Principles of Mathematical Analysis 2

Paolo Guiotto

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CHAPTER 1

Basic Differential Equations

Ordinary Differential Equations (ODEs) is a wide topic of Mathematical Analysis. Their relevance is due to applications in Physics, Engineering, Biology, Economy, etc. An ODE is first of all an *equation*, that is an identity with an unknown. Differently from algebraic equations, the unknown is a *function* of one variable, let say $y = y(t)$ ($t \in \mathbb{R}$). The equation expresses a relation between $y(t)$ and some of its derivatives $y'(t), y''(t), \dots$ up to a certain maximum order, called *order of the equation*. This explains the D and the E of ODE. The O is to distinguish these equations to similar equations but with unknown function depending by several variables (this type of equations is called *Partial Differential Equations*, PDEs).

In this Chapter we introduce to the simplest type of ODEs: first and second order linear equations, first order separable variables equations. To understand importance of these equations, we will accompany theory with several applied examples.

1.1. Differential Equations in Applications

In this Section we show how ODEs arise in applied problems. Reader should focus not only to the mathematical side of the story, rather on the *modelling process* leading to an ODE.

EXAMPLE 1.1.1. *A water vessel looses, by percolation, a fraction of the volume of water therein contained at constant time rate ν. The vessel is refilled by a constant flux ϕ. Discuss the time behaviour of the volume of water contained the vessel. In particular, what happens in long time? Does the volume of water reach some stable level?*

SOL. — Let's call $V(t)$ the volume of the water inside the vessel, V_{max} the maximum volume and $V(0)$ the initial volume. The variation of volume $V(t)$ on a small time interval, that is $V(t+dt) - V(t)$ is given by a reduction $-\nu V(t)\,dt$ due to percolation and by an increase $\phi\,dt$ due to the flux. Therefore

$$V(t+dt) - V(t) = -\nu V(t)\,dt + \phi\,dt.$$

Dividing by dt and letting $dt \longrightarrow 0$ we get the equation

$$V'(t) = -\nu V(t) + \phi.$$

Leaving apart the formal details, we could notice that the equation is equivalent to

$$\frac{V'(t)}{-\nu V(t) + \phi} = 1.$$

At the left hand side we recognize, more or less, a derivative. Indeed

$$(\log|-\nu V(t) + \phi|)' = \frac{-\nu V'(t)}{-\nu V(t) + \phi} = -\nu,$$

by the equation. So we could say that

$$\log|-\nu V(t)+\phi| = -\nu t + C,$$

where C is some constant that can be found imposing that $V(0)=V_0$ that is $C=\log|-\nu V_0+\phi|$. Now, we would have

$$|-\nu V(t)+\phi| = e^{-\nu t+C}, \iff V(t)=\frac{\phi}{\nu}\pm\frac{1}{\nu}e^{-\nu t+C},$$

But $+$ or $-$? According the value for C, $V_0=\frac{\phi}{\nu}\pm\frac{1}{\nu}|-\nu V_0+\phi| = \frac{\phi}{\nu}\pm\left|\frac{\phi}{\nu}-V_0\right|$. We see that if $\frac{\phi}{\nu}\geqslant V_0$ we have to take $-$ otherwise $+$. Therefore

$$V(t)=\begin{cases} \frac{\phi}{\nu}-\frac{\phi-\nu V_0}{\nu}e^{-\nu t} = \frac{\phi}{\nu}\left(1-e^{-\nu t}\right)+V_0e^{-\nu t}, & \text{if } \frac{\phi}{\nu}\geqslant V_0, \\ \\ \frac{\phi}{\nu}+\frac{\nu V_0-\phi}{\nu}e^{-\nu t} = \frac{\phi}{\nu}\left(1-e^{-\nu t}\right)+V_0e^{-\nu t}, & \text{if } \frac{\phi}{\nu}<V_0. \end{cases} = \frac{\phi}{\nu}\left(1-e^{-\nu t}\right)+V_0e^{-\nu t}.$$

As time goes on, V stabilizes through a limit level $V(+\infty)=\frac{\phi}{\nu}$. ■

The equation $V'(t)=-\nu V(t)+\phi$ is an example of *first order linear equation*. Roughly, linear means that the dependence on V and V' is through a first degree polynomial. As we will see for ODE this means a particularly simple structure for solutions of the equations and explicit formulas for the solution.

Example 1.1.2 (Newton's Equations). *The most classical example of ODE is given by* Newton's equations*, direct consequence of* **Newton's second law**. *A particle of mass m in movement under the effect of some force $\vec{F}$ fulfils,*

$$m\vec{a}=\vec{F},$$

where $\vec{a}$ is the acceleration of the particle. For sake of simplicity, we assume the mass moving on a straight rail, we can characterize its position in terms of a function $x=x(t)$, t representing time. Acceleration is then $x''(t)$. As for acceleration, force $\vec{F}$ can be identified with a scalar F. In general, Physical forces depend by position $x(t)$ *(as in the case of* gravitational forces *or* elastic forces*),* velocity $x'(t)$ *(as in the case of* friction*) or directly by time t (if intensity of applied force is changing in time). Therefore, Newton's second law assumes the form*

$$mx''(t)=F(t,x(t),x'(t)).$$

A classical example is that one of a mass m moving under the action of a elastic force and subject to friction. If $\kappa\geqslant 0$ is the elastic constant and assuming the origin as the rest position, first force is

$$-\kappa x(t).$$

The minus means that when the particle is out of the origin the force tends to move the particle toward the origin. Second component of applied force is friction, which depends on velocity. For simplicity, we will assume the rail be homogeneous and friction be proportional to velocity in a way to decelerate the mass. This means the force is

$$-\nu x'(t),$$

($\nu\geqslant 0$ is called viscosity). Putting all together we obtain the equation

$$mx''(t)=-\kappa x(t)-\nu x'(t).$$

If the viscosity vanishes (*harmonic oscillator*), we expect a perfect infinite oscillatory motion, that is $x(t)$ like a sinusoid, whereas if we set off the spring and we leave only the viscosity we expect that if the mass has some initial velocity, it will tends to stop sometime later. Combining the two we expect an attenuated oscillation motion. But: how to prove it? And what about if we would have a precise measure of the attenuation?

If an external force $f(t)$ (that is, independent by the mass) acts on the mass the equation, the equation assumes the form

$$mx''(t) = -\kappa x(t) - \nu x'(t) + f(t).$$

With this simple equation we describe lots of phenomena like *forced oscillations*. An interesting and surprising example is the case of *resonance*. Imagine a periodic external force is applied to an harmonic oscillator,

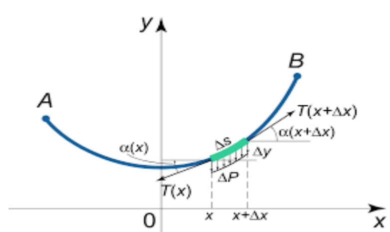
$$mx''(t) = -\kappa x(t) + F_0 \sin(\omega t).$$

It turns out that if $\omega = \sqrt{\kappa}$ external force enters in resonance with elastic force leading to an x with oscillation amplitude increasing in time. A model like this was used to provide a simple explanation of the famous Takoma bridge collapse. ■

This is a second order *linear* equation, linear because the dependence by x, x', x'' is by a so called linear function, that is a first degree polynomial. In the two previous Examples, the unknown function had time as independent variable. Of course this is not the unique possible parametrization.

Example 1.1.3 (Catenary Problem). *A chain is suspended by two fixed points: what is the curve the hanging chain assumes under its own weight when supported only at its ends? In his Two New Sciences (1638), Galileo says that a hanging cord is an approximate parabola. But what precisely is this curve?*

Sol. — Let xy be the plane containing the curve, we use x as parameter in such a way that $\alpha = \alpha(x)$ is the ordinate corresponding to x on the catenary. Our goal is to determine the function α. Let's see how the Mechanics of the problem enters to determine a Differential Equation for α.



Consider a small portion of the catenary included between points $(x, \alpha(x))$ and $(x + \Delta x, \alpha(x + \Delta x))$ ($\Delta x > 0$ be "small"). On this portion of catenary the following forces act: the *tension* exercised at two extremities by the remaining parts of the catenary and the *gravitation*. This last is easy because pulls downward as $m\vec{g}$. Here $\vec{g} = (0, -g)$ ($g = 9.8m/s^2$) while m is the mass of the small portion of catenary. Let's say that ϱ is *mass linear density*, $m = \varrho \cdot ds$ where ds =length of the portion. By Pythagorean Thm

$$ds \sim \sqrt{(\Delta x)^2 + (\alpha(x + \Delta x) - \alpha(x))^2},$$

the approximation being precise as $\Delta x \approx 0$. Because this is the case we're considering, $\alpha(x + \Delta x) - \alpha(x) = \alpha'(x)\Delta x + o(\Delta x) \sim \alpha'(x)\Delta x$, whence

$$m = \varrho\sqrt{1 + \alpha'(x)^2}\Delta x.$$

In conclusion

$$m\vec{g} = \left(0, -\varrho g\sqrt{1 + \alpha'(x)^2}\Delta x\right).$$

About the tension, let's denote by $\vec{T}(y)$ the force exercises by the part of the catenary included between $(y, \alpha(y))$ and B (final point). Therefore, the force exercised by the part of the catenary included between A and $(y, \alpha(y))$ must be $-\vec{T}(y)$. Therefore, in $(x, \alpha(x))$ is acting $-\vec{T}(x)$, in $(x+\Delta x, \alpha(x+\Delta x))$ is acting $\vec{T}(x+\Delta x)$ and these two are in equilibrium with $m\vec{g}$, that is

$$-\vec{T}(x) + \vec{T}(x+\Delta x) + m\vec{g} = \vec{0},$$

or, in components $\vec{T} = (\tau, \sigma)$,

$$\begin{cases} \tau(x+\Delta x) - \tau(x) = 0, \\ \\ \sigma(x+\Delta x) - \sigma(x) - \varrho g\sqrt{1+\alpha'(x)^2}\Delta x = 0. \end{cases}$$

The first one says that $\tau(x) \equiv \tau_0$ is constant. The second one, dividing by Δx and letting this to 0, says

$$\sigma'(x) = \varrho g\sqrt{1+\alpha'(x)^2}.$$

There's still one more information we need to use: of course $\vec{T}$ is tangent to the catenary at each of its points. Now, because $\vec{T} = (\tau_0, \sigma(x))$ we need that

$$\frac{\sigma(x)}{\tau_0} = \alpha'(x), \quad \sigma(x) = \tau_0\alpha'(x).$$

By this we obtain finally $\sigma'(x) = \tau_0\alpha''(x)$, thus

$$\alpha''(x) = \frac{\varrho g}{\tau_0}\sqrt{1+\alpha'(x)^2}. \tag{1.1.1}$$

This is the *catenary equation* and it is a *non linear* equation. ■

1.2. First order linear equations

The first type of ODE we consider is the following

$$y'(t) = a(t)y(t) + b(t), \quad t \in I. \tag{1.2.1}$$

Here $a, b : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are known function (called *coefficients*). If $b \equiv 0$ we say that the equation is *homogeneous*. In this case the set of solutions has a *linear structure*. Indeed we can notice that if φ and ψ are solutions, then also $\alpha\varphi + \beta\psi$ is a solution (here $\alpha, \beta \in \mathbb{R}$). Indeed

$$(\alpha\varphi + \beta\psi)'(t) = \alpha\varphi'(t) + \beta\psi'(t) = \alpha a(t)\varphi(t) + \beta a(t)\psi(t) = a(t)(\alpha\varphi + \beta\psi)(t), \quad t \in I.$$

Homogeneous equations are simpler. Roughly,

$$y'(t) = a(t)y(t) \overset{\text{if } y\neq 0}{\Longleftrightarrow} \frac{y'(t)}{y(t)} = a(t), \iff (\log|y(t)|)' = a(t),$$

thus, at least if $y \neq 0$,

$$\log|y(t)| = \int a(t)\,dt + c,$$

where c is a constant. Therefore

$$|y(t)| = e^c e^{\int a(t)\,dt}, \iff y(t) = \pm e^c e^{\int a(t)\,dt} \equiv ce^{\int a(t)\,dt}, \; c \in \mathbb{R}.$$

Notice that this formula produces, for $c = 0$, $y \equiv 0$ which is a solution for the homogeneous equation. Formula we just obtained could work for $y \neq 0$ and $y \equiv 0$ solutions. But who says that these are all the possible solutions? Rather than fixing this argument, we provide a simpler but less "intuitive" proof that shows this characterization holds true:

Proposition 1.2.1. *Let $a \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. All the solutions of the homogeneous equation*

$$y'(t) = a(t)y(t),\ t \in I,$$

are

$$y(t) = ce^{\int a(t)\, dt},\ c \in \mathbb{R}. \tag{1.2.2}$$

Proof — Define $A(t) = \int a(t)\, dt$. Then

$$\left(e^{-A}y\right)' = -A'e{-}Ay + e^{-A}y' = e^{-A}\left(-ay + y\right).$$

Now, because y is a solution iff $y' - ay = 0$, we have also,

$$y \text{ solution} \iff \left(e^{-A}y\right)' \equiv 0,$$

and because I is an interval, we deduce

$$y \text{ solution} \iff e^{-A}y \equiv c, \iff y(t) = ce^{-A(t)} = ce^{-\int a(t)\, dt}. \quad \blacksquare$$

Let's move to the general case of a non homogeneous equation,

$$y'(t) = a(t)y(t) + b(t).$$

We prove now that the general solution is obtained by summing to (1.2.2) a *particular solution* of the non homogeneous equation.

Proposition 1.2.2. *Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. If $U = U(t)$ is a particular solution of the non homogeneous equation, then the general solution of*

$$y'(t) = a(t)y(t) + b(t),\ t \in I,$$

is

$$y(t) = ce^{\int a(t)\, dt} + U(t),\ t \in I. \tag{1.2.3}$$

Proof — Just notice that y is a solution of the non homogeneous equation iff

$$(y - U)' = y' - U' = (ay + b) - (aU + b) = a(y - U), \overset{(1.2.2)}{\iff} y(t) - U(t) = ce^{\int a(t)\, dt}. \quad \blacksquare$$

Thus, to determine the general solution for the non homogeneous equation it remains to determine a particular solution. This may be determined through the so called *method of variation of constants*:

Theorem 1.2.3. *Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. Then, a particular solution for*

$$y'(t) = a(t)y(t) + b(t), \quad t \in I,$$

is

$$U(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt} b(t)\, dt,\ t \in I. \tag{1.2.4}$$

Thus, the general solution of the non homogeneous equation is

$$y(t) = e^{\int a(t)dt} \left[\int e^{-\int a(t)dt} b(t)\, dt + c \right], \quad t \in I, \tag{1.2.5}$$

where $c \in \mathbb{R}$ is a constant. The (1.2.5) is also called **general integral**.

