martedì 10 dicembre 2019 12:31

Exam Sim

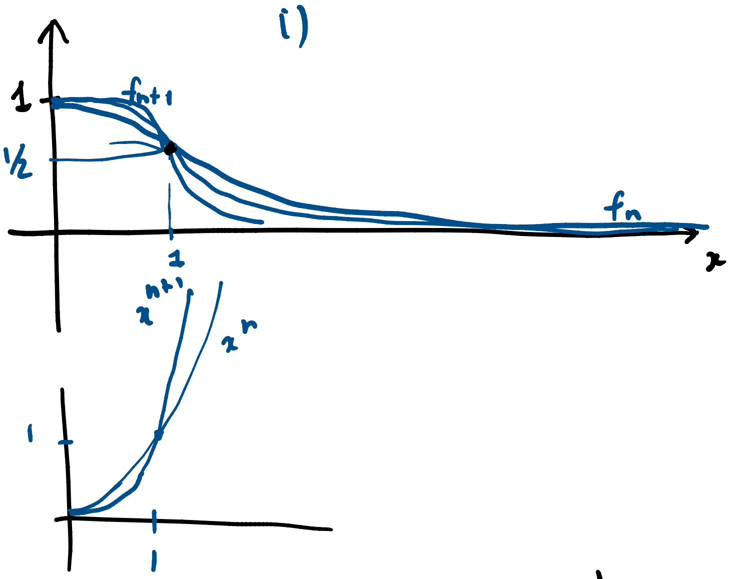
Ex 1 $\quad f_n(x) = \dfrac{1}{1+x^n}, \quad x \in [0,+\infty[\quad n \in \mathbb{N}, \ n \geq 2$

i) Plot f_n

ii) $f_n \in L^1([0,+\infty[)$

iii) (f_n) conv (to what?) in $L^1([0,+\infty[)$?

i)



$f_n \searrow$

$f_n \geq 0$

$f_n(0) = 1$

$f_n(+\infty) = 0$

$x = 0$ max for f_n

ii) $f_n \in L^1 \iff \displaystyle\int_0^{+\infty} |f_n| < +\infty$

$$\parallel$$

$$\int_0^{+\infty} \frac{1}{1+x^n} \, dx$$

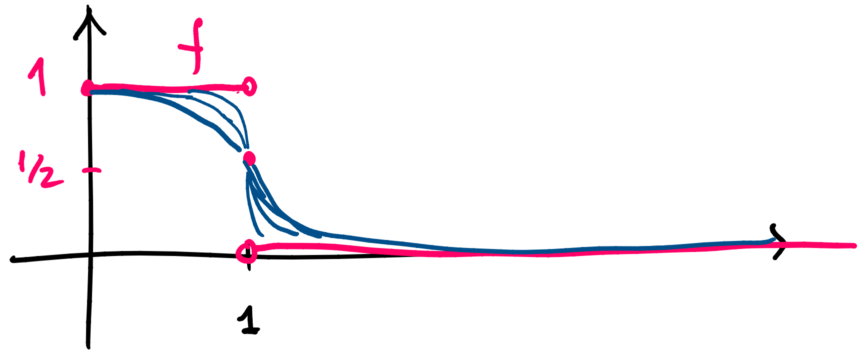
Because $f_n(x) = \dfrac{1}{1+x^n} \underset{+\infty}{\sim} \dfrac{1}{x^n}$ and $n \geq 2$

int at $+\infty$

$$\Rightarrow \int_0^{+\infty} f_n < +\infty.$$

iii) $(f_n) \subset L^1([0,1])$ is conv and to what?

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{1}{1+\underbrace{x^n}} = \begin{cases} 1 & 0 \leq x < 1 \quad x^n \to 0 \\ 1/2 & x = 1 \\ 0 & x > 1 \quad x^n \to +\infty \end{cases}$$



Guess: $f_n \xrightarrow{L^1} f$ To check this we have to prove

$$\|f_n - f\|_1 \to 0$$

$$\|f_n - f\|_1 = \int_0^{+\infty} |f_n - f|\, dx = \int_0^1 \underbrace{|f_n - 1|}_{1-f_n} + \int_1^{+\infty} \underbrace{|f_n - 0|}_{= f_n}$$

$$= \int_0^1 \left(1 - \frac{1}{1+x^n}\right) dx + \int_1^{+\infty} \frac{1}{1+x^n}\, dx$$

$$= \int_0^1 \frac{x^n}{1+x^n}\, dx + \int_1^{+\infty} \frac{1}{1+x^n}\, dx \quad (*)$$

