lambda \geq 0$$

iii) $\exists g = g(x) \in L^1([0,1]) : \quad |\partial_\lambda f(x,\lambda)| \leq g(x)$ a.e. $x \in [0,1]$, $\lambda \geq 0$

Because $|\partial_\lambda f(x,\lambda)|$

$$= |x^\lambda| \underset{x \in]0,1[}{=} x^\lambda \leq \boxed{1} =: g(x) \in L^1([0,1])$$

$$\Rightarrow \partial_\lambda I(\lambda) = \int_0^1 \partial_\lambda f(x,\lambda)\, dx$$

$$= \int_0^1 x^\lambda \, dx \qquad \lambda \geq 0$$

$$= \frac{x^{\lambda+1}}{\quad} \Big|_0^1 \quad \ldots$$

$$= \frac{x^{\lambda+1}}{\lambda+1}\Big|_0^1 = \frac{1}{\lambda+1} \qquad (\lambda \geq 0)$$

$$\Rightarrow \partial_\lambda I = \frac{1}{\lambda+1} \qquad \forall \lambda \geq 0$$

$$\Rightarrow I(\lambda) = \log(\lambda+1) + C$$

$$\left.\int_0^1 \frac{x^\lambda - 1}{\log x}\,dx\,\right|_{\lambda=0} = \int_0^1 0 = 0$$

$$\Rightarrow \log 1 + C = 0 \quad \Rightarrow C = 0. \qquad \blacksquare$$

Abstract Measure and Integral

The Leb meas / int is the starting point for a general def of meas and int.

A measure is a function defd on a family of sets

$$\mu : \mathcal{F} \subset P(X) \longrightarrow [0, +\infty]$$

fulfilling certain conds.

<u>Def:</u> (σ - algebra of sets)

... $P(X)$ be the

Def: (σ - algebra of sets)

Let X be a generic set, $\mathcal{P}(X)$ be the family of all the subsets of X (parts of X)

A family $\mathcal{F} \subset \mathcal{P}(X)$ is called a <u>σ-algebra</u> if

i) $\emptyset, X \in \mathcal{F}$

ii) $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$

iii) $(E_n)_{n\in\mathbb{N}} \subset \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{F}$

<u>Ex 1:</u> • The Lebesgue class $\mathcal{M}_d$ is a σ-alg.

• $\mathcal{F} = \mathcal{P}(X)$. is trivially a σ-alg. of sets.

• $\mathcal{F} = \{\emptyset, X\}$ is a σ-alg.

• $X = \{a, b, c\}$

$\mathcal{F} = \{\emptyset, X, \{a\}, \{b, c\}, \ldots \}$

$\mathcal{F} = \{\emptyset, X, \{a\}, \{a,b\}, \{b,c\}, \{c\}, \{b\}, \{a,c\}\}$

<u>Rmk:</u> A σ-alg must be always closed resp. to set operations:

$\boxed{(E_n) \subset \mathcal{F}} \Rightarrow \bigcap_n E_n \in \mathcal{F}$

iii)

$$\left(\bigcap_n E_n\right)^c = \bigcup_n \underbrace{E_n^c}_{\in \mathcal{F}\ ii)} \overset{iii)}{\in} \mathcal{F}$$

$\Downarrow$

$$\left(\bigcap_n E_n\right)^c \in \mathcal{F} \underset{ii)}{\Rightarrow} \bigcap_n E_n \in \mathcal{F}$$

$E, F \in \mathcal{F}$, $\quad E \setminus F = E \cap F^c \in \mathcal{F}$

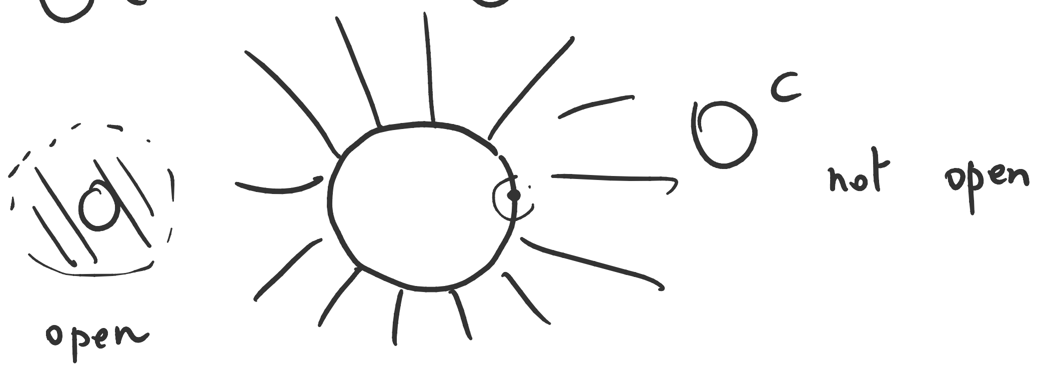
$\Rightarrow E \triangle F \in \mathcal{F}$

$$\mathcal{F} = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c\}, \{a,c\}, \{b,c\}\} = \mathcal{P}(X)$$

<u>Ex:</u> $\mathcal{F} = \{O \subset \mathbb{R}^d : O \text{ is open}\} \ni \emptyset, \mathbb{R}^d$

Is $\mathcal{F}$ a σ-alg? No!

ii) $O \in \mathcal{F} \Rightarrow O^c \in \mathcal{F}$?



iii) if $(O_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ (each O_n is open)

$\Rightarrow \bigcup_n O_n$ is open $(\Rightarrow \in \mathcal{F})$ $\boxtimes$

In general is not easy to give direct def's of a σ-alg.

<u>Def:</u> Let $\mathcal{S} \subset P(X)$ be a family of subsets of X. (not nec. a σ-alg). We call σ-alg generated by $\mathcal{S}$ the <u>minimal</u> σ-alg $\mathcal{F}$ containing $\mathcal{S}$:

$$\mathcal{F} := \bigcap_{\mathcal{G}\,:\,\sigma\text{-alg} \supset \mathcal{S}} \mathcal{G}$$

$$\mathcal{F} = \sigma(\mathcal{S})$$

(σ-alg gen. by $\mathcal{S}$)

<u>Prop:</u> <u>$\sigma(\mathcal{S})$ is a σ-alg</u>, it contains $\mathcal{S}$ and if $\mathcal{G}$ is σ-alg $\supset \mathcal{S} \Rightarrow \mathcal{G} \supset \sigma(\mathcal{S})$

<u>Ex:</u> $\sigma(\text{rectangles in } \mathbb{R}^d) = \mathcal{M}_d$.

<u>Ex:</u> $X = \{a,b,c,d\}$

$\mathcal{S} = \{\{a\},\{b\}\}$

$\sigma(\mathcal{S}) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{b,c,d\},$

[illegible]

$$\sigma(S) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{b,c,d\},$$
$$\{a,c,d\}\}$$

∎