Proof — We start by searching for $U = U(t)$ particular solution. The idea is to look at

$$U(t) = c(t)e^{\int a(t)\,dt}, \text{ where } c = c(t) \text{ has to be determined by imposing } U' = aU + b.$$

Now, being

$$U'(t) = \left(c(t)e^{\int a(t)\,dt}\right)' = c'(t)e^{\int a(t)\,dt} + c(t)e^{\int a(t)\,dt}a(t) = e^{\int a(t)\,dt}\left(c'(t) + a(t)c(t)\right),$$

we have

$$U' = aU + b, \iff e^{\int a(t)\,dt}\left(c'(t) + a(t)c(t)\right) = a(t)c(t)e^{\int a(t)\,dt} + b(t),$$

that is

$$c'(t)e^{\int a(t)\,dt} = b(t), \iff c'(t) = e^{-\int a(t)\,dt}b(t), \iff c(t) = \int e^{-\int a(t)\,dt}b(t)\,dt + c,\ c \in \mathbb{R}.$$

Because any of these $c(t)$ is good, we may take $c = 0$, this leading to (1.2.4). Together with (1.2.3), finally we obtain

$$y(t) = ce^{\int a(t)\,dt} + \left(\int e^{-\int a(t)\,dt}b(t)\,dt\right)e^{\int a(t)\,dt} = e^{\int a(t)\,dt}\left(\int e^{-\int a(t)\,dt}b(t)\,dt + c\right),$$

which is (1.2.5). ■

Example 1.2.4. *Find the general integral for the equation*

$$y'(t) - \frac{2}{t}y(t) = 1, \quad t \in]0, +\infty[.$$

Sol. — We have

$$y'(t) = \frac{2}{t}y(t) + 1 = a(t)y(t) + b(t), \text{ where } a(t) = \frac{2}{t},\ b(t) = 1.$$

Therefore

$$y(t) = e^{\int \frac{2}{t}dt}\left(\int e^{-\int \frac{2}{t}dt}dt + c\right) = e^{2\log t}\left(\int e^{-2\log t}dt + c\right) = t^2\left(\int \frac{1}{t^2}dt + c\right) = t^2\left(-\frac{1}{t} + c\right) = -t + ct^2. \ ■$$

You shouldn't be surprised because uniqueness does not hold for ODEs. Just the simplest of equations

$$y' = 0,$$

has infinitely many solutions (all constants). However, further conditions may lead to a unique solution. A very important case is the so called *Cauchy Problem* or *passage problem* or, again, *initial condition* problem. This consists in finding a solution of an ODE fulfilling a passage/initial value condition. Formally, this problem may be stated in the following form:

$$\mathsf{CP}(t_0, y_0) \begin{cases} y'(t) = a(t)y(t) + b(t), & t \in I, \\ \\ y(t_0) = y_0. \end{cases} \tag{1.2.6}$$

Here, of course, $t_0 \in I$. It is easy to check that this problem has a unique solution:

Corollary 1.2.5. *Let $a, b \in \mathscr{C}(I)$, $I \subset \mathbb{R}$ interval. For every $t_0 \in I$, Cauchy Problem $\mathsf{CP}(t_0, y_0)$ admits a unique solution.*

Proof — Because general solution has the form

$$y(t) = ce^{\int a(t)\,dt} + U(t) \equiv ce^{A(t)} + U(t), \text{ where } A(t) = \int a(t)\,dt,$$

being U the particular solution, we have that y solves $\mathsf{CP}(t_0, y_0)$ iff

$$y_0 = ce^{A(t_0)} + U(t_0), \iff c = \frac{y_0 - U(t_0)}{e^{-A(t_0)}}.$$

This c clearly exists ($e^{-A(t_0)} \neq 0$) and it is unique, thus we have existence and uniqueness for $\mathsf{CP}(t_0, y_0)$. ■

Example 1.2.6. *Solve the Cauchy Problem*

$$\begin{cases} y'(t) - \dfrac{2y(t)}{1-t^2} = t, \quad t > 1 \\ y(2) = 0. \end{cases}$$

Sol. — Rewriting the equation in the canonical form

$$y'(t) = \frac{2}{1-t^2} y(t) + t, \implies y(t) = e^{\int \frac{2}{1-t^2}\,dt} \left(\int e^{-\int \frac{2}{1-t^2}\,dt} t\,dt + C \right).$$

Now

$$\int \frac{2}{1-t^2} dt = \int \frac{2}{(1-t)(1+t)} dt = \int \frac{1}{1-t} + \frac{1}{1+t} dt = -\int \frac{1}{t-1} dt + \log|1+t| = \log\left|\frac{t+1}{t-1}\right|.$$

Because $t \in]1, +\infty[$, $\frac{t+1}{t-1} > 0$, therefore

$$y(t) = e^{\log \frac{t+1}{t-1}} \left(\int e^{-\log \frac{t+1}{t-1}} t\,dt + C \right) = \frac{t+1}{t-1} \left(\int \frac{t-1}{t+1} t\,dt + C \right).$$

Now

$$\int t \frac{t-1}{t+1}\,dt = \int t \frac{t+1-2}{t+1}\,dt = \int t\,dt - 2\int \frac{t}{t+1}\,dt = \frac{t^2}{2} - 2\int dt + 2\int \frac{1}{t+1}\,dt = \frac{t^2}{2} - 2t + 2\log|t+1|,$$

and finally

$$\varphi(t) = \frac{t+1}{t-1} \left(\frac{t^2}{2} - 2t + 2\log(1+t) + C \right), \quad t \in]0, +\infty[.$$

Imposing $\varphi(2) = 0$ we have

$$2(2 - 4 + 2\log 3 + C) = 0, \iff C = 2(1 - \log 3). \quad ■$$

1.3. First order separable variables equations

An equation of type

$$y'(t) = a(t)f(y(t))$$

is called *separable variables equation*. This type of equations is, in some sense, an extension of first order homogeneous equations

$$y'(t) = a(t)y(t), \text{ assuming } f(y) := y.$$

This remark suggests a possible method to solve the equation, the so called *method of separation of variables*. To understand better the argument let's consider the Cauchy problem

$$\mathsf{CP}(t_0, y_0) \quad \begin{cases} y'(t) = a(t)f(y(t)), \ t \in I, \\ \\ y(t_0) = y_0. \end{cases}$$

Here we assume

$$a \in \mathscr{C}(I), \quad f \in \mathscr{C}(J), \quad I, J \subset \mathbb{R}, \text{ intervals.}$$

In order $\mathsf{CP}(t_0, y_0)$ make sense, here we need $t_0 \in I$ and $y_0 \in J$. Indeed, setting $t = t_0$ in the equation we must have

$$y'(t_0) = a(t_0)f(y(t_0)) = a(t_0)f(y_0),$$

that is both $a(t_0)$ (thus $t_0 \in I$) and $f(y_0)$ (thus $y_0 \in J$) must be well defined. Now we have the following alternative:

- if $f(y_0) = 0$, then clearly $y(t) \equiv y_0$ is a solution because

  $$y'(t) = 0, \quad a(t)f(y(t)) = a(t)f(y_0) = 0.$$

  In this case $y(t) \equiv y_0$ (constant solution) is a solution of the equation.
- if $f(y_0) \neq 0$, then $f(y(t_0)) \neq 0$ and, by continuity, $f(y(t)) \neq 0$ in some neighbourhood of t_0. In such a neighbourhood, we may write

  $$y'(t) = a(t)f(y(t)), \iff \frac{y'(t)}{f(y(t))} = a(t).$$

  This step is called *separation of variables* because it consists in writing all the terms containing y on one side, leaving on the other an explicit expression of t. As in the case of first order linear homogeneous equations, we may see at lhs as the derivative of something. In other terms, taking side by side the primitives, we have

  $$\int \frac{y'(t)}{f(y(t))}\, dt = \int a(t)\, dt + c,$$

  where c is an arbitrary constant. The rhs can be explicitly computed. Conversely, the lhs contains the unknown y. However, we may notice that

  $$\int \frac{y'(t)}{f(y(t))}\, dt \overset{u:=y(t),\ du=y'(t)\,dt}{=} \int \frac{1}{f(u)}\, du =: G(u)\Big|_{u=y(t)} = G(y(t)).$$

Therefore, assuming $G = \int \frac{1}{f}$ known, we have an algebraic equation for y,

$$G(y(t)) = \int a(t)\,dt + c. \tag{1.3.1}$$

This is called **implicit form of the solution**. If G is also invertible, we may go on to obtain

$$y(t) = G^{-1}\left(\int a(t)\,dt + c\right). \tag{1.3.2}$$

This is the **explicit form**. The value of c is determined imposing the passage condition $y(t_0) = y_0$. Apparently, this argument shows that **either** y is constant (if $f(y_0) = 0$) **or** y is given by (1.3.1) in implicit form or (1.3.2) in explicit form. There's, however, a subtle point: these two alternatives hold up $f(y(t)) \neq 0$. So the question is whether it is possible or less that $f(y(t)) \neq 0$ always. This turns out to be true if we require something more on f:

THEOREM 1.3.1. *Let $a \in \mathscr{C}(I)$ and $f \in \mathscr{C}^1(J)$ with $I, J \subset \mathbb{R}$ intervals. Then, for any $(t_0, y_0) \in I \times J$ there exists a solution y of*

$$\begin{cases} y'(t) = a(t)f(y(t)), \\ \\ y(t_0) = y_0. \end{cases}$$

In particular

- *if $f(y_0) = 0$ then $y(t) \equiv y_0$ is a (stationary) solution.*
- *if $f(y_0) \neq 0$ then, setting $G(z) := \int \frac{1}{f(z)}\,dz$ a primitive of $1/f$, we have*

$$G(y(t)) = \int a(t)\,dt + c, \tag{1.3.3}$$

for a suitable c. If G is invertible we have

$$y(t) = G^{-1}\left(\int a(t)\,dt + c\right). \tag{1.3.4}$$

The (1.3.3) is called **implicit form** *for the solution, the (1.3.4) is called* **explicit form** *for the solution.*

In particular: existence and uniqueness for the Cauchy problem holds true.

PROOF — Omitted. ■

EXAMPLE 1.3.2. *Solve the Cauchy problem*

$$\begin{cases} y'(t) = 1 + y(t)^2, \\ \\ y(0) = y_0. \end{cases}$$

SOL. — This is a separable variables equation $y' = a(t)f(y)$ with $a(t) \equiv 1$ and $f(y) = 1 + y^2$ clearly fulfills the hypotheses of the previous Thm. Of course because $f(y_0) = 1 + y_0^2 > 0$ we don't have stationary solutions and we can separate variables:

$$y'(t) = 1 + y(t)^2, \iff \frac{y'(t)}{1+y(t)^2} = 1, \iff \int \frac{y'(t)}{1+y(t)^2}\,dt = \int 1 \; + C = t + C.$$

Now,

$$\int \frac{y'(t)}{1+y(t)^2}\,dt = \int \frac{1}{1+u^2}\,du\Big|_{u=y(t)} = \arctan u|_{u=y(t)} = \arctan y(t),$$

so

$$\arctan(y(t)) = t + C, \iff y(t) = \tan(t+C).$$

Imposing $y(0) = y_0$ we get $y_0 = \tan C$, that is $C = \arctan y_0$ and finally

$$y(t) = \tan(t + \arctan y_0).$$

Evidently this solution "lives" up to the time when $t + \arctan y_0 = \frac{\pi}{2}$ (in the future) and $t + \arctan y_0 = -\frac{\pi}{2}$ (in the past), that is in this case

$$I_{t_0} = \left] -\frac{\pi}{2} - \arctan y_0, \frac{\pi}{2} - \arctan y_0 \right[. \quad \blacksquare$$

Example 1.3.3 (Catenary). *Solve the catenary equation (1.1.1).*

Sol. — Setting $\alpha'(x) =: y(x)$ we have the first order equation

$$y' = \frac{\varrho g}{\tau_0}\sqrt{1+y^2},$$

which is a particular case of *separable variables equation*. Here $a(x) \equiv \frac{\varrho g}{\tau_0}$, $f(y) = \sqrt{1+y^2}$. Clearly $a \in \mathscr{C}(\mathbb{R})$ and $f \in \mathscr{C}^1(\mathbb{R})$. Notice also that $f \neq 0$ always. Thus, to determine solutions we can separate variables,

$$\frac{y'}{\sqrt{1+y^2}} = \frac{\varrho g}{\tau_0}, \iff \int \frac{y'}{\sqrt{1+y^2}}\,dx = \frac{\varrho g}{\tau_0}x + \gamma.$$

Now,

$$\int \frac{y'}{\sqrt{1+y^2}}\,dt \overset{u=y(x)}{=} \int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1} u = \sinh^{-1} y(x).$$

In conclusion

$$\sinh^{-1} y(x) = \frac{\varrho g}{\tau_0}x + \gamma, \iff y(x) = \sinh\left(\frac{\varrho g}{\tau_0}x + \gamma\right).$$

Finally, because $y = \alpha'$,

$$\alpha(x) = \int \sinh\left(\frac{\varrho g}{\tau_0}x + \gamma\right)\,dx + \widetilde{\gamma} = \frac{\tau_0}{\varrho g}\cosh\left(\frac{\varrho g}{\tau_0}x + \gamma\right) + \widetilde{\gamma}.$$

For instance assume that $A = (-\ell/2, h)$, $B = (\ell/2, h)$. Parameters $\gamma, \widetilde{\gamma}$ are determined by solving

$$\begin{cases} \frac{\tau_0}{\varrho g}\cosh\left(-\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right) + \widetilde{\gamma} = h, \\ \\ \frac{\tau_0}{\varrho g}\cosh\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0} + \gamma\right) + \widetilde{\gamma} = h. \end{cases}$$

Taking the difference, $\cosh\left(-\frac{\ell}{2}\frac{\varrho g}{\tau_0}+\gamma\right)=\cosh\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0}+\gamma\right)$, and because cosh is even, the unique possibility if that $-\frac{\ell}{2}\frac{\varrho g}{\tau_0}+\gamma=-\left(\frac{\ell}{2}\frac{\varrho g}{\tau_0}+\gamma\right)$ that is, $\gamma=0$. Therefore, $\widetilde{\gamma}=h-\frac{\tau_0}{\varrho g}\cosh\frac{\ell}{2}\frac{\varrho g}{\tau_0}$, whence

$$\alpha(x)=\frac{\tau_0}{\varrho g}\cosh\frac{\varrho g}{\tau_0}x+h-\frac{\tau_0}{\varrho g}\cosh\frac{\ell}{2}\frac{\varrho g}{\tau_0}. \quad \blacksquare$$

1.4. Second Order Linear Equations

More involved the theory for second order linear ODE, that is for equations of type

$$y''(t)=a(t)y'(t)+b(t)y(t)+f(t). \tag{1.4.1}$$

If $f\equiv 0$, the equation is said **homogenous** and if $a(t)\equiv a$, $b(t)\equiv b$ the equation is said to have *constant coefficients*. We will limit to this case for simplicity that, for convenience, we will rewrite as

$$y''+ay'+by=f(t).$$

To begin we will consider the homogeneous case

$$y''(t)+ay'(t)+by(t)=0.$$

To solve in general this equation notice the following: if we call D the derivative, then previous equation can be rewritten as

$$(D^2+aD+b)y=0.$$

The polynomial

$$\lambda^2+a\lambda+b$$

is called **characteristic polynomial** and basically it contains all the information to look for solutions.

Theorem 1.4.1. *The general integral of $y''+ay'+by=0$ is*

$$c_1w_1(t)+c_2w_2(t),\quad c_1,c_2\in\mathbb{R},$$

where

- *if $\Delta=a^2-4b>0$, $(w_1,w_2)=(e^{\lambda_1 t},e^{\lambda_2 t})$ with $\lambda_{1,2}$ are the roots of the char. pol.;*
- *if $\Delta=0$, $(w_1,w_2)=(e^{\lambda_1 t},te^{\lambda_1 t})$ with λ_1 is the unique root of the char. pol.;*
- *if $\Delta<0$, $(w_1,w_2)=(e^{\alpha t}\cos(\beta t),e^{\alpha t}\sin(\beta t))$ with $\lambda_{1,2}=\alpha\pm i\beta$ are the complex roots of the char. pol.*

The couple (w_1,w_2) is called **fundamental system of solutions**.