Pb: Compute lim of

Pb: Compute $\overline{\lim_{n\to+\infty}}$ of

$$0 \leq \int_0^1 \frac{x^n}{\underbrace{1+x^n}_{\geq 1}}\,dx \leq \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \to 0$$

another way:

$$\lim_n \int_0^1 \overset{= g_n(x)}{\underset{1-f_n}{\frac{x^n}{1+x^n}}} \overset{?}{=} \int_0^1 \lim_n \frac{x^n}{1+x^n} = \int_0^1 0 = 0$$

$0 \leq x < 1$: $\frac{x^n}{1+x^n} \to 0$

$f_n < f_{n+1}$ on $[0,1]$

$\Downarrow$

$1-f_n > 1-f_{n+1} > 0$ we can apply monot conv provided

$1-f_2 \in L^1([0,1])$

true.

in alternative we could use dom conv:

i) $\frac{x^n}{1+x^n} \longrightarrow 0$ $\forall\, 0 \leq x < 1$

(a.e. $x \in [0,1]$)

ii) $|g_n| = \frac{x^n}{1+x^n} \leq 1 \in L^1([0,1])$

$\rightarrow$ concl. follows.

About $\displaystyle\int_1^{+\infty} \frac{1}{1+x^n}\,dx$

$f_n = \dfrac{1}{1+x^n} \searrow^n \qquad 0 \leq f_{n+1} \leq f_n \leq \ldots \quad f_2 \in L^1$

$\Rightarrow$ by monot conv for decr. seqs.

$$\lim_n \int_1^{+\infty} \frac{1}{1+x^n} = \int_1^{+\infty} \lim_n \frac{1}{1+x^n_{\searrow +\infty}} = \int_1^{+\infty} 0 = 0$$

Alternative: by dom conv

i) $f_n(x) = \dfrac{1}{1+x^n} \longrightarrow 0 \qquad 1 < x < +\infty$

ii) $|f_n(x)| \leq \dfrac{1}{1+x^2} =: g \in L^1([1,+\infty[)$

$\Rightarrow$ concl follows.

Conclusion: $f_n \xrightarrow{L^1} f = 1_{[0,1]}$. ☒

Ex 2: $H = L^2([0,\pi])$. Solve

$$\operatorname*{arg\,min}_{\boxed{a,b} \in \mathbb{R}} \| x - \underbrace{(a\cos x + b\sin x)}_{} \|_2$$

Sol: Let

$U = \text{Span} \langle \cos x, \sin x \rangle$

(rmk: cos, sin are lin indep vectors because

$$a \cos x + b \sin x = 0 \quad \text{a.e.} \quad x \in [0, \pi]$$

$$\Downarrow$$

$\cos x \propto \sin x$ impossible.)

In part U has dim $= 2 \Rightarrow U$ is closed.

Sol to the pb consists in finding the el of U at min dist to $f(x) = x$. Optimal a^*, b^* are such that

$$a^* \cos x + b^* \sin x = \Pi_U f$$

To build an orthonormal base for U we start with $\cos x, \sin x$ and apply the Gram-Schmidt orthogonalization:

$$e_0 = \frac{\cos}{\|\cos\|_2}$$

$$\|\cos\|_2^2 = \int_0^{\pi} |\cos x|^2 \, dx = \int_0^{\pi} (\cos x)^2 \, dx$$

$\| \cos \|_2$

$$= \int_0^\pi (\cos x)(\sin x)' = \underbrace{[\cos x \sin x]_0^\pi}_{0} + \int_0^\pi (\sin x)^2$$

$$= \int_0^\pi (\sin x)^2 = \int_0^\pi 1-(\cos x)^2 \, dx$$

$$= \pi - \int_0^\pi (\cos x)^2 \, dx$$

$$\cancel{2} \int_0^\pi (\cos x)^2 = \pi/2$$

$$e_0 = \frac{\cos}{\sqrt{\frac{\pi}{2}}} = \sqrt{\frac{2}{\pi}} \cos .$$

$$e_1 = \frac{\sin - \overbrace{\langle \sin, e_0 \rangle}^{0} e_0}{\| \quad \|_2}$$

$$\langle \sin, e_0 \rangle = \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^\pi \underbrace{2 \sin x \cos x}_{\sin(2x)} \, dx$$

$$= \frac{1}{4} \sqrt{\frac{2}{\pi}} \underbrace{\int_0^{2\pi} \sin t \, dt}_{= 0}$$

$$= \frac{\sin}{\| \quad \|_2}$$

$$e_1 = \sqrt{\frac{2}{\pi}} \sin$$

$$\Rightarrow \quad \Pi_U f = \langle f, e_0 \rangle e_0 + \langle f, e_1 \rangle e_1$$

$$\Rightarrow \quad \Pi_U x = \underbrace{\cancel{\sqrt{\tfrac{2}{\pi}}}\left(\int_0^\pi t \cos t \, dt\right)}_{\substack{\| \\ a^*}} (\cos x) + \underbrace{\frac{2}{\pi}\left(\int_0^\pi t \sin t \, dt\right)}_{b^*} \sin x$$

$\boxtimes$

<u>Ex 3</u>

$$F(t) = \int_0^{+\infty} \underbrace{e^{-tx} \frac{1 - \cos x}{x}} \, dx$$

i) Domain of F

ii) Is F cont on its domain?

iii) $\partial_t F$?

iv) det F

<u>Sol</u>: i) $\text{Domain}(F) = \{ t \in \mathbb{R} : F(t) \text{ makes sense} \}$

$$\Updownarrow$$

$$e^{-t\#} \frac{1 - \cos \#}{\#} \in L^1([0, +\infty[)$$

$$\Updownarrow$$

$$\int_0^{+\infty} \left| e^{-tx} \frac{1 - \cos x}{x} \right| dx < +\infty$$

$$\Updownarrow$$

$$\int_0^{+\infty} e^{-tx} \frac{1 - \cos x}{x} dx < +\infty .$$

$$\int_0 \underbrace{e \quad \frac{1-\cos x}{x}}_{f(t,x).} dx < +\infty .$$

$$\frac{1-\cos x}{x^2} \to 1/2$$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

Notice that

$$f(t,x) = \underset{\swarrow 1}{e^{-tx \nearrow 0}} \quad \frac{1-\cos x \nearrow 0}{x \swarrow 0}$$

$$= \underset{\substack{\uparrow \\ \text{something going to } 1}}{1_x} \quad \frac{x^2/2 + o(x^2)}{x} = 1_x \left(\frac{x}{2} + o(x) \right) \underset{x \to 0}{\longrightarrow} 0$$

$\Rightarrow f(t, \#)$ is int at $x = 0$ $\forall t \in \mathbb{R}$

At $\underline{\underline{+\infty}}$

$$0 \leq f(t,x) \leq e^{-tx} \frac{2}{x} \underset{x>1}{\leq} 2e^{-tx}$$

int at $+\infty$ $\underline{t > 0}$

What if $t = 0$ or $t < 0$?

$t = 0$ $\quad f(0,x) = \dfrac{1-\cos x}{x}$ is integrable.

$$\int_{.}^{+\infty} \frac{1-\cos x}{x} = \int^{+\infty} \frac{1}{x} - \frac{\cos x}{x}$$

$$\int_1 \frac{1-\cos x}{x} = \int_1 \frac{1}{x} - \frac{\cos x}{x}$$

$$\int_1^{+\infty} \frac{\cos x}{x} = \int_1^{+\infty} \frac{1}{x} (\sin x)'$$

$$= \frac{1}{x}\sin x \Big|_1^{+\infty} + \int_1^{+\infty} \frac{1}{x^2} \sin x \, dx$$

$$= -\sin 1 + \int_1^{+\infty} \frac{1}{x^2} \sin x \, dx$$

$$\left|\frac{1}{x^2}\sin x\right| \leq \frac{1}{x^2} \quad \text{int.}$$

so if $\boxed{\exists \int_1^{+\infty} \frac{1-\cos x}{x} \in \mathbb{R}}$ +

$$\int_1^{+\infty} \frac{\cos x}{x} \in \mathbb{R}$$

$$\int_1^{+\infty} \frac{1}{x}\, dx \in \mathbb{R} \quad \text{(false!)}$$

If $t<0$ $\quad f(t,x) = \underbrace{e^{-tx}}_{\substack{x\to+\infty \\ \downarrow \\ +\infty}} \frac{1-\cos x}{x}$ situation is worse.

$D(F) =]0,+\infty[.$

ii) F cont on $D =]0,+\infty[$?