Proof — We develop the proof on three cases, according $\Delta>0$, $\Delta=0$, $\Delta<0$.
Case $\Delta>0$: the characteristic polynomial can be factorized as

$$\lambda^2+a\lambda+b=(\lambda-\lambda_1)(\lambda-\lambda_2),$$

so we may expect that

$$D^2+aD+b=(D-\lambda_1)(D-\lambda_2),$$

hence

$$(D^2+aD+b)y=0,\iff (D-\lambda_1)(D-\lambda_2)y=0.$$

Now call $\psi=(D-\lambda_2)y$. Then

$$(D-\lambda_1)\psi=0,\iff \psi'=\lambda_1\psi,\iff \psi=ce^{\lambda_1 t}.$$

But then

$$(D-\lambda_2)y = c_1 e^{\lambda_1 t}, \iff y' = \lambda_2 y + c_1 e^{\lambda_1 t}.$$

This is a first order linear equation that may be easily solved by the general formula (1.2.5), obtaining

$$y(t) = e^{\lambda_2 t}\left(\int e^{-\lambda_2 t} c_1 e^{\lambda_1 t}\, dt + c_2\right) = e^{\lambda_2 t}\left(c_1 \int e^{(\lambda_1-\lambda_2)t}\, dt + c_2\right) = \frac{c_1}{\lambda_1-\lambda_2} e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

and being c_1, c_2 arbitrary, we get finally

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Case $\Delta = 0$: we can repeat the same computations as before, just to the point

$$y(t) = e^{\lambda_2 t}\left(c_1 \int e^{(\lambda_1-\lambda_2)t}\, dt + c_2\right),$$

but now $\lambda_1 = \lambda_2$, therefore

$$y(t) = e^{\lambda_1 t}\left(c_1 \int dt + c_2\right) = c_1 t e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

Case $\Delta < 0$: in this case the characteristic polynomial is irreducible. However

$$\lambda^2 + a\lambda + b = \left(\lambda + \frac{a}{2}\right)^2 + \frac{4b-a^2}{4} = (\lambda-\alpha)^2 + \beta^2 = 0, \iff (\lambda-\alpha)^2 = -\beta^2.$$

Therefore

$$(D^2 + aD + b)y = 0, \iff (D-\alpha)^2 y = -\beta^2 y.$$

Now, notice that

$$(D-\alpha)y = y' - \alpha y = e^{\alpha t} D\left(e^{-\alpha t} y\right),$$

whence

$$(D-\alpha)^2 y = e^{\alpha t} D\left(e^{-\alpha t}\left(e^{\alpha t} D\left(e^{-\alpha t} y\right)\right)\right) = e^{\alpha t} D^2\left(e^{-\alpha t} y\right).$$

By this

$$(D-\alpha)^2 y = -\beta^2 y, \iff e^{\alpha t} D^2\left(e^{-\alpha t} y\right) = -\beta^2 y, \iff D^2\left(e^{-\alpha t} y\right) = -\beta^2 e^{-\alpha t} y.$$

Finally, setting for a while $\phi = e^{-\alpha t} y$, the previous equation becomes

$$D^2 \phi = -\beta^2 \phi,$$

and two solutions of this are $\phi_1(t) = \cos(\beta t)$, $\phi_2(t) = \sin(\beta t)$. This means that

$$y_1(t) = e^{\alpha t}\cos(\beta t), \quad y_2(t) = e^{\alpha t}\sin(\beta t)$$

are two solutions of the initial equation and the general integral is in this case

$$c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t). \quad \blacksquare$$

As for linear first order equations, the general solution can be obtained by the general solution of the homogeneous equation by adding a particular solution:

PROPOSITION 1.4.2. *Let (w_1, w_2) be a fundamental system of solutions for the homogeneous equation*

$$y'' + ay' + by = 0, \tag{1.4.2}$$

and U a particular solution of the equation

$$y'' + ay' + by = f(t). \tag{1.4.3}$$

Then, the general integral of (1.4.3) is

$$y(t) = c_1 w_1(t) + c_2 w_2(t) + U(t), \quad c_1, c_2 \in \mathbb{R}.$$

Proof — Just notice that if y solves (1.4.3), then $y - U$ solves (1.4.2). Indeed

$$(y-U)'' + a(y-U)' + b(y-U) = y'' + ay' + by - (U'' + aU' + bU) = f - f = 0, \implies y - U = c_1 w_1 + c_2 w_2. \quad \blacksquare$$

To complete the solution of the second order non homogeneous equation, we need to determine an its particular solution. As for the first order case, this may be determined through the *method of variation of constants*. This consists in looking to U of type

$$U(t) = c_1(t) w_1(t) + c_2(t) w_2(t), \; t \in I.$$

To find these coefficients, we impose that U be a solution of the equation.

Theorem 1.4.3 (Lagrange). *Let (w_1, w_2) a fundamental system of solutions of (1.4.2). Define*

$$W(t) := \det \begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix},$$

the **wronskian** *of (w_1, w_2). Then, $W(t) \neq 0$ for all t and*

$$U(t) = -\left(\int \frac{w_2(t)}{W(t)} f(t)\, dt\right) w_1(t) + \left(\int \frac{w_1(t)}{W(t)} f(t)\, dt\right) w_2(t), \quad t \in I, \tag{1.4.4}$$

is a particular solution of (1.4.3).

Proof — Let $U = c_1 w_1 + c_2 w_2$ with $c_j \equiv c_j(t)$ $j = 1, 2$. Then

$$U' = c_1' w_1 + c_1 w_1' + c_2' w_2 + c_2 w_2'.$$

To simplify computations we impose the condition

$$c_1' w_1 + c_2' w_2 = 0.$$

Then

$$U'' = c_1' w_1' + c_1 w_1'' + c_2' w_2' + c_2 w_2''.$$

Hence

$$U'' = aU' + bU + f, \iff c_1' w_1' + c_2' w_2' = f.$$

We may conclude that U is a solution iff

$$\begin{cases} c_1' w_1 + c_2' w_2 = 0, \\ c_1' w_1' + c_2' w_2' = f. \end{cases} \tag{1.4.5}$$

This can be seen as a 2×2 linear system in the unknown (c_1', c_2') and coefficients the matrix

$$\begin{bmatrix} w_1 & w_2 \\ w_1' & w_2' \end{bmatrix}.$$

Now, to find c_1', c_2' we apply the Cramer rule. Calling $W(t)$ the determinant of the previous matrix,

$$W(t) := w_1 w_2' - w_2 w_1', \quad \textbf{(wronskian of } (w_1, w_2)\textbf{)}$$

it is easy to check that in all cases $W(t) \neq 0$ for any t:

$$\det\begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ \lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \end{bmatrix} = e^{(\lambda_1+\lambda_2)t}(\lambda_2-\lambda_1). \quad \det\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ \lambda e^{\lambda t} & (1+\lambda t)e^{\lambda t} \end{bmatrix} = e^{2\lambda t},$$

and

$$\det\begin{bmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}(\alpha\cos(\beta t)-\beta\sin(\beta t)) & e^{\alpha t}(\alpha\sin(\beta t)+\beta\cos(\beta t)) \end{bmatrix} = \beta e^{2\alpha t}.$$

Therefore, by Cramer rule,

$$c_1'(t) = \frac{-w_2(t)f(t)}{W(t)}, \quad c_2'(t) = \frac{w_1(t)f(t)}{W(t)}.$$

that is

$$c_1(t) = -\int \frac{w_2(t)}{W(t)} f(t)\, dt, \quad c_2(t) = \int \frac{w_1(t)}{W(t)} f(t)\, dt, \tag{1.4.6}$$

so we get just the (1.4.4). ∎

Example 1.4.4. *Find the general integral of the equation*

$$y''(t) + y'(t) - 6y(t) = 2e^{-t}, \ t \in \mathbb{R}.$$

Sol. — We start computing the fundamental system of solutions of the homogeneous equation. The characteristic polynomial is

$$\lambda^2 + \lambda - 6 = 0, \quad \Delta = 1 + 24 = 25 > 0, \quad \lambda_\pm = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2} = 2, -3.$$

Therefore the fundamental solutions are $w_1(t) = e^{2t}$, $w_2(t) = e^{-3t}$ with wronskian

$$W(t) = (-3-2)e^{-t} = -5e^{-t}.$$

By Lagrange formula (1.4.4) we have

$$\begin{aligned} U(t) &= -\int \frac{e^{-3t}}{-5e^{-t}} 2e^{-t}\, dt\, e^{2t} + \int \frac{e^{2t}}{-5e^{-t}} 2e^{-t}\, dt\, e^{-3t} = \frac{2}{5}\int e^{-3t}\, dt\, e^{2t} - \frac{2}{5}\int e^{2t}\, dt\, e^{-3t} \\ &= -\frac{2}{15}e^{-t} - \frac{2}{10}e^{-t} = -\frac{1}{3}e^{-t}. \end{aligned}$$

Therefore, the general integral is

$$y(t) = c_1 e^{2t} + c_2 e^{-3t} - \frac{1}{3}e^{-t}, \quad c_1, c_2 \in \mathbb{R}. \ ∎$$

In the case of second order equations we see an interesting phenomenon: general integral depends on two free constants c_1, c_2. It is therefore clear that a unique condition on y is not sufficient to determine uniquely a solution. That's why the Cauchy problem for a second order ODE is different respect to the first order case. Intuitively, we need a second condition on the solution. There're two interesting cases:

- the **Cauchy problem**, that consists in finding a solution y fulfilling two initial conditions as

$$\mathsf{CP}(t_0, y_0, y_0') \begin{cases} y'' + ay' + by = f(t), \\ y(t_0) = y_0, \\ y'(t_0) = y_0'. \end{cases}$$

- the **boundary value problem**, that consists in finding a solution y fulfilling two passage conditions as

$$\mathsf{BV}(t_0, y_0; t_1, y_1) \begin{cases} y'' + ay' + by = f(t), \\ y(t_0) = y_0, \\ y(t_1) = y_1. \end{cases}$$

These are two entirely different problems. The first one has a strict interest in Physics, for instance. Here, the ODE corresponds to Newton's second law and initial conditions consist in assigning initial position and velocity of the mass in movement. The second has also an interest, it might correspond to the problem of finding a trajectory of a mass that at some initial time t_0 is in some position y_0 and at some future time t_1 reaches a position y_1. The different nature of these problems is also reflected by different results we may prove concerning existence and uniqueness. This holds true in general for the Cauchy Problem, but might be false for the Boundary Value Problem.

Theorem 1.4.5. *The Cauchy Problem* $\mathsf{CP}(t_0, y_0, y_0')$ *has a unique solution for any* $t_0 \in I$ *and* $y_0, y_0' \in \mathbb{R}$.

Proof — It is easy: we have to prove that there exists a unique c_1, c_2 such that

$$y = c_1 w_1 + c_2 w_2 + U,$$

is a solution of $\mathsf{CP}(t_0, y_0, y_0')$. Just impose the two conditions: we get

$$\begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) + U(t_0) = y_0, \\ \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) + U'(t_0) = y_0', \end{cases} \iff \begin{cases} c_1 w_1(t_0) + c_2 w_2(t_0) = y_0 - U(t_0), \\ \\ c_1 w_1'(t_0) + c_2 w_2'(t_0) = y_0' - U'(t_0). \end{cases}$$

Now, look at this as a system 2×2. The coefficient matrix is the wronskian matrix which is, in our assumption, invertible. Therefore the system has a unique solution c_1, c_2. ■

Example 1.4.6. *Find the solution of the Cauchy Problem*

$$\begin{cases} y''(t) + y(t) = e^t, \ t \in \mathbb{R}, \\ y(0) = 0, \\ y'(0) = 1. \end{cases}$$

Sol. — The characteristic equation is $\lambda^2 + 1 = 0$, that is $\lambda = \pm i$. Therefore $w_1(t) = \cos t$, $w_2(t) = \sin t$ is a fundamental system of solutions for the homogenous equation. The wronskian is

$$W(t) = \det \begin{bmatrix} \cos t & \sin t \\ \\ -\sin t & \cos t \end{bmatrix} = (\cos t)^2 + (\sin t)^2 = 1.$$

Therefore a particular solution, by the Lagrange formula, is

$$\begin{aligned} U(t) &= -\left(\int \frac{\sin t}{1} e^t\, dt\right)\cos t + \left(\int \frac{\cos t}{1} e^t\, dt\right)\sin t = -\left(\int e^t \sin t\, dt\right)\cos t + \left(\int e^t \cos t\, dt\right)\sin t \\ &= -\frac{e^t}{2}(\sin t - \cos t)\cos t + \frac{e^t}{2}(\cos t + \sin t)\sin t = \frac{e^t}{2}. \end{aligned}$$

Hence the general integral is

$$\varphi(t) = c_1 \cos t + c_2 \sin t + \frac{e^t}{2}.$$

Now, imposing the initial conditions we get the system

$$\begin{cases} c_1 + \frac{1}{2} = 0, \\ \\ c_2 + \frac{1}{2} = 1, \end{cases} \iff c_1 = -\frac{1}{2},\ c_2 = \frac{1}{2}, \implies \varphi(t) = \frac{1}{2}\left(\sin t - \cos t + e^t\right). \quad \blacksquare$$

We close this Section by illustrating few examples of applicative use of second order linear equations.

Example 1.4.7 (Damped Oscillations). *Consider the equation of motion of a mass m subjected to an elastic force (of elastic constant κ) on a viscous media (viscosity ν):*

$$mx''(t) = -\kappa x(t) - \nu x'(t), \iff mx''(t) + \nu x'(t) + \kappa x(t) = 0.$$

Describe the behavior of the solutions.

Sol. — The equation is just a linear second order equation with constant coefficients. Its characteristic equation is

$$m\lambda^2 + \nu\lambda + \kappa = 0.$$

Because $\Delta = \nu^2 - 4m\kappa$ we have that if $\Delta \geqslant 0$, that is if $\nu^2 \geqslant 4m\kappa$, $\nu \geqslant \sqrt{4m\kappa}$, the fundamental solutions are of exponential type, so we haven't oscillations. Also as $\Delta = 0$ we have the same. To have oscillations we need $\Delta < 0$, that is $\nu < \sqrt{4m\kappa}$. In this case

$$\lambda_\pm = -\frac{\nu}{2m} \pm i\frac{\sqrt{-\Delta}}{2m},$$

and the fundamental system of solutions is

$$w_+(t) = e^{-\frac{\nu}{2m}t}\cos\left(\frac{\sqrt{-\Delta}}{2m}t\right), \quad w_-(t) = e^{-\frac{\nu}{2m}t}\sin\left(\frac{\sqrt{-\Delta}}{2m}t\right).$$

In this case the oscillations are attenuated by the exponential $e^{-\frac{\nu}{2m}t}$ that goes to 0 as $t \longrightarrow +\infty$. The heavier the mass is, the stronger the attenuation is.

Example 1.4.8 (Resonance). *Describe long time behavior of the solutions of*

$$y''(t) = -k^2 y(t) + \sin(kt). \tag{1.4.7}$$

This equation is often used as model for the so–called phenomenon of **resonance**. *For instance it was used in the case of Tacoma bridge replacing the equation for the angle $\theta(t)$ (non linear) with an its linear approximation. The interesting aspect of this equation is that presents unbounded solutions.*

SOL. — The characteristic equation is $\lambda^2 = -k^2$, that is $\lambda = \pm ik$, therefore the fundamental system of solutions for the homogeneous equations is $w_1(t) = \cos(kt)$, $w_2(t) = \sin(kt)$. The wronskian is $W(t) \equiv k$ and a particular solution is

$$U(t) = -\left(\int \frac{\sin(kt)}{k} \sin(kt)\, dt\right) \cos(kt) + \left(\int \frac{\cos(kt)}{k} \sin(kt)\, dt\right) \sin(kt).$$

Now

$$\begin{aligned}\int \sin(kt)^2\, dt &= \int \sin(kt)\sin(kt)\, dt = -\frac{1}{k}\int \sin(kt)\,(\cos(kt))' = -\frac{1}{k}\left[\sin(kt)\cos(kt) - k\int \cos(kt)^2\, dt\right]\\ &= -\frac{1}{2k}\sin(2kt) + \int (1-\sin(kt)^2)\, dt = -\frac{1}{2k}\sin(2kt) + t - \int \sin(kt)^2\, dt\end{aligned}$$

and by this we have

$$\int \sin(kt)^2\, dt = \frac{t}{2} - \frac{\sin(2kt)}{4k}.$$

Moreover

$$\int \frac{\cos(kt)}{k}\sin(kt)\, dt = \frac{1}{2k} upupupupup \sin(2kt)\, dt = -\frac{\cos(2kt)}{4k^2}.$$

In conclusion

$$U(t) = \left(\frac{\sin(2kt)}{4k^2} - \frac{t}{2k}\right)\cos(kt) - \frac{\cos(2kt)}{4k^2}\sin(kt).$$

By this the conclusion is evident.

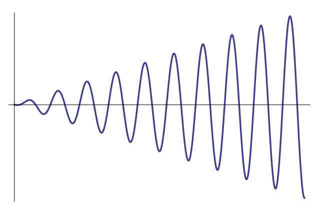


FIGURE 1. Plot of U.

1.5. Exercises

EXERCISE 1.5.1. Find the general integral of the following equations:

1. $y' + (\cos t)y = \frac{1}{2}\sin(2t),\ t \in \mathbb{R}$.
2. $y' - \frac{t}{1-t^2}y = t,\ t \in]-1, 1[$.
3. $y' + 2ty = 2t^3,\ t \in \mathbb{R}$.
4. $y' - \frac{1}{t}y + \frac{\log t}{t} = 0,\ t \in]0, +\infty[$.
5. $y' + (\tan t)y = t^3,\ t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.
6. $y' + 2ty = te^{-t^2},\ t \in \mathbb{R}$.
7. $y' + y = \sin t,\ t \in \mathbb{R}$.
8. $y' + (\cos t)y = (\cos t)^2,\ t \in \mathbb{R}$.
9. $y' = \frac{2t}{t^2+1}y + 2t(t^2+1),\ t \in \mathbb{R}$.

EXERCISE 1.5.2. Solve the Cauchy Problem

$$\begin{cases} y'(t) + \frac{3t^2}{t^3+5}y(t) = \sqrt[3]{t},\\ y(0) = 1.\end{cases}$$

Exercise 1.5.3. Consider the equation

$$y' - (\tan t)y = \frac{1}{\sin t}, \quad t \in \left]0, \frac{\pi}{2}\right[.$$

i) Find the general integral. ii) Is it true that for every solution $\lim_{t\to 0+} y(t) = -\infty$? iii) Are there solutions such that $\exists \lim_{t\to\frac{\pi}{2}-} y(t) \in \mathbb{R}$. In this case, what is the value of the limit?

Exercise 1.5.4. Consider the equation

$$y' + (\sin t)y = \sin t, \quad t \in \mathbb{R}.$$

i) Find its general integral. ii) Are there solutions y such that $\exists \lim_{t\to+\infty} y(t) \in \mathbb{R}$. iii) Find the solution of the Cauchy Problem $y\left(\frac{\pi}{2}\right) = 1$.

Exercise 1.5.5. Consider the equation

$$y'(t) = -\frac{1}{t} y(t) + \arctan t.$$

Find its general integral on $]-\infty, 0[$ and on $]0, +\infty[$. Does it exists a $y : \mathbb{R} \longrightarrow \mathbb{R}$ solution on both $]-\infty, 0[$ and $]0, +\infty[$. In this case, what is $y(0)$?

Exercise 1.5.6. Solve the Cauchy problems

1. $\begin{cases} y' = \dfrac{y^2 - y - 2}{3} \arcsin t, \\ y(0) = 3. \end{cases}$ 2. $\begin{cases} y' = \dfrac{y\sqrt{2y-1}}{\cosh t}. \\ y(0) = 1. \end{cases}$ 3. $\begin{cases} y' = \dfrac{\cos^2(2y)}{t(2 - \log^2 t)} \\ y(1) = \frac{\pi}{2}. \end{cases}$ 4. $\begin{cases} y' = \dfrac{(e^t + 1)y\sqrt{1-y}}{e^t + 2}, \\ y(0) = 1/2. \end{cases}$

Exercise 1.5.7. Find, in function of the initial condition $y(0) = y_0$ the solution of the Cauchy problem

$$\begin{cases} y' = 4y(1-y), \\ y(0) = y_0. \end{cases}$$

Plot quickly a qualitative graph of the various solutions.

Exercise 1.5.8. Consider the Cauchy problem

$$\begin{cases} y' = y(1-y^2), \\ y(0) = 1/2. \end{cases}$$

Determine the implicit form for the solution. Is it true that the solution is defined for all times $t \in \mathbb{R}$?

Exercise 1.5.9. For each of the following equations find a fundamental system of solutions and write the general integral.

1. $y'' - 3y' + 2y = 0$. 2. $y'' - 2y' + 2y = 0$. 3. $y'' - 4y + 3y = 0$. 4. $y'' + y' = 0$. 5. $y'' - y' + y = 0$.

Exercise 1.5.10. Find the general integral of the following equations:

1. $y''(t) + y'(t) - 6y(t) = 2e^{-t}$. 2. $y'' - y' + y = e^t$. 3. $y'' + 4y' + 2y = t^2$. 4. $y'' + 2y' = e^t$.

5. $y'' - y = \cos t$. 6. $y'' + y = \frac{1}{\cos t}$. 7. $y'' + 2y' + 2y = 2t + 3 + e^{-t}$. 8. $y'' - 2y' + 2y = e^t \cos t$.

Exercise 1.5.11. For each of the following equations find the general integral and the solution of the Cauchy Problem with initial conditions $y(0) = y'(0) = 0$.

1. $y'' - y = t$.
2. $y'' + 4y = e^t$.
3. $y'' + y = t$.
4. $y'' + y' - 6y = -4e^t$.
5. $y'' - 8y' + 17y = 2t + 1$.
6. $y'' + y = \frac{1}{\cos t}$.

Exercise 1.5.12. Consider the following differential equation

$$y''(t) - y'(t) = te^t, \quad t \in \mathbb{R}.$$

i) Find its general integral. ii) Are there solutions such that $\lim_{t\to+\infty} y(t) \in \mathbb{R}$? iii) Find the solution of the Cauchy Problem $y(0) = 1$, $y'(0) = 0$.

Exercise 1.5.13. Find the general integral of the equation

$$y''(t) - 5y'(t) - 6y(t) = 16e^{-2t},\ t \in \mathbb{R}.$$

Hence, say if there exists a solution such that $y(0) = 0$ and $\lim_{t\to+\infty} y(t) = 0$.

Exercise 1.5.14. Consider the equation

$$y''(t) + y'(t) = t + \cos t, \quad t \in \mathbb{R}.$$

Find its general integral. Say it there are solutions y such that $\exists \lim_{t\to+\infty} y(t) \in \mathbb{R}$. Say it there are solutions y such that $y(0) = 0$ and $\lim_{t\to-\infty} y(t) = +\infty$.

Exercise 1.5.15. Consider the equation

$$y''(t) + y(t) = \frac{1}{\cos t}, \quad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[.$$

Find its general integral. Is it true that any solution of the equation is such that $\lim_{t\to\frac{\pi}{2}-} y(t) = +\infty$? Say if there are solutions such that $y(t) \sim Ct^2$ per $t \to 0$ (for some $C \neq 0$).

Exercise 1.5.16. Find the general integral of the equation

$$y''(t) + 4y'(t) + 4 = \frac{e^{-2t}}{t^2},\ \ t \in]0, +\infty[.$$

Are there solutions of the equation such that $\exists \lim_{t\to0+} y(t)$?

Exercise 1.5.17. A radioactive material decay of 20% in 10 days. Find his halving time.

Exercise 1.5.18. In an hospital a radioactive substance is accumulated into a vessel at rate of 2m^3 each month. The radioactivity has a decay rate estimated to be proportional to the quantity present in the vessel according a constant of proportionality $k = -1$. Knowing that initially the vessel is empty find the total amount of radioactive substance contained when the vessel is full.

Exercise 1.5.19. The water in a pool with squared base of side 10m and depth 2m evaporates at rate of 5 liters each hour. The bottom of the pool is porous with a certain speed ph expressed in liter/minute, where p is a constant and h is the level of the water into the pool. Once the pool is full of water, it takes 24 hours in order to have the pool completely empty. The problem is: determine p.

Exercise 1.5.20. A vessel has capacity 10 liters. A valve is opened on the bottom of the vessel with flow per hour proportional with constant 3 to the $\sqrt[3]{}$ of the total volume present inside the vessel. If initially the vessel is full, compute how long it takes the vessel to be completely empty. Suppose now that a constant flux F of fluid is infused into the vessel. Are there values of F such that the fluid into the vessel reach, at long times, an equilibrium?

Exercise 1.5.21. The queue formed after a car accident on an highway reduces at some rate inversely proportional to the square root of the length of the queue with some constant c of proportionality. Knowing that to reduce to the half a queue of 1km it takes 10min, how long it takes to reduce to the half a queue of 2km? Do a queue reduces to 0 in a finite time?

Exercise 1.5.22. In a fish breeding the population of fishes is assumed to follows a logistic evolution

$$y'(t) = 0.1y(t) - by(t)^2,$$

where b is to be determined. You know that initially there're 500kg of fishes and after one year there're 1.250kg of fishes. Determine b.

Exercise 1.5.23 (⋆). A particle of mass m fall down under action of gravity and air friction in such a way that the equation of motion is

$$ma(t) = -mg - mkv(t).$$

Find an equation for v as function of the quote x and find $v = v(x)$. This is the case of a light particle. If we have an heavy particle the friction changes as $-mkv(t)^2$. What can you say in this case?

Exercise 1.5.24 (⋆⋆). A swimmer want to cross a river of section ℓ. He starting point and the arriving point are aligned orthogonally to the direction of the river. The water into the river flow at constant speed v. The swimmer want to follow a trajectory always directed to his destination with constant speed $V < v$. Does he will reach the other side of the river? Describe the trajectory of the swimmer through a suitable differential equation and find under which conditions the swimmer we be succeful.

Exercise 1.5.25 (⋆). A ship of mass m moves from rest under a constant propelling force mf and against a resistance mkv^2. Determine the speed $v = v(a)$ as function of the covered distance a. Suppose that, fixed a, the engines are reversed. What is the distance necessary to stop the ship?

Exercise 1.5.26. A particle of unit mass moves along the $x-$axis under the attraction of a forse of magnitude $4x$ towards the point $x = 0$ and a resistance equal in magnitude to twice the velocity. The particle is released at rest at $x = a$. Determine all the positions at instantaneous rest.

Exercise 1.5.27. A mass m is attached to two springs along the vertical with same elastic constant k. Initially the mass is at rest positions for the springs. Determine the motion of the mass taking account of gravity.

CHAPTER 2

Topology in $\mathbb{R}^d$

In a wide part of this course we will study Analysis in a multidimensional context,

$$\mathbb{R}^d := \{(x_1, \dots, x_d) \ : \ x_i \in \mathbb{R}, \ i = 1, \dots, d\} .$$

The basic tool of Mathematical Analysis is the concept of *limit* with its applications: continuous functions, differential calculus, integral calculus and more. The definition of limit passes through a suitable definition of *distance* between points. In $\mathbb{R}^d$ there's a natural notion of distance, namely the *Euclidean distance*

$$\mathrm{dist}(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_d - y_d)^2}, \ \ x = (x_1, \dots, x_d), \ y = (y_1, \dots, y_d).$$

The euclidean distance is actually a function of the difference between the coordinates of the two points x, y or, in other words, is translation invariant. The distance $d(x, 0)$ (where $0 = (0, \dots, 0)$ is the origin) can be interpreted as the length of a *vector* and plays the same role of the *modulus* for numbers. This is the fundamental concept of *norm* by wich we will begin.

2.1. Euclidean norm

We start recalling that $\mathbb{R}^d$ is a *vector space on* $\mathbb{R}$ with the operations of *sum* and *product*

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d), \ \ \lambda(x_1, \dots, x_d) := (\lambda x_1, \dots, \lambda x_d).$$

DEFINITION 2.1.1. *Given $x = (x_1, \dots, x_d) \in \mathbb{R}$ we call* **norm of** *x the quantity*

$$\|x\| := \sqrt{\sum_{i=1}^{d} x_i^2}.$$

The norm plays the same role of the modulus in $\mathbb{R}$. Precisely

PROPOSITION 2.1.2. *The norm fulfills the following properties:*

i) *positivity: $\|x\| \geqslant 0$, $\forall x \in \mathbb{R}^d$, and $\|x\| = 0$ iff $x = 0_d := (0, \dots, 0)$.*
ii) *homogeneity: $\|\lambda x\| = |\lambda| \|x\|$, $\forall \lambda \in \mathbb{R}$, $\forall x \in \mathbb{R}^d$.*
iii) *triangular inequality: $\|x + y\| \leqslant \|x\| + \|y\|$, $\forall \ x, y \in \mathbb{R}^d$.*

PROOF — $\|x\| \geqslant 0$ for eny $x \in \mathbb{R}^d$ is evident. Let's check the vanishing:

$$\|x\| = 0, \iff \sum_{i=1}^{d} x_i^2 = 0, \iff x_i^2 = 0, \forall i, \iff x_i = 0, \forall i.$$

The homogeneity is very easy:

$$\|\lambda x\| = \sqrt{\sum_i (\lambda x_i)^2} = \sqrt{\lambda^2 \sum_i x_i^2} = |\lambda| \sqrt{\sum_i x_i^2} = |\lambda| \|x\|.$$

Finally the triangular inequality: for convenience let's square everything and notice that

$$\|x+y\|^2 = \sum_i (x_i + y_i)^2 = \sum_i (x_i^2 + y_i^2 + 2x_i y_i) = \|x\|^2 + \|y\|^2 + 2\sum_i x_i y_i.$$

Lemma 2.1.3 (Cauchy–Schwarz inequality).

$$\sum_i x_i y_i \leqslant \left(\sum_i x_i^2\right)^{1/2} \left(\sum_i y_i^2\right)^{1/2}. \tag{2.1.1}$$

Proof — (Lemma) Excluding the trivial cases when $\|x\| = 0$ or $\|y\| = 0$ we may assume $\|x\|, \|y\| \neq 0$ and prove

$$\sum_i \frac{x_i}{\|x\|} \frac{y_i}{\|y\|} \leqslant 1.$$

This simple trick make easy to conclude: recall the elementary inequality

$$ab \leqslant \frac{1}{2}(a^2 + b^2), \ \left(\Longleftrightarrow \ 2ab \leqslant a^2 + b^2, \ \Longleftrightarrow \ (a-b)^2 \geqslant 0 \right).$$

Then using $a = \frac{x_i}{\|x\|}$ and $b = \frac{y_i}{\|y\|}$ we have

$$\sum_i \frac{x_i}{\|x\|} \frac{y_i}{\|y\|} \leqslant \frac{1}{2} \sum_i \left(\frac{x_i^2}{\|x\|^2} + \frac{y_i^2}{\|y\|^2} \right) = \frac{1}{2} \left(\frac{\|x\|^2}{\|x\|^2} + \frac{\|y\|^2}{\|y\|^2} \right) = 1. \quad \blacksquare$$

By Cauchy–Schwarz,

$$\|x+y\|^2 \leqslant \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2, \ \Longleftrightarrow \ \|x+y\| \leqslant \|x\| + \|y\|. \quad \blacksquare$$

Through the norm we define the concept of *limit for a sequence* and, later, the limit of a function:

Definition 2.1.4. *Let $(x_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that*

$$x_n \longrightarrow \xi \in \mathbb{R}^d, \ \Longleftrightarrow \ \|x_n - \xi\| \longrightarrow 0 \ (\textit{in } \mathbb{R}).$$

A little bit of care is needed for $x_n \longrightarrow \infty$ because, differently by $\mathbb{R}$, there's not a $+\infty$ and a $-\infty$:

Definition 2.1.5. *Let $(x_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that*

$$x_n \longrightarrow \infty_d, \ \Longleftrightarrow \ \|x_n\| \longrightarrow +\infty.$$

A *neighborhood*, that is *a set of points closed to a given point x*:

Definition 2.1.6. *Let $x \in \mathbb{R}^d$ and $r > 0$. We call*

closed ball centered in x **of radius** r *the set* $B(x,r] := \{y \in \mathbb{R}^d : \|y - x\| \leqslant r\}$,

open ball centered in x **of radius** r *the set* $B(x,r[:= \{y \in \mathbb{R}^d : \|y - x\| < r\}$.