Because $\quad F(t) = \int_0^{+\infty} f(t,x)\,dx$

we apply the cont thm: recall that if

- $f(t,\#) \in L^1([0,+\infty[) \quad \forall\, t \in \Lambda$
- $f(\#,x) \in \mathcal{C}(\Lambda) \quad$ a.e. $x \in [0,+\infty[$
- $|f(t,x)| \leq g(x) \quad g \in L^1([0,+\infty[) \ \forall t \in \Lambda$

$\Rightarrow F \in \mathcal{C}(\Lambda)$

Take $\Lambda = D$. First cond automatically fulfilled

It is clear that $\quad f(\#,x) \in \mathcal{C}(]0,+\infty[)$

a.e. $x \in]0,+\infty[$.

Finally:

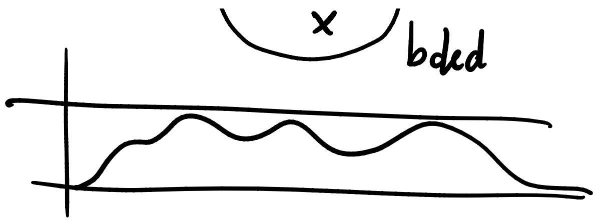
$$|f(t,x)| = f(t,x) = e^{-tx}\,\frac{1-\cos x}{x}$$

$$\leq \frac{1-\cos x}{x} \notin L^1$$

Modify $\Lambda = [t_0,+\infty[\quad t_0 > 0$ fixed

If $t \in \Lambda \quad (t \geq t_0)$

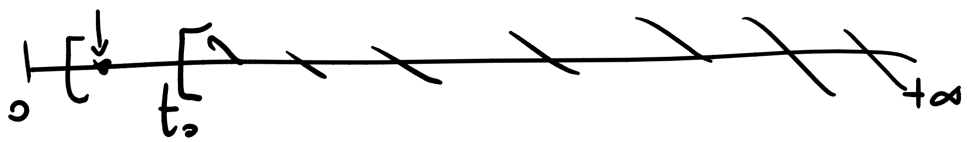
$$|f(t,x)| \leq e^{-t_0 x}\,\underbrace{\frac{1-\cos x}{x}}_{\text{bded}} \qquad x > 0$$



$\leq C e^{-t_0 x} \in L^1([0,+\infty[)$

$\forall t \in [t_0, +\infty[\quad$ a.e. $\quad x \in [0,+\infty[$

$\Rightarrow F \in \mathcal{C}([t_0,+\infty[) \quad \forall t_0 > 0 \Rightarrow$



$\Rightarrow F \in \mathcal{C}(]0,+\infty[)$.

iii) We want to apply diffentiation thm

$$\partial_t F(t) = \partial_t \int_0^{+\infty} f(t,x)\,dx = \int_0^{+\infty} \partial_t f(t,x)\,dx.$$

To do this we need

- $f(t,\#) \in L^1([0,+\infty[) \quad \forall t \in \Lambda$
- $\exists\, \partial_t f(t,x) \quad \forall t \in \Lambda,$ a.e. $x \in [0,+\infty[$
- $|\partial_t f(t,x)| \leq g(x) \in L^1([0,+\infty[) \quad \forall t \in \Lambda$

  a.e. $x \in [0,+\infty[$

First cond: checked above $\Lambda =]0,+\infty[$

First cond: checked above $\Lambda =]0, +\infty[$

Second cond: $\partial_t f(t,x) = \partial_t \left(e^{-tx}\right) \frac{1-\cos x}{x}$

$$= -x e^{-tx} \frac{1-\cos x}{x}$$

$$= -e^{-tx}(1-\cos x)$$

$\forall t \in \Lambda =]0, +\infty[$ a.e. $x \in [0, +\infty[$

<u>Third:</u> $|\partial_t f(t,x)| = e^{-tx}|1-\cos x|$

$$\leq 2e^{-tx} \leq 2e^{-t_0 x} \in L^1([0,+\infty[)$$

$\Lambda \Longleftrightarrow [t_0, +\infty[\quad t_0 > 0 \qquad \forall t_0 > 0$

$\Rightarrow \quad \partial_t F = \int_0^{+\infty} -e^{-tx}(1-\cos x)\, dx \qquad \forall t \in [t_0, +\infty[$

$\forall t_0 > 0$

$\Updownarrow$

$\forall t > 0$

$$= + \int_0^{+\infty} \underbrace{(-e^{-tx})}_{\partial_x\left(\frac{e^{-tx}}{t}\right)} dx + \int_0^{+\infty} e^{-tx} \cos x\, dx$$

$$= \left[\frac{e^{-tx}}{t} \right]^{x=+\infty} + \int_0^{+\infty} e^{-tx} \cos x\, dx$$

$$= \left[\frac{e^{-tx}}{t} \right]_{x=0} + \int_0 e^{-tx} \cos x \, dx$$

$$\| $$

$$- \frac{1}{t}$$

$$\int_0^{+\infty} \underset{\substack{\| \\ \partial_x\left(\frac{e^{-tx}}{-t}\right)}}{e^{-tx}} \cos x \, dx = \left[\frac{e^{-tx}}{-t} \cos x \right]_{x=0}^{x=+\infty} - \frac{1}{t} \int_0^{+\infty} e^{-tx} \sin x dx$$

$$= + \frac{1}{t} - \frac{1}{t} \left[\left(\frac{e^{-tx}}{-t} \right) \sin x \Big|_{x=0}^{x=+\infty} \nearrow 0 + \frac{1}{t} \int_0^{+\infty} e^{-tx} \cos x \, dx \right]$$

$$= \frac{1}{t} - \frac{1}{t^2} \int_0^{+\infty} e^{-tx} \cos x \, dx$$

$$\Rightarrow \left(1 + \frac{1}{t^2} \right) \int_0^{+\infty} e^{-tx} \cos x \, dx = \frac{1}{t}$$

$$\Rightarrow \int_0^{+\infty} e^{-tx} \cos x \, dx = \frac{t^2}{t^2+1} \cdot \frac{1}{t} = \frac{t}{t^2+1}$$

$$\partial_t F(t) = -\frac{1}{t} + \frac{t}{t^2+1} .$$

iv) $F(t) = \int -\frac{1}{t} + \frac{1}{2} \frac{2t}{t^2+1} \, dt + c$

$$\underset{t>0}{=} -\log t + \frac{1}{2}\log(t^2+1) + c$$

$$= \log \frac{\sqrt{t^2+1}}{t} + c$$

$$F(t) = \log\sqrt{1+\frac{1}{t^2}} + c$$

$$\| $$

$$\int_0^{+\infty} e^{-tx}\,\frac{1-\cos x}{x}\,dx$$

$$\lim_{t\to+\infty} F(t) \overset{?}{=} c$$

$$\|$$

$$\lim_{t\to+\infty}\int_0^{+\infty} e^{-tx}\,\frac{1-\cos x}{x}\,dx \overset{?}{=} \int_0^{+\infty} \underbrace{\lim_{t\to+\infty} e^{-tx}}_{=0}\,\frac{1-\cos x}{x}$$

$$= \int_0^{+\infty} 0 = 0$$

$$\rightarrow \boxed{c=0}$$

$$\lim_{n\to+\infty} F(n) = \lim \int_0^{+\infty} \underbrace{e^{-nx}\,\frac{1-\cos x}{x}}_{f_n(x)}\,dx$$

Let's apply dom conv:

$f_n(x) \longrightarrow 0$ a.e $x \in [0,+\infty[$

. $f_n(x) \xrightarrow[n\to+\infty]{} 0$ a.e $x \in [0,+\infty[$

. $|f_n(x)| = f_n(x) \leq \underbrace{e^{-x}\,\frac{1-\cos x}{x}}_{\in L^1} \quad \forall n \geq 1$

$\Rightarrow$ conclusion : $F(t) = \log\sqrt{1+1/t^2}$. $\blacksquare$

Ex 4: $f(x) = \frac{1}{1+x^4}$.

i) Does $\hat{f}$ exists ?

$\hat{f} \in L^1$?

$\hat{f} \in L^2$?

$\hat{f} \in \mathcal{C}^1$?

$\hat{f} \in \mathcal{S}$?