Every set U_x containing a ball centered at $x \in \mathbb{R}^d$ its called **neightborhood of** x. *We call also* **neighborhood of** ∞_d *any set U_∞ containing the exterior of a ball, that is $U_{\infty_d} \supset B(0,r]^c$.*

Remark 2.1.7. As $d = 1$ we have $B(x,r] = [x-r, x+r]$; as $d = 2$, $B(x,r]$ is the disk centered in x with radius r. We used the square bracket in the notation $B(x,r]$ to recall that $B(x,r]$ contains all the points distant exactly r from the center (that is the "skin" of the ball). ■

Definition 2.1.8 (open set). *A set $S \subset \mathbb{R}^d$ is said*

- **open**, *if* $\forall\, x \in S\ \exists\, r(x) > 0\ :\ B(x, r(x)[\subset S$.
- **closed**, *if its complementary* $S^c = \mathbb{R}^d \backslash S$ *is open.*

By definition $\emptyset$ is assumed to be open.

Remark 2.1.9. A common error consists in thinking that *every set S is open or closed*. This a wrong idea probably due to the meaning of "closed" and "open" in the common language. For instance:

- in $\mathbb{R}$, $[a,b[$ is neither open or closed;
- $\emptyset$ is open by definition; according to the definition it is also closed: indeed $\emptyset^c = \mathbb{R}^d$ which is clearly open. Same for $\mathbb{R}^d$

There are no other simultaneously closed and open subsets in $\mathbb{R}^d$, but this is not easy to prove. ■

Proposition 2.1.10. *If $A, B \subset \mathbb{R}^d$ are open (closed) sets then $A \cup B, A \cap B$ are open (closed).*

Proof — Exercise. ■

An important characterization of closed sets is the following:

Theorem 2.1.11 (Cantor). *S is closed iff it contains all the finite limits of all its sequences that is*

$$S \text{ closed} \iff \forall (x_n) \subset S,\ :\ x_n \longrightarrow \xi \in \mathbb{R}^d, \text{ then } \xi \in S. \tag{2.1.2}$$

Proof — $\Longrightarrow$ Assume that S is closed and let's prove that if $(x_n) \subset S$ with $x_n \longrightarrow \xi \in \mathbb{R}^d$ then $\xi \in S$. Assume that this is false: then $\xi \in S^c$. But S^c is open (being S closed) and because $\xi \in S^c$,

$$\exists\, B(\xi, r[\subset S^c.$$

But $x_n \longrightarrow \xi$, that is $\|x_n - \xi\| \longrightarrow 0$ hence $\|x_n - \xi\| < r$ definitively: this means that $x_n \in B(\xi, r[\subset S^c$ definitively, and this is a contradiction being $(x_n) \subset S$.

$\Longleftarrow$ Assume the the property (2.1.2) is true and let's prove that S is closed, that is S^c is open. Take $\xi \in S^c$ and assume that, by contradiction,

$$\nexists\, B(\xi, r[\subset S^c.$$

Then

$$\forall r > 0,\ B(\xi, r[\not\subset S^c, \iff \forall r > 0,\ \exists x \in B(\xi, r[\ :\ x \in S.$$

Take $r = \frac{1}{n}$ and call $x_n \in B(\xi, \frac{1}{n}[$ such that $x_n \in S$. The sequence $(x_n) \subset S$ and because $\|x_n - \xi\| < \frac{1}{n} \longrightarrow 0$ we deduce $x_n \longrightarrow \xi$. But then, by (2.1.2), $\xi \in S$, and this is a contradiction. ■

Let's introduce two useful concepts:

Definition 2.1.12 (interior and boundary). *Let $S \subset \mathbb{R}^d$. We call*

- $\mathrm{Int}(S)$ (**interior of** S) *the set of points of $x \in S$ such that there exists U_x (neighborhood of x) such that $U_x \subset S$;*

- ∂S (**boundary of** S) *the set of points* $x \in \mathbb{R}^d$ *such that every neighborhood* U_x *of* x *contains point of S and of* S^c, *that is* $U_x \cap S \neq \emptyset$, $U_x \cap S^c \neq \emptyset$.

In particular, S is open if and only if $S = \text{Int}(S)$.

2.2. Limit

In this section we want to define the notion of limit

$$\lim_{x \to x_0} f(x) = \ell,$$

for a function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. As in one variable Calculus, to set this Defnition we need the concept *accumulation point*:

Definition 2.2.1. *Let* $S \subset \mathbb{R}^d$. *We say that*

- $\xi \in \mathbb{R}^d$ *is* **accumulation point for** S *if* $\exists (x_n) \subset S \backslash \{\xi\}$ *such that* $x_n \longrightarrow \xi$;
- ∞_d *is* **accumulation point for** S *if* $\exists (x_n) \subset S$ *such that* $x_n \longrightarrow \infty_d$.

The set of all accumulation points of S will be denoted by Acc(D).

By this and importing the same idea introduced for limits of one real variable functions we have the:

Definition 2.2.2. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ *and* $x_0 \in \text{Acc}(D)$. *We say that*

$$\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}^m \cup \{\infty_m\}, \iff f(x_n) \longrightarrow \ell,\ \forall (x_n) \subset D \backslash \{x_0\},\ x_n \longrightarrow x_0. \tag{2.2.1}$$

This Definition has the advantage to cover all the possibilities: limit at a finite point (when $x_0 \in \mathbb{R}^d$), at infinite (when $x_0 = \infty_d$) as well as finite limit (when $\ell \in \mathbb{R}^m$) or infinite limit ($\ell = \infty_m$). Despite this, the Definition is not helpful to compute practically a limit. Let's see some useful techniques.

2.2.1. Sections.

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a function such that $\lim_{x \to x_0} f(x) = \ell$. We may imagine that taking a "road" into D going to x_0, f will drive us just to ℓ. With "road" we mean a line in the space.

Definition 2.2.3 (curve). *A function* $\gamma : [a, b] \longrightarrow \mathbb{R}^d$ *is called* **curve** *in* D *if* $\gamma(t) \in D$ *for every* $t \in [a, b]$ *(notation* $\gamma \subset D$*). We call* **support** *of* γ *the set* $\text{Supp}(\gamma) := \gamma([a, b])$. *The curve* γ *is said to be* **continuous** *if* $\gamma \in \mathscr{C}([a, b])$.

Proposition 2.2.4. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x_0 \in \text{Acc}(D)$ *be such that* $\lim_{x \to x_0} f(x) = \ell$. *Let* $\gamma \subset D$ *be a curve such that* $\lim_{t \to t_0} \gamma(t) = x_0$ *and* $\gamma(t) \neq x_0$ *forall* t. *Then*

$$\lim_{t \to t_0} f(\gamma(t)) = \ell.$$

Proof — It's just an application of the definitions. Take $t_n \longrightarrow t_0$: then, because

$$\lim_{t \to t_0} \gamma(t) = x_0, \implies \gamma(t_n) \longrightarrow x_0.$$

By iii) we know that $x_n := \gamma(t_n) \neq x_0$ and we come to see that $x_n \longrightarrow x_0$. Therefore, by the Definition (2.2.1) $f(x_n) = f(\gamma(t_n)) \longrightarrow \ell$. So we proved that

$$\forall t_n \longrightarrow t_0, \implies f(\gamma(t_n)) \longrightarrow \ell,$$

and this is nothing but the conclusion. ■

COROLLARY 2.2.5. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x_0 \in \mathrm{Acc}(D)$. If there exists γ_1, γ_2 curves in D fulfilling ii) and iii) of the Proposition 2.2.4 and such that*

$$\lim_{t\to t_0} f(\gamma_1(t)) \neq \lim_{t\to t_0} f(\gamma_2(t))$$

then **the** $\lim_{x\to x_0} f(x)$ **doesn't exists**.

EXAMPLE 2.2.6. *Show that*

$$\lim_{(x,y)\to 0_2} \frac{xy}{x^2+y^2}$$

doesn't exists.

SOL. — Let

$$f(x,y) = \frac{xy}{x^2+y^2},\ (x,y) \in D = \mathbb{R}^2\backslash\{(0,0)\}.$$

Let's check what happens along the two sections along the axes. These are

$$f(t,0) = 0,\ \ f(0,t) = 0.$$

Here $\gamma_1(t) = (t,0) \longrightarrow (0,0)$ as $t \longrightarrow 0 =: t_0$ and clearly $\gamma_1(t) \neq (0,0)$ for all $t \neq t_0$. Hence

$$f(\gamma_1(t)) = f(t,0) = 0 \longrightarrow 0, \text{ as } t \longrightarrow 0.$$

Similarly $f(0,t) \longrightarrow 0$. Is this enough to conclude that the limit exists? NO! Because we checked just two of the infinitely many sections. Let consider a new section, that is a point moving along a straight line $y = mx$. The curve describing this is simply

$$\gamma(t) := (t, mt),\ m \in \mathbb{R}.$$

Notice that the corresponding section of f is

$$f(\gamma(t)) = f(t,mt) = \frac{mt^2}{t^2+m^2t^2} = \frac{m}{1+m^2} \longrightarrow \frac{m}{1+m^2}, \text{ as } t \longrightarrow 0.$$

We conclude that the behavior of f along the axes is different to that one along straight lines through the origin with angular coefficient $m \neq 0$. The limit doesn't exists. ■

EXAMPLE 2.2.7. *Show that*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

doesn't exists.

SOL. — Let

$$f(x,y) = \frac{xy^2}{x^2+y^4},\ (x,y) \in D = \mathbb{R}^2\backslash\{(0,0)\}.$$

The sections along the axes are $f(t,0) \equiv 0$ and $f(0,t) \equiv 0$. Notice that this says, in particular, that **if the limit exists, it must be equal to** 0. Now if we take a section along the line $y = mx$,

$$f(t,mt) = \frac{m^2t^3}{t^2+m^2t^4} = \frac{m^2t}{1+m^2t^2} \longrightarrow 0, \text{ as } t \longrightarrow 0.$$

So apparently again no contradictions! But if we consider the line $x = ay^2$ we have

$$f(at^2,t) = \frac{at^2t^2}{a^2t^4+t^4} = \frac{a}{a^2+1} \longrightarrow \frac{a}{a^2+1}, \text{ as } t \longrightarrow 0.$$

This is different from 0 if $a \neq 0$: so we have found a family of curves on which the limit of f exists but is different on any family: we deduce that the limit doesn't exists. ■

EXAMPLE 2.2.8. *Show that*

$$\lim_{(x,y)\to\infty_2} \left(x^2 + y^2 - 4xy\right)$$

doesn't exists.

SOL. — Let $f(x, y) := x^2 + y^2 - 4xy$. Sections along the axes are $f(t, 0) = t^2$, $f(0, t) = t^2$. Clearly the points $(t, 0), (0, t)$ go to ∞_2 iff $t \longrightarrow \pm\infty$. In any case $f(t, 0), f(0, t) \longrightarrow +\infty$. So the candidate to be the eventual limit is $+\infty$. However, along the line $y = x$,

$$f(t, t) = t^2 + t^2 - 4t^2 = -2t^2,$$

and because $(t, t) \longrightarrow \infty_2$ iff $t \longrightarrow \pm\infty$ we have immediately that $f(t, t) \longrightarrow -\infty$. We conclude that the limit doesn't exists. ■

We have seen then that sections may be used to

- *guess the possible limit* (because **if** the limit exists **then** along any section the limit exists and it is the same);
- *exclude existence of the limit* (if there're two different sections along which the limits are different the global limit cannot exists).

Of course to guess what the "right" sections are is not an easy business.

2.2.2. Methods of calculus for scalar functions. Sections are useful to find a candidate or to exclude existence of the limit, but are useless to prove that a function has a limit. In the following Example we will introduce an interesting method to answer to this problem.

EXAMPLE 2.2.9. *Compute*

$$\lim_{(x,y)\to 0_2} \frac{xy^2}{x^2 + y^2}.$$

SOL. — We have to begin with to guess a candidate. We remember that **if** the limit exists must coincide with the limit along any section. Now $f(x, 0) = 0 \longrightarrow 0$, so if the limit exists must be 0. This is confirmed, by the way, by the $y-$axis section $f(0, y) = 0$ and by sections along $y = mx$, because

$$f(x, mx) = \frac{xm^2x^2}{x^2 + m^2x^2} = x\frac{m^2}{1 + m^2} \longrightarrow 0, \quad x \longrightarrow 0.$$

Ok, if the limit exists it must be 0. How can we check that this is actually the case? Notice that by using polar coordinates

$$\begin{cases} x = \rho\cos\theta, \\ \\ y = \rho\sin\theta, \end{cases}$$

we have

$$f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho^3(\cos\theta)(\sin\theta)^2}{\rho^2} = \rho(\cos\theta)(\sin\theta)^2,$$

so

$$|f(\rho\cos\theta, \rho\sin\theta)| \leqslant |\rho(\cos\theta)(\sin\theta)^2| \leqslant \rho.$$

Returning to euclidean coordinates this last says that

$$|f(x,y)| \leqslant \|(x,y)\|.$$

It seems now evident that if $(x,y) \longrightarrow 0_2$, that is $\|(x,y)\| \longrightarrow 0+$ then, by some argument similar to the two-policemen thm, we should have also $|f(x,y)| \longrightarrow 0+$, that is $f(x,y) \longrightarrow 0$. ■

What is the argument invoked at the end of the previous example? We found a numerical function $g = g(\rho) : [0,+\infty[\longrightarrow \mathbb{R}$ such that $g(\rho) \longrightarrow 0$ as $\rho \longrightarrow 0$ and

$$|f(x,y)| \leqslant g(\|(x,y)\|).$$

It is clear that, as $\|(x,y)\| \longrightarrow 0$ (that is $(x,y) \longrightarrow 0_2$) by the two policemen Lemma we have easily that $f(x,y) \longrightarrow 0$. We can extend this to a more general setting:

Proposition 2.2.10. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. Suppose that there exist g such that*

i) $|f(x) - \ell| \leqslant g(\|x - \xi\|)$ *in some* $U_\xi \setminus \{\xi\}$*;*

ii) $\lim_{\rho\to 0+} g(\rho) = 0$.

Then $\exists \lim_{x\to\xi} f(x) = \ell$.

Proof — Let $x_n \longrightarrow \xi$, that is $\|x_n - \xi\| \longrightarrow 0$. Then

$$|f(x_n) - \ell| \leqslant g(\|x_n - \xi\|) \longrightarrow 0, \implies f(x_n) \longrightarrow \ell. \quad ■$$

Example 2.2.11. *Compute*

$$\lim_{(x,y,z)\to 0_3} \frac{\sin(xyz)}{x^2+y^2+z^2}.$$

Sol. — Let $f(x,y,z) := \frac{\sin(xyz)}{x^2+y^2+z^2}$ defined on its natural domain $D = \mathbb{R}^3\setminus\{0_3\}$. The sections on the axes $f(x,0,0) = f(0,y,0) = f(0,0,z)$ vanish, so the eventual candidate to be the limit is 0. Using spherical coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi, \end{cases}$$

we have

$$f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) = \frac{\sin\left(\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)\right)}{\rho^2}.$$

Recalling that $\sin(\xi) = \xi + o(\xi)$ we have

$$\begin{aligned} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi) &= \frac{\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)}{\rho^2} + \frac{o(\rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi))}{\rho^2} \\ &= \rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi) + o\left(\rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)\right). \end{aligned}$$

Clearly,

$$|\rho(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi)| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0+,$$

hence $o(\ldots) \longrightarrow 0$. Therefore the limit exists and is 0. ■

We have a similar strategy in the case $\ell = +\infty$ (or $-\infty$):

Proposition 2.2.12. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\xi \in \mathrm{Acc}(D)$. *Suppose that there exist* g *such that*

i) $|f(x)| \geqslant g(\|x-\xi\|)$ *in some* $U_\xi \backslash \{\xi\}$;

ii) $\lim_{\rho \to 0+} g(\rho) = +\infty$.

Then $\exists \lim_{x \to \xi} f(x) = +\infty$.

Proof — Exercise. ■

A final important case is when $x \longrightarrow \infty_d$. We just quote the following

Proposition 2.2.13. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\infty_d \in \mathrm{Acc}(D)$. *Suppose that there exist* g *such that*

i) $f(x) \geqslant g(\|x\|)$ *in some* U_{∞_d};

ii) $\lim_{\rho \to +\infty} g(\rho) = +\infty$.

Then $\exists \lim_{x \to \infty_d} f(x) = +\infty$.

Proof — Exercise. ■

Example 2.2.14. *Compute*

$$\lim_{(x,y) \to \infty_2} \left(x^4 + y^4 - xy \right).$$

Sol. — Looking at the sections along the axes we have $f(x,0) = x^4 \longrightarrow +\infty$ and $f(0,y) = y^4 \longrightarrow +\infty$. So, if the limit exists must be $+\infty$. This seems reasonable because $x^4 + y^4$ should dominate xy. In this case we need just a "lower" policemen $g = g(\rho)$ such that

$$f(\rho\cos\theta, \rho\sin\theta) \geqslant g(\rho) \longrightarrow +\infty, \ \rho \longrightarrow +\infty.$$

We have

$$f(\rho\cos\theta,\rho\sin\theta) = \rho^4(\cos\theta)^4 + \rho^4(\sin\theta)^4 - \rho^2(\cos\theta)(\sin\theta) = \rho^4\left[(\cos\theta)^4 + (\sin\theta)^4\right] - \frac{1}{2}\rho^2\sin(2\theta).$$

Now: notice that the quantity $K(\theta) := (\cos\theta)^4 + (\sin\theta)^4$ is always positive and has a minimum as $\theta \in [0, 2\pi]$. Indeed: we don't need any computation because K is clearly continuous, hence K has a minimum by Weierstrass's theorem. Moreover $K(\theta) = 0$ iff $\cos\theta = \sin\theta = 0$, and this in impossible. We call C the minimum value of K: $K(\theta) \geqslant C > 0$ for any $\theta \in [0, 2\pi]$. Recalling also that $|\sin(2\theta)| \leqslant 1$ we have

$$f(x,y) \geqslant C\rho^4 - \frac{1}{2}\rho^2\sin(2\theta) \geqslant C\rho^4 - \frac{1}{2}\rho^2 =: g(\rho) \longrightarrow +\infty.$$

By this the conclusion follows. ■

Example 2.2.15. *Compute*

$$\lim_{(x,y,z) \to \infty_3} \left[(x^2+y^2+z^2)^2 - xyz \right].$$

Sol. — A quick check on the sections along the axes show that they tend to $+\infty$. Again: it seems reasonable that the fourth order term $(x^2+y^2+z^2)^2$ dominates on xyz. Passing to spherical coordinates

$$f = (\rho^2)^2 - \rho^3(\cos\theta)(\sin\theta)(\sin\varphi)^2(\cos\varphi) = \rho^4 - \frac{1}{4}\rho^3(\sin(2\theta))(\sin(2\varphi))(\sin\varphi).$$

Now, because

$$|(\sin(2\theta))(\sin(2\varphi))(\sin\varphi)| \leqslant 1,$$

we have

$$f \geqslant \rho^4 - \frac{1}{4}\rho^3 =: g(\rho) \longrightarrow +\infty,$$

from which the conclusion follows. ■

Example 2.2.16. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2+y^2)^2 + z^2 - xy\right]$$

Sol. — Easily the sections are all convergent to $+\infty$ (e.g. $f(x,0,0) = x^4 \longrightarrow +\infty$ when $\|(x,0,0)\| = |x| \longrightarrow +\infty$). In this case it is convenient to introduce *cylindrical coordinates*

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases}$$

because $x^2 + y^2 = \rho^2$. But be careful: $(x,y,z) \longrightarrow \infty_3$ means $\|(x,y,z)\| = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} \longrightarrow +\infty$, and this doesn't mean necessarily that $\rho \longrightarrow +\infty$. However,

$$f_{cil} = (\rho^2)^2 + z^2 - \rho^2\cos\theta\sin\theta \geqslant \rho^4 + z^2 - \rho^2, \quad (|\cos\theta\sin\theta| \leqslant 1).$$

Now: if we had $f(x,y,z) \geqslant \rho^2 + z^2 = \|(x,y,z)\|^2$ we would be done. To this aim we may hope that $\rho^4 - \rho^2 \geqslant \rho^2$ and indeed this is actually true if ρ is big enough but not for every ρ. To get a lower bound true for any ρ we may notice that

$$\exists K \ : \ \rho^4 - \rho^2 \geqslant \rho^2 + K, \ \forall \rho.$$

Indeed: this is equivalent to say that $\rho^4 - 2\rho^2 \geqslant K$, that is the function $\rho \longmapsto \rho^4 - 2\rho^2$ is bounded below. But a quick check shows that this function has a global minimum: so, if we call K the minimum of the function $\rho \longmapsto \rho^4 - 2\rho^2$ we have the conclusion. ■

2.3. Continuity

One of the major application of the concept of limit is the definition of continuity:

Definition 2.3.1 (Continuous function). *Let $f : D \subset \mathbb{R}^d \to \mathbb{R}^m$, $x_0 \in D \cap \mathrm{Acc}(D)$. We say that*

$$f \textbf{ is continuous in } x_0 \textbf{ iff } \lim_{x\to x_0} f(x) = f(x_0).$$

If f is continuous in any point of D we say that f **is continuous over** *D and we write $f \in \mathscr{C}(D)$.*

The usual properties of continuity for one variable functions remain true. For instance: sum, difference and products of continuous functions at some point (or in some domain) is a continuous function at that point (or in that domain). The same for the ratio with the extra requirement that the denominator is different from 0. It is quite easy to prove (we omit this) that

any polynomial in the $(x_1, \ldots, x_d)$ variable is continuous on $\mathbb{R}^d$.

By polynomial we mean a finite sum of monomials of type

$$a x_1^{k_1} x_2^{k_2} \cdots x_d^{k_d}, \ \ k_1, \ldots, k_d \in \mathbb{N}, \ a \in \mathbb{R}.$$

Quite useful is the chain rule:

if f is continuous in x_0, g is continuous in $f(x_0)$ then $g \circ f$ is continuous in x_0.

For instance: *any continuous scalar function of a polynomial is continuous where defined.*

EXAMPLE 2.3.2. *Where is continuous the function* $f(x,y) := \log(1-x^2-y^2)$*?*

SOL. — The function is defined on

$$D = \left\{(x,y) \in \mathbb{R}^2 \ : \ 1-x^2-y^2 > 0\right\} = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 < 1\right\} = B(0,1[.$$

We may write $f = \log \circ p$ where p is the polynomial $p(x,y) := 1-x^2-y^2$. Therefore $f \in \mathscr{C}(B(0,1[)$. ■

EXAMPLE 2.3.3. *The euclidean norm is continuous on* $\mathbb{R}^d$.

SOL. — Remind that

$$\|x\| = \sqrt{x_1^2 + \ldots + x_d^2} \equiv \sqrt{\ } \circ p, \text{ where } p(x_1,\ldots,x_d) = x_1^2 + \ldots + x_d^2.$$

Now: $\sqrt{\ }$ is continuous where defined and $p \geqslant 0$. It follows $\|\cdot\| \in \mathscr{C}(\mathbb{R}^d)$. ■

It is easy to check that continuity *component wise*:

PROPOSITION 2.3.4. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $f = (f_1,\ldots,f_m)$. *Then* f *is continuous in* x_0 *iff any* f_j *is continuous in* x_0, $j = 1,\ldots,m$.

In particular, if A is a linear transformation, that is

$$x \longmapsto Ax = \begin{bmatrix} a_{11} & \ldots & a_{1d} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{md} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1d}x_d \\ \vdots \\ a_{m1}x_1 + \ldots + a_{md}x_d \end{pmatrix},$$

because every component is a first order polynomial, we get by the last proposition that A is continuous:

COROLLARY 2.3.5. *Any linear transformation* $T \in \mathscr{L}(\mathbb{R}^d;\mathbb{R}^m)$ *is continuous.*

2.4. Properties of continuous functions

2.4.1. Weierstrass Theorem. We recall that any $f \in \mathscr{C}([a,b];\mathbb{R})$ has minimum/maximum over $[a,b]$. This is the well known Wierstrass Thm. The conclusion is false if the interval $[a,b]$ is not closed and bounded. These two properties are the key properties to extend the Thm to the case of functions of several variables. We need first to give the

DEFINITION 2.4.1. *A set S is said* **bounded** *if*

$$\exists M, \ : \ \|x\| \leqslant M, \ \forall x \in S.$$

THEOREM 2.4.2 (WEIERSTRASS). *Any continuous function* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ *on a domain D closed and bounded has global minimum and global maximum on D, that is*

$$\exists\, x_{min}, x_{max} \in D, \ : \ f(x_{min}) \leqslant f(x) \leqslant f(x_{max}), \ \forall x \in D.$$

Weierstrass thm points out the importance of the class of closed and bounded subsets of $\mathbb{R}^d$:

DEFINITION 2.4.3. *A set* $S \subset \mathbb{R}^d$ *is called* **compact** *if it is closed and bounded.*

So we may quickly say that *continuous functions on compact sets have min/max*. If we remove compactness we cannot assure the existence of global extreme points. There're however cases when the domain D is still closed but *unbounded* (as for instance when $D = \mathbb{R}^d$) in which something can be said. We notice that

$$S \text{ unbounded} \iff \forall n\ \exists x_n \in S\ :\ \|x_n\| \geqslant n, \iff \exists (x_n) \subset S,\ x_n \longrightarrow \infty_d.$$

In particular, S is unbounded iff $\infty_d \in \mathrm{Acc}(S)$.

Corollary 2.4.4. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on D, closed and unbounded, such that*

$$\lim_{x \to \infty_d} f(x) = +\infty\ (-\infty).$$

Then f has a global minimum (maximum).

Proof — Fix a point $x_0 \in D$: by hypotheses, there exists R such that

$$f(x) \geqslant f(x_0) + 1,\ \forall x \in D\ :\ \|x\| \geqslant R.$$

Indeed: if such R wouldn't exists, for any $R = n \in \mathbb{N}$ then there should be a point $x_n \in D$ such that $f(x_n) \leqslant f(x_0)+1$ and $\|x_n\| \geqslant R = n$. But then $x_n \longrightarrow \infty_d$ hence by assumption $f(x_n) \longrightarrow +\infty$ which is impossible being $f(x_n) \leqslant f(x_0) + 1$ (that is bounded).

Now, with such R we can notice that **if** the minimum exists it must belongs to $D \cap B(0, R]$ and also that $x_0 \in B(0, R]$ (otherwise $f(x_0)$ should be grater than $f(x_0) + 1$ which is impossible). But $D \cap B(0, R]$ is closed (intersection of two closed set) and bounded (because contained in $B(0, R]$). Therefore, by Weierstrass's theorem applied to f on $D \cap B(0, R]$, there exists $x_{min} \in D \cap B(0, R]$ such that

$$f(x_{min}) \leqslant f(x),\ \forall x \in D \cap B(0, R].$$

In particular, also, $f(x_{min}) \leqslant f(x_0) < f(x_0) + 1 \leqslant f(x)$ for all $x \in D \cap B(0, R]^c$. By this follows that

$$f(x_{min}) \leqslant f(x),\ \forall x \in D. \quad \blacksquare$$

Remark 2.4.5. *Of course, because $\lim_{x \to \infty_d} f(x) = +\infty$ the function* **cannot have a maximum***!*

Example 2.4.6. *Show that the function $f(x, y) := x^4 + y^4 - xy$ has global minimum on $\mathbb{R}^2$. What about global maximum?*

Sol. — Of course $f \in \mathscr{C}(\mathbb{R}^2)$ (because it is a polynomial) and $\mathbb{R}^2$ is closed (its complementary is empty, so open by definition) and unbounded. We have also seen (see Example 2.2.14) that

$$\lim_{(x,y) \to \infty_2} f(x, y) = +\infty.$$

Therefore, by the Corollary of Weierstrass's thm we have that there exists a global minimum for f on $\mathbb{R}^2$. On the other side, because f is upper unbounded (by the limit at ∞_2) the global maximum doesn't exists. $\blacksquare$

2.4.2. Domains defined by continuous functions. Weiertrass' Thm shows how important is to say if a set S is closed or less. A natural way to define subsets in $\mathbb{R}^d$ is through equalities or inequalities: S is the intersection of the closed unit ball $\left\{x^2 + y^2 + z^2 \leqslant 1\right\}$ with the cylinder $\left\{(x-1)^2 + y^2 \leqslant \frac{1}{4}\right\}$ with axis parallel to z−axis passing at the point $(1/2, 0, 0)$ with radius $\frac{1}{2}$. A quite general setting is to define a set as intersection of sets of type

$$\{f \leqslant 0\},\ \text{ or }\ \{f = 0\},\ \text{ or }\ \{f < 0\}.$$

An important question is to know if these sets are open or closed.