$\hat{f}$ exists because $f \in L^1 \cap L^2$

$\hat{f} \in L^1$? Recall that if $\underline{f, f', f'' \in L^1} \Rightarrow \hat{f} \in L^1$

$f \in L^1$

$f' = -\frac{4x^3}{(1+x^4)^2} \in \mathcal{C}(\mathbb{R}) \underset{\pm\infty}{\sim} C\,\frac{x^3}{x^8} = \frac{C}{x^5}$

$\rightarrow f' \in L^1$

$f'' = -\frac{12x^2(1+x^4)^2 - 4x^3\,2(1+x^4)\,4x^3}{(1+x^4)^4} \in \mathcal{C}$

$$\sim_{\pm\infty} \frac{C x^{10}}{x^{16}} = \frac{C}{x^6} \quad \text{int at } \pm\infty$$

$$\Rightarrow f'' \in L^1$$

$\hat{f} \in \mathcal{C}^1$? $\quad \partial_\xi \hat{f} = \widehat{\underbrace{i2\# f}}$

$$\# f \in L^1 \iff \frac{x}{1+x^4} \in L^1 \quad \text{yes because}$$

it is cont on $\mathbb{R}$ and $\sim_{\pm\infty} \frac{x}{x^4} = \frac{1}{x^3}$

int at $\pm\infty$.

$\hat{f} \in \mathcal{S}$? No because $\hat{f} \in \mathcal{S} \iff f \in \mathcal{S}$

$$\mathcal{S} = \{ f \in \mathcal{C}^\infty : |x|^n \partial_x^k f \xrightarrow[|x|\to\infty]{} 0 \}$$

$$x^4 f(x) = \frac{x^4}{1+x^4} \xrightarrow[|x|\to\pm\infty]{} 1.$$

$$\Rightarrow f \notin \mathcal{S}.$$

ii) Compute FT

$$\frac{1}{x^2 \pm \sqrt{2}\, x + 1}$$

Recall that

$$\widehat{\ 1\ } \qquad 2\pi a|x|$$

$$\widehat{\frac{1}{\#^2+a^2}}(\xi) = \frac{\pi}{a} e^{-2\pi a|\xi|}$$

$$\frac{1}{x^2+a^2}$$

$$\frac{1}{x^2 \pm \sqrt{2}x+1} = \frac{1}{\underbrace{x^2 \pm 2\left(\frac{\sqrt{2}}{2}\right)x + \frac{1}{2}}_{\left(x \pm \frac{1}{\sqrt{2}}\right)^2} + \frac{1}{2}}$$

$$\left(x \pm \frac{1}{\sqrt{2}}\right)^2 + 1/2$$

$$= \frac{1}{\left(x \pm \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\widehat{\frac{1}{x^2 \pm \sqrt{2}x+1}}(\xi) = \widehat{\frac{1}{\left(x \pm \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}}(\xi)$$

$$\frac{1}{\sqrt{2}} = a$$

$$\widehat{f(\# \pm c)}(\xi) = \int_{\mathbb{R}} f(\underbrace{x \pm c}_{y})\, e^{-i2\pi\xi x}\, dx$$

$$f(y)\, e^{-i2\pi\xi(y-c)}\, dy$$

$$= e^{i2\pi\xi c}\, \hat{f}$$

$$= e^{\pm i2\pi\xi\frac{1}{\sqrt{2}}}\, \widehat{\frac{1}{\#^2+\left(1/\sqrt{2}\right)^2}}(\xi)$$

$$\pm i\sqrt{2}\pi\xi \qquad \pi \qquad -\sqrt{2}\pi\, \frac{1}{}\, |\xi|$$

$$= e^{\pm i\sqrt{2}\pi\xi} \; \frac{\pi}{1/\sqrt{2}} \; e^{-\sqrt{2}\pi \frac{1}{\sqrt{2}} |\xi|}$$

$$= \sqrt{2}\pi \; e^{\pm i\sqrt{2}\pi\xi} \; e^{-\sqrt{2}\pi|\xi|}.$$

iii) Notice $1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$

express $f = \dfrac{1}{1+x^4}$ in terms of $\dfrac{1}{x^2 \pm \sqrt{2}x + 1}$

and use this to det $\hat{f}$.

$$\frac{1}{1+x^4} = \frac{1}{(\quad)(\quad)}$$

$$\overset{?}{=} \frac{1}{x^2+\sqrt{2}x+1} - \frac{1}{x^2-\sqrt{2}x+1}$$

$$= \frac{\cancel{x^2} - \sqrt{2}x + \cancel{1} - (\cancel{x^2} + \sqrt{2}x + \cancel{1})}{(\quad)(\quad)} = \frac{-2\sqrt{2}x}{(\quad)(\quad)}$$

$$\widehat{\frac{1}{1+x^4}} = \widehat{\frac{1}{(\;)(\;)}}$$

$$\Rightarrow \frac{-2\sqrt{2}x}{1+x^4} = \frac{-2\sqrt{2}x}{(\quad)(\quad)} = \frac{1}{} - \frac{1}{}$$

$$\Rightarrow \widehat{-\frac{\cancel{2}\sqrt{2}}{i\pi}\, i2\pi\#\, f(\#)} = \widehat{\frac{1}{\#^2+\sqrt{2}\#+1}} - \widehat{\frac{1}{\#^2-\sqrt{2}\#+1}}$$

$$-\frac{\sqrt{2}}{i\pi}\,\partial_\xi \hat{f} = \sqrt{2}\pi\, e^{+i\sqrt{2}\pi\xi}\, e^{-\sqrt{2}\pi|\xi|}$$

$$-\sqrt{2}\pi\, e^{-i\sqrt{2}\pi\xi}\, e^{-\sqrt{2}\pi|\xi|}$$

$$= \sqrt{2}\pi\, e^{-\sqrt{2}\pi|\xi|}\, 2i \left(\frac{e^{i\sqrt{2}\pi\xi} - e^{-i\sqrt{2}\pi\xi}}{2i} \right)$$

$$\frac{e^{i\sqrt{2}\pi\xi} - e^{-i\sqrt{2}\pi\xi}}{2i} = \sin(\sqrt{2}\pi\xi)$$

$$\cancel{+\frac{\sqrt{2}}{i\pi}}\,\partial_\xi \hat{f} = \cancel{i}\,2\cancel{\sqrt{2}}\pi^2\, e^{-\sqrt{2}\pi|\xi|} \sin(\sqrt{2}\pi\xi)$$

$$\boxed{\partial_\xi \hat{f}(\xi) = 2\pi^2\, e^{-\sqrt{2}\pi|\xi|} \sin(\sqrt{2}\pi\xi)}$$

$$\partial_\xi \hat{f}(\xi) = \begin{cases} 2\pi^2\, e^{-\sqrt{2}\pi\xi} \sin(\sqrt{2}\pi\xi) & \xi \geq 0 \\ 2\pi^2\, e^{+\sqrt{2}\pi\xi} \sin(\sqrt{2}\pi\xi) & \xi < 0 \end{cases}$$

$$\left(2\pi \quad e \qquad \sin(\sqrt{2}\pi\xi) \qquad \xi < 0 \right.$$

$$\hat{f}(\xi) = \begin{cases} 2\pi^2 \int e^{-\sqrt{2}\pi\xi} \sin(\sqrt{2}\pi\xi)\, d\xi + c & \xi \geq 0 \\ 2\pi^2 \int e^{+\sqrt{2}\pi\xi} \sin(\sqrt{2}\pi\xi)\, d\xi + \tilde{c} & \xi < 0 \end{cases}$$

$$\int e^{\alpha\xi} \sin(\beta\xi)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha\xi} \left(\sin(\beta\xi) - \frac{\beta}{\alpha} \cos(\beta\xi) \right)$$

$$\hat{f}(\xi) = \begin{cases} \cancel{2}\pi^{\cancel{2}} \frac{-\sqrt{2}\pi}{2\cdot\cancel{2}\cancel{\pi}^{\cancel{2}}} e^{-\sqrt{2}\pi\xi} \left(-\sin(\sqrt{2}\pi\xi) \mp \cos(\sqrt{2}\pi\xi) \right) + c \\ \cancel{\ominus} \frac{\pi}{\sqrt{2}} \quad \text{''} \quad \text{''} \quad \text{''} \\ \cancel{2}\pi^{\cancel{2}} \frac{\sqrt{2}\pi}{\cancel{2}\, 2\cancel{\pi}^{\cancel{2}}} e^{\sqrt{2}\pi\xi} \left(\sin(\sqrt{2}\pi\xi) - \cos(\sqrt{2}\pi\xi) \right) + \tilde{c} \\ + \frac{\pi}{\sqrt{2}} \end{cases}$$

$$\hat{f}(+\infty) = c \qquad \hat{f} \in L^1 \implies c = 0, \ \tilde{c} = 0$$

$$\hat{f}(\xi) = \frac{\pi}{\sqrt{2}} e^{-\sqrt{2}\pi|\xi|} \left(-\sin(\sqrt{2}\pi|\xi|) - \cos(\sqrt{2}\pi\xi) \right)$$

□