FIGURE 1. The set $S := \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 \leqslant 1,\; \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\right\}$

THEOREM 2.4.7. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on R^d. Then*

i) $\{f < 0\}$ *is open;*
ii) $\{f \leqslant 0\}$ *and* $\{f = 0\}$ *are closed.*

PROOF — i) Let's show that $\{f < 0\}^c = \{f \geqslant 0\}$ is closed. To this aim let's use the characterization (2.1.2): let $(x_n) \subset \{f \geqslant 0\}$ such that $x_n \longrightarrow \xi \in \mathbb{R}^d$. The goal is to show that $\xi \in \{f \geqslant 0\}$. We know that

$$(x_n) \subset \{f \geqslant 0\}, \implies f(x_n) \geqslant 0.$$

But f is continuous at ξ hence $f(x_n) \longrightarrow f(\xi)$, and because $f(x_n) \geqslant 0$ for any n, by permanence of sign it follows that $f(\xi) \geqslant 0$, that is $\xi \in \{f \geqslant 0\}$, and this concludes the proof.
ii) The proof is similar to that one of the previous point. ■

COROLLARY 2.4.8. *If $f_1, \ldots, f_k \in \mathscr{C}(\mathbb{R}^d; \mathbb{R})$ then $\{f_1 \geqslant 0, \ldots, f_k \geqslant 0\}$ and $\{f_1 = 0, \ldots, f_k = 0\}$ are closed, $\{f_1 > 0, \ldots, f_k > 0\}$ is open.*

PROOF — Just each of the sets is intersection of sets like $\{f_j \geqslant 0\}$, $\{f_j = 0\}$ and $\{f_j > 0\}$. ■

2.5. Exercises

EXERCISE 2.5.1. Discuss the following questions:

i) $B(\xi, \rho]$ is closed.
ii) $S \subset \mathbb{R}^d$ is closed iff $\partial S \subset S$.
iii) S is open iff $S = \mathrm{Int}(S)$.
iv) $\mathrm{Int}(S)$ is open for every set $S \subset \mathbb{R}^d$.
v) A set S is open iff $\partial S \subset S^c$.
vi) Are there cases of sets S such that $\mathrm{Int}(S) = \emptyset$? And sets such that $\partial S = \emptyset$?
vii) Prove the proposition 2.1.10.

EXERCISE 2.5.2. Looking to suitable sections, prove that the following limits don't exist:

1. $\lim_{(x,y)\to 0_2} \frac{x^2 - y^2}{x^2 + y^2}$.
2. $\lim_{(x,y)\to 0_2} \frac{x^2 + y^3}{x^2 + y^2}$.
3. $\lim_{(x,y)\to 0_2} \frac{xy}{|x| + |y|}$.
4. $\lim_{(x,y)\to 0_2} \frac{y^2 - xy}{x^2 + y^2}$.
5. $\lim_{(x,y,z)\to 0_3} \frac{x + y^2 + z^3}{\sqrt{x^2 + y^2 + z^2}}$.
6. $\lim_{(x,y)\to 0_2} \frac{xy + \sqrt{y^2 + 1} - 1}{x^2 + y^2}$.
7. $\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^4 + y^2 + z^2}$.

EXERCISE 2.5.3. Compute the following limits:

1. $\lim_{(x,y)\to 0_2} \frac{xy}{\sqrt{x^2 + y^2}}$.
2. $\lim_{(x,y)\to 0_2} \frac{x^2 y^3}{(x^2 + y^2)^2}$.
3. $\lim_{(x,y)\to 0_2} \frac{x^3 - y^3}{x^2 + y^2}$.
4. $\lim_{(x,y)\to 0_2} \frac{x\sqrt{|y|}}{\sqrt[3]{x^4 + y^4}}$.
5. $\lim_{(x,y)\to 0_2} \frac{xy}{|x| + |y|}$.

EXERCISE 2.5.4. *An open ball is an open set.*

EXERCISE 2.5.5. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\lim_{(x,y)\to 0_2} \frac{e^{4y^3} - \cos(x^2 + y^2)}{x^2 + y^2}$.
2. $\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^2 + y^2 + z^2}$.
3. $\lim_{(x,y,z)\to 0_3} \frac{(x^2 + yz)^2}{\sqrt{(x^2 + y^2)^2 + z^4}}$.
4. $\lim_{(x,y)\to 0_2} \frac{\log(1 + 2x^3)}{\sinh(x^2 + y^2)}$.
5. $\lim_{(x,y)\to (0,1)} \frac{x^3 \sinh(y - 1)}{x^2 + y^2 - 2y + 1}$.
6. $\lim_{(x,y)\to (1,1)} \frac{(x - 1)^2 (y - 1)^7}{\left((x - 1)^2 + (y - 1)^2\right)^{5/2}}$.

EXERCISE 2.5.6. For each of the following limit, say if it exists (and in the case compute it) or less:

1. $\lim_{(x,y)\to \infty_2} \left(x^3 + xy^2 - y^2\right)$.
2. $\lim_{(x,y)\to \infty_2} (x^4 - y^4 + y^2 - x^2)$.
3. $\lim_{(x,y)\to \infty_2} \left(x^2 y^2 + x^2 + y^2 - xy\right)$.
4. $\lim_{(x,y,z)\to \infty_3} \left(x^4 + y^4 + z^4 - xyz\right)$.
5. $\lim_{(x,y,z)\to \infty_3} \left(x^2 + y^2 + z^4 - xz\right)$.
6. $\lim_{(x,y,z)\to \infty_3} \left(\sqrt{x^2 + y^2} + z^2 - z\right)$.
7. $\lim_{(x,y,z)\to \infty_3} \left(\sqrt{(x^2 + y^2)^2 + z^4} - xyz\right)$.

CHAPTER 3

Differential Calculus

In this Chapter we extend the *Differential Calculus* to the general setting of *vector valued functions of several variables*,

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m,$$

where D is some domain in $\mathbb{R}^d$. The case $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is very important in view of the *optimization problems*, that is *to find min/max of f on the set D*. This problem is one of the main reasons to introduce the Differential Calculus because the derivative should give informations useful to search extreme points.

The extension to the multi variable setting is not at all straightforward. Just beginning with the definition, in our setting we can't write

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

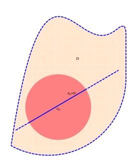
The problem is the domain: if the dimension $d > 1$, $x, h \in \mathbb{R}^d$ and we do not have an operation of "division" between vectors. As we will see, there're several possible definitions of derivative, but just precisely one is the more appropriate.

3.1. Directional derivative

In all the section we will assume that

$$\boxed{f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m,\ x \in \mathrm{Int}(D),\ d > 1}$$

Fix a vector $v \neq 0_d$. The set $\{x + tv \ : \ t \in \mathbb{R}\}$ is the straight line passing by x and with direction v. It is clear that being $x \in \mathrm{Int}(D)$ at least for t small $x + tv \in D$ (see the figure). Precisely: if $B(x, r] \subset D$ then $x + tv \in B(x, r]$ iff $\|tv\| \leqslant r$, iff $|t| \leqslant \frac{r}{\|v\|}$. Now, in a natural way we define



Definition 3.1.1. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \mathrm{Int}(D)$. We call* **directional derivative of f in the point x along** $v \neq 0_d$ *the limit (if it exists finite)*

$$D_v f(x) := \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.$$

The directional derivative works as an ordinary derivative but, unfortunately, is not a good concept for derivative: it may happens that all the $D_v f$ exists but f is not even continuous!

EXAMPLE 3.1.2. *Let*

$$f(x,y) := \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq 0_2, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

Then f has all the directional derivatives in the point 0_2 but it is not continuous in 0_2.

SOL. — Let's start by the continuity. Looking at the sections along axes we have $f(x,0) = f(0,y) \equiv 0 \longrightarrow 0$. But along the section $y = x^2$ we have

$$f(x,x^2) = \frac{x^2 x^2}{x^4 + x^4} = \frac{1}{2} \longrightarrow \frac{1}{2} \neq f(0,0) = 0.$$

Therefore $\nexists \lim_{(x,y)\to 0_2} f(x,y)$ and consequently the function cannot be continuous! Let's prove now that $\exists D_v f(0,0)$ for any v. Let $v = (a,b) \neq 0_2$. We have

$$D_v f(0,0) = \lim_{t\to 0} \frac{f((0,0) + t(a,b)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(ta,tb)}{t} = \lim_{t\to 0} \frac{\frac{t^3 a^2 b}{t^2(t^2 a^4 + b^2)}}{t} = \lim_{t\to 0} \frac{a^2 b}{t^2 a^4 + b^2},$$

that is

$$D_v f(0,0) = \begin{cases} 0, & \text{if } b = 0 \text{ (and of course } a \neq 0\text{)}, \\ \\ \frac{a^2}{b^2}, & \text{if } b \neq 0. \end{cases} \qquad \blacksquare$$

Directional derivatives are just variations on one variable derivatives. A particularly important case is the following:

DEFINITION 3.1.3 (PARTIAL DERIVATIVE). *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \text{Int}(D)$ and let $e_1, \dots, e_d$ be the canonical base of $\mathbb{R}^d$, that is $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the $j-$th place. If it exists, we define* **partial derivative of f with respect to the $j-$th variable in the point** *x the*

$$\partial_j f(x) := D_{e_j} f(x).$$

REMARK 3.1.4. *Partial derivative ∂_j is nothing but an ordinary derivatives w.r.t x_j considering all other variables x_i $i \neq j$ as fixed parameters.* Indeed

$$\begin{aligned} \partial_j f(x) &= \lim_{t\to 0} \frac{f\left((x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) + t(0, \dots, 0, 1, 0, \dots, 0)\right) - f(x_1, \dots, x_d)}{t} \\ &= \lim_{t\to 0} \frac{f\left(x_1, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_d\right) - f(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d)}{t} \end{aligned}$$

So, for instance

$$\partial_x (y \sin x) = y \cos x, \quad \partial_y (y \sin x) = \sin x. \quad \blacksquare$$

3.2. Differential

Let's take the definition of derivative from another side. Notice that, for real functions of real variable,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \iff \lim_{h\to 0}\frac{f(x+h)-f(x)-f'(x)h}{h} = 0,$$

that is

$$f(x+h)-f(x)-f'(x)h = o(h).$$

If $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $f'(x)$ is a number and $f'(x)h$ is the algebraic product between $f'(x)$ and h. If now $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ we expect that

i) $f'(x)h$ is of the same nature of $f(x+h)$ and $f(x)$, that is $f'(x)h\mathbb{R}^m$ (this because the quantity $f(x+h)-f(x)-f'(x)h$ should make sense);
ii) $f'(x)h$ depends linearly by h.

In other words, $f'(x)$ should be something that works linearly on vectors $h \in \mathbb{R}^d$ to vectors of $f'(x)h\mathbb{R}^m$. The natural object for $f'(x)$ is therefore an $m \times d$ *matrix*. This motivates the

Definition 3.2.1. *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $x \in \mathrm{Int}(D)$. We say that f* **is differentiable in** *x iff there exists a $m \times d$ matrix denoted by $f'(x)$ and called* **jacobian matrix** *such that*

$$f(x+h)-f(x)-f'(x)h = o(h), \tag{3.2.1}$$

in the sense that

$$\lim_{h\to 0}\frac{\|f(x+h)-f(x)-f'(x)h\|}{\|h\|} = 0. \tag{3.2.2}$$

The natural question is: *what are the entries of the jacobin matrix?*

Proposition 3.2.2. *If f is differentiable in x then there exists all the directional derivatives of f in x and*

$$D_v f(x) = f'(x)v, \ \forall v \in \mathbb{R}^d. \tag{3.2.3}$$

In particular, if the components of f are $f = (f_1, \dots, f_m)$ then

$$f'(x) = \begin{bmatrix} \partial_1 f_1(x) & \partial_2 f_1(x) & \dots & \partial_d f_1(x) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_m(x) & \partial_2 f_m(x) & \dots & \partial_d f_m(x) \end{bmatrix}. \tag{3.2.4}$$

Proof — Let's prove the (3.2.3). Fix $v \neq 0$. Then, by (3.2.2)

$$f(x+tv)-(f(x)+f'(x)(tv)) = o(tv), \implies \frac{f(x+tv)-f(x)}{t} = f'(x)v + \frac{o(tv)}{t}.$$

Now, because

$$\lim_{t\to 0}\left\|\frac{o(tv)}{t}\right\| = \lim_{t\to 0}\frac{\|o(tv)\|}{|t|} = \|v\|\lim_{t\to 0}\frac{\|o(tv)\|}{\|tv\|} = \|v\|\lim_{h\to 0}\frac{\|o(h)\|}{\|h\|} = 0,$$

we conclude that

$$\lim_{t\to 0}\frac{f(x_0+tv)-f(x_0)}{t} = f'(x_0)v.$$

Let's prove now the (3.2.4): if we call $f'(x_0) = [a_{ij}]$, it is well known by Linear Algebra that

$$f'(x_0)e_j$$

gives the $j-$th column of the matrix $f'(x_0)$. So the element a_{ij} of $f'(x_0)$ is obtained by taking the $i-$th component of the vector $f'(x_0)e_j$. But: by (3.2.3) we have

$$f'(x_0)e_j = D_{e_j} f(x_0) = \partial_j f(x_0) = \left(\partial_j f_1(x_0), \partial_j f_2(x_0), \dots, \partial_j f_m(x_0)\right),$$

hence the $i-$th component is $\partial_j f_i(x_0)$, and this proves (3.2.4). ■

REMARK 3.2.3. *In particular, differentiable $\Longrightarrow$ directionally derivable $\Longrightarrow$ partially derivable.* ■

Two important cases are

- $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$: in this case $f'(x)$ is a $1 \times d$ matrix, precisely
$$f'(x) = [\partial_1 f(x)\ \partial_2 f(x)\ \dots\ \partial_d f(x)] =: \nabla f(x),$$
called **gradient of** f **in** x. In this case
$$f'(x)h = \nabla f(x) \cdot h,$$
where we denoted by $\cdot$ the scalar product of vectors in $\mathbb{R}^d$.
- $\gamma : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d$: in this case $\gamma'(t)$ is a $d \times 1$ matrix, precisely
$$\gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_d'(t) \end{bmatrix}.$$

EXAMPLE 3.2.4. *Discuss the differentiability at* $(0, 0)$ *of*

$$f(x, y) := \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0), \\ \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

SOL. — We know that **the** candidate for $f'(0, 0,) = \nabla f(0, 0)$ if it exists. Notice that we cannot simply compute partial derivatives and evaluate in $(0, 0)$ because, for instance,

$$\partial_x f(x, y) = \partial_x \frac{x^2y^2}{x^2 + y^2} = \frac{2xy^2(x^2 + y^2) - x^2y^2 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2},$$

is of course not defined in $(0, 0)$. In this case we have to proceed directly in the computation of $\partial_x f(0, 0)$, that is

$$\partial_x f(0, 0) = D_{(1,0)} f(0, 0) = \lim_{t \to 0} \frac{f((0, 0) + t(1, 0)) - f(0, 0)}{t} = \lim_{t \to 0} \frac{f(t, 0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0,$$

and similarly $\partial_y f(0, 0) = 0$. We deduce $\nabla f(0, 0) = (0, 0)$. *What it remains to do?* We have to check that

$$f(0_2 + h) - f(0_2) - \nabla f(0_2) \cdot h = o(h), \iff \lim_{h \to 0_2} \frac{\|f(0_2 + h) - f(0_2) - \nabla f(0_2) \cdot h\|}{\|h\|} = 0.$$

Now: call $h = (u, v)$:

$$f(0_2 + h) - f(0_2) - \nabla f(0_2) \cdot h = f(u, v) - 0 - (0, 0) \cdot (u, v) = f(u, v).$$

We have therefore to prove that

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\|(u,v)\|} = 0.$$

This is a limit in $\mathbb{R}^2$ that we will compute by using the methods of previous chapter. Notice that

$$\frac{f(u,v)}{\|(u,v)\|} = \frac{\frac{u^2v^2}{u^2+v^2}}{\sqrt{u^2+v^2}} = \frac{u^2v^2}{(u^2+v^2)^{3/2}} \overset{u=\rho\cos\theta,\ v=\rho\sin\theta}{=} \frac{\rho^4(\cos\theta)^2(\sin\theta)^2}{\rho^3} = \rho(\cos\theta)^2(\sin\theta)^2,$$

hence

$$\left|\frac{f(u,v)}{\|(u,v)\|}\right| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0.$$

This finishes the exercise and prove that f is differentiable in 0_2 and $f'(0_2) = (0,0)$. ■

To check differentiability by using the definition is long and complex. A useful test is the following

THEOREM 3.2.5 (TOTAL DIFFERENTIAL THEOREM). *Let $f = (f_1, \dots, f_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, D open. If*

$$\partial_j f_i \in \mathscr{C}(D),\ \forall i, j,\ \Longrightarrow\ \exists f \text{ is differentiable in any } x \in D$$

A function f fulfilling this hypothesis is called a $\mathscr{C}^1(D)$ function.

Above we proved that to be differentiable implies that all the directional derivatives exists. Differentiability is actually a stronger concept. This follows as by product of the following

PROPOSITION 3.2.6. *If f is differentiable at x it is therein continuous.*

PROOF — By (3.2.2) we have $f(y) = f(x) + f'(x)(y-x) + o(y-x) \longrightarrow f(x)$, as $y \longrightarrow x$. ■

The rules of calculus of differentials basically are the same of those of ordinary calculus. For instance

$$(f+g)'(x) = f'(x) + g'(x).$$

if f, g are differentiable at x. Similarly it holds the important

THEOREM 3.2.7 (CHAIN RULE). *Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $g : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$, $x \in \mathrm{Int}(D)$ and $f(x) \in \mathrm{Int}(E)$ such that $\exists\, f'(x)$ and $\exists\, g'(f(x))$. Then*

$$\exists\,(g\circ f)'(x) = g'(f(x))f'(x). \tag{3.2.5}$$

A special important case of (3.2.5) is the following: suppose we want to compute

$$\frac{d}{dt}g(\gamma(t)), \text{ where } \gamma : I \subset \mathbb{R} \longrightarrow \mathbb{R}^m,\ \ g : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}.$$

Assuming all the hypotheses fulfilled we have

$$\frac{d}{dt}g(\gamma(t)) = g'(\gamma(t))\gamma'(t) = [\partial_1 g(\gamma(t))\ \dots\ \partial_m g(\gamma(t))]\begin{bmatrix}\gamma_1'(t)\\ \vdots \\ \gamma_m'(t)\end{bmatrix} = \nabla g(\gamma(t))\cdot\gamma'(t). \tag{3.2.6}$$

This is also called *total derivative of g along γ*.

3.3. Extrema

In this section we study the problem of finding min/max points of a function. Let's introduce first the important

DEFINITION 3.3.1. *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. *We say that a point* $x_0 \in D$ *is*

- **global maximum (minimum) of** f **on** D *if* $f(x) \leqslant f(x_0)$ *(*$f(x) \geqslant f(x_0)$*) for any* $x \in D$*;*
- **local maximum (minimum) of** f **on** D *if there exists a neighborhood* $U_{x_0} \subset D$ *of* x_0 *such that* $f(x) \leqslant f(x_0)$ *(*$f(x) \geqslant f(x_0)$*) for any* $x \in U_{x_0}$.

The goal of this section is to show how differential calculus can be used to determine local min/max points. Let's start by the extension of a well known property of Calculus in one variable:

THEOREM 3.3.2 (FERMAT). *Let* $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ *and* $x_0 \in \text{Int}(D)$ *be a local min/max. If* f *is differentiable at* x_0 *then* $\nabla f(x_0) = 0$.

PROOF — The idea is quite easy: if x_0 is (for instance) a local maximum for f, it is also local maximum on any section. Let's translate formally this idea: assuming, for instance, that x_0 is a local maximum for f on D,

$$\exists\, r > 0, : \; f(x) \leqslant f(x_0), \; \forall x \in B(x_0, r] \cap D.$$

Because $x_0 \in \text{Int}(D)$ we may assume directly that $B(x_0, r] \subset D$. Now, consider f on a straight line passing by x_0 with direction v: $f(x_0 + tv)$. Once $x_0 + tv \in B(x_0, r]$ (and this happens iff $|t| \leqslant \frac{r}{\|v\|}$) we will have

$$f(x_0 + tv) \leqslant f(x_0), \; \forall t \in I_0 := \left[-\frac{r}{\|v\|}, \frac{r}{\|v\|} \right].$$

This condition says that $t \longmapsto f(x_0 + tv)$ has a maximum at $t = 0$. By classical Fermat thm for one variable case, we deduce that

$$0 = \frac{d}{dr} f(x_0 + tv)\Big|_{t=0} \overset{(3.2.6)}{=} \nabla f(x_0) \cdot v.$$

This must be true for any $v \in \mathbb{R}^d$, that is $\nabla f(x_0) \cdot v = 0$, $\forall v \in \mathbb{R}^d$. This means that $\nabla f(x_0)$ is orthogonal to any vector v of $\mathbb{R}^d$. Hence $\nabla f(x_0) = 0$. ■

Be careful! First: the condition $\nabla f(x_0) = 0$ works only if $x_0 \in \text{Int}(D)$.

EXAMPLE 3.3.3. *Let* $f(x, y) = x^2 + y^2$ *on* $D = \{x^2 + y^2 \leqslant 1\}$. Clearly $\min_D f = 0$ and the minimum is attained at $(0, 0)$, $\max_D f = 1$ and every point on $\{x^2 + y^2 = 1\}$ is a maximum point for f on D. Of course these points are not in $\text{Int}(D) = \{x^2 + y^2 < 1\}$ and being $\nabla f(x, y) = (2x, 2y)$ we see that $\nabla f(x, y) = 0_2$ iff $(x, y) = (0, 0)$, so $\nabla f \neq 0$ at every $(x, y) \in \{x^2 + y^2 = 1\}$. ■

Second: the condition $\nabla f = 0$ not necessarily identifies min/max.

EXAMPLE 3.3.4. *Let* $f(x, y) = x^2 - y^2$ *on* $D = \mathbb{R}^2$. Clearly, $\nabla f(x, y) = (2x, -2y)$, therefore $\nabla f(0, 0) = (0, 0)$. However $(0, 0)$ is not an extremum because if we section f along the x axis we get $f(x, 0) = x^2$, and this says that $(0, 0)$ is a minimum (even global) for this section, whereas when we take the y axis section $f(0, y) = -y^2$ we have that $(0, 0)$ is a global maximum for the section. Again, taking $f(x, x) = 0$ we get that the function is constant! ■

There's no surprise with the previous example because the same happens for functions of real variable. For instance, $f(x) = x^2$ on $[-1,1]$ has max at $x = \pm 1$ but $f'(\pm 1) \neq 0$ and $f(x) = x^3$ on $\mathbb{R}$ has no min/max but $f'(0) = 0$. In particular, points where $\nabla f = 0$ are not necessarily min/max, but we will give to them a name:

DEFINITION 3.3.5. *If* $\nabla f(x_0) = 0$ *we say that* x_0 **is a stationary point for** f.

EXAMPLE 3.3.6. *Let*

$$f(x,y) = xye^{x^2+y^2}, \quad (x,y) \in D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2 + y^2 \leqslant 1\}.$$

Draw D*. What can you say about* D*: is open? closed? compact? connected? Show that* f *admits global extrema on* D *and find these points. Finally, determine* $f(D)$*.*

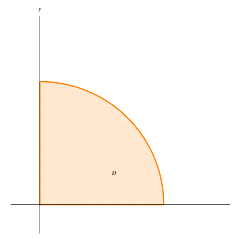


FIGURE 1. The boundary of D is colored in thick orange.

SOL. — Being D defined by large inequalities it is closed. It is not open because the points on the boundary belongs to D. Moreover, D is a subset of unit disk, therefore is bounded, hence is compact. A picture of D is easy.

Existence. Because f is continuous and D compact, Weierstrass's thm says that f has global min and max.

Determination. Because D is not open the argument (which is, by the way, a standard argument) requires some care. Assume (x,y) be a min/max. We have two cases: either $(x,y) \in \text{Int}(D)$ (hence it is necessarily a stationary point according to Fermat Theorem) or $(x,y) \in \partial D$ (then it is not necessarily a stationary point and we have to discuss directly).

If $(x,y) \in \text{Int}(D)$ then, by Fermat thm, $\nabla f(x,y) = 0$. We have

$$\nabla f(x,y) = \left(e^{x^2+y^2}(y + 2x^2 y), e^{x^2+y^2}(x + 2y^2 x)\right) = 0, \iff \begin{cases} y(1+2x^2) = 0, \\ \\ x(1+2y^2) = 0, \end{cases} \iff x = y = 0.$$

But this means that there aren't stationary points of f in $\text{Int}(D)$. In particular, the extrema belongs to ∂D.

Hence, $(x,y) \in \partial D$. We have to proceed by direct inspection. First of all

$$\partial D = \{(x,0) \ : \ 0 \leqslant x \leqslant 1\} \cup \{(0,y) \ : \ 0 \leqslant y \leqslant 1\} \cup \left\{(x,y) \ : \ x^2 + y^2 = 1,\ x \geqslant 0,\ y \geqslant 0\right\} =: A \cup B \cup C.$$

On A we have $f(x,0) = 0$, so f is constant; on B we have the same $f(0,y) = 0$. Let see what happens on C. It's better to describe C in the way

$$C = \left\{(\cos\theta, \sin\theta) \ : \ \theta \in \left[0, \frac{\pi}{2}\right]\right\}.$$

In this way

$$f(\cos\theta, \sin\theta) = (\cos\theta)(\sin\theta)e^1 = \frac{e}{2}\sin(2\theta),$$

and clearly this quantity is maximum as $2\theta = \frac{\pi}{2}$, that is $\theta = \frac{\pi}{4}$, that is in the point $\frac{1}{\sqrt{2}}(1,1)$, whereas the minimum is 0 (as $\theta = 0, \frac{\pi}{2}$, corresponding to the points $(1,0)$ and $(0,1)$). In conclusion: on A and B f is constant 0 whereas on C the minimum is 0 (taken on $(1,0) \in A$ and $(0,1) \in B$) whereas the maximum is $\frac{e}{2}$ reached in the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The moral is:

$$\min_D f = 0, \text{ minimum points: } A \cup B, \quad \max_D f = \frac{e}{2}, \text{ maximum points: } \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

Finally: because D is connected, $f(D)$ is connected in $\mathbb{R}$, hence interval. Of course, by what we have said before, $f(D) = [0, \frac{e}{2}]$. ■

Example 3.3.7. *Find the eventual global extrema of the function*

$$f : D = B(0,1] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) := \sqrt{|x+y|}e^{-(x^2+y^2)}.$$

Sol. — *Existence.* Notice first that $f \in \mathscr{C}(D)$ and clearly D is compact. By Weierstrass's Thm f admits global min/max.

Determination. Let $(x,y) \in D$ be an extreme point. We have the following alternative: either $(x,y) \in \text{Int}(D)$ or $(x,y) \in \partial D$.

Case $(x,y) \in \text{Int}(D)$. If f is differentiable at (x,y) then $\nabla f = 0$. We have to specify *if* f is differentiable because f is not differentiable where $x + y = 0$ (because of the term $|x+y|$). On that points however $f(x,y) \equiv 0$. If $x + y \neq 0$ we need $\nabla f = 0$. Let's compute the gradient.

$$\partial_x f = \frac{\text{sgn}(x+y)}{2\sqrt{|x+y|}}e^{-(x^2+y^2)} + \sqrt{|x+y|}e^{-(x^2+y^2)}(-2x) = \frac{e^{-(x^2+y^2)}}{2\sqrt{|x+y|}}\left(\text{sgn}(x+y) - 4x|x+y|\right),$$

$$\partial_y f = \frac{e^{-(x^2+y^2)}}{2\sqrt{|x+y|}}\left(\text{sgn}(x+y) - 4y|x+y|\right).$$

Then $\nabla f(x,y) = 0_2$ iff

$$\begin{cases} \text{sgn}(x+y) - 4x|x+y| = 0, \\ \\ \text{sgn}(x+y) - 4y|x+y| = 0, \end{cases} \iff \begin{cases} \text{sgn}(x+y)(1 - 4x(x+y)) = 0, \\ \\ \text{sgn}(x+y)(1 - 4y(x+y)) = 0, \end{cases} \iff \begin{cases} x(x+y) = \frac{1}{4}, \\ \\ y(x+y) = \frac{1}{4}. \end{cases}$$

By this, easily $x, y \neq 0$ and $\frac{x}{y} = 1$, that is $x = y$, and plugging again into the system be get $x(2x) = \frac{1}{4}$, that is $x^2 = \frac{1}{2}$, $x = \pm\frac{1}{\sqrt{2}}$. Therefore we find

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \in \text{Int}(D) \backslash \{y = -x\}.$$

Notice that

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{\frac{2}{\sqrt{2}}}e^{-\frac{1}{2}} = \frac{\sqrt[4]{2}}{\sqrt{e}}.$$

Case $(x,y) \in \partial D = \{x^2 + y^2 = 1\} \cup \{y = -x\}$**.** On $\{y = -x\}$ we have $f(x,-x) = 0$, and because $f \geqslant 0$ clearly for every $(x,y) \in \mathbb{R}^2$, the points $\{y = -x\}$ are surely global minimum points; on $\{x^2 + y^2 = 1\}$ it is convenient to use the standard parametrization: we have

$$f(\cos\theta, \sin\theta) = \sqrt{|\cos\theta + \sin\theta|}e^{-1}.$$

It is easy to check that $\cos\theta + \sin\theta$ gets its maximum value as $\theta = \frac{\pi}{4}, \frac{3}{4}\pi$. Therefore

$$\max_{x^2+y^2=1} f(x,y) = \sqrt{\frac{2}{\sqrt{2}}}e^{-1} = \sqrt[4]{2}e^{-1} < \sqrt[4]{2}e^{-1/2} = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

By this follows that $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ e $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are global maximum. ■

3.4. Exercises

Exercise 3.4.1. Compute the following directional derivatives (if they exists):

1. $D_{(\sqrt{3},1)} \log(1 + x^2y^2)$, at $(1,1)$. 2. $D_{(2,2)} \arctan(x + y)$, at $(1,0)$. 3. $D_{(1,1)} \dfrac{x^2y}{|x| + y^2}$, at $(0,0)$.

4. $D_{(-1,1)} \dfrac{xy}{x^2 + y^4}$, at $(0,0)$. 5. $D_{(-1,-2)} \dfrac{y(e^x - 1)}{\sqrt{x^2 + y^2}}$, at $(0,0)$.

Exercise 3.4.2. For each of the following functions say if a) is continuous at point $(0,0)$; ii) there exist $\partial_x f(0,0)$, $\partial_y f(0,0)$; iii) is differentiable in $(0,0)$.

1. $f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$ 2. $f(x,y) := \begin{cases} \frac{x^2+y^2}{|x|+|y|}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

3. $f(x,y) := \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$ 4. $f(x,y) := \begin{cases} \frac{x^2y}{x^2+y^2} + x - y, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

Exercise 3.4.3. Show that the function $f(x,y) = x\sqrt{x^2 + y^2}$, $(x,y) \in \mathbb{R}^2$ is differentiable on $\mathbb{R}^2$.

Exercise 3.4.4. Determine the stationary points of each of the following functions:

1. $f(x,y) = xy(x + 1)$. 2. $f(x,y) = x^2 + y^2 + xy$. 3. $f(x,y) = x^3 + y^3 + 2x^2 + 2y^2 + x + y$.

4. $f(x,y) = xe^y + ye^x$. 5. $f(x,y,z) = (x^3 - 3x - y^2)z^2 + z^3$.

Exercise 3.4.5. For each of the following functions a) find the stationary points, b) find eventual min/max on the domain, c) find the image of the domain.

(1) $f(x,y) = x^4 + y^4 - xy$, on $D = \mathbb{R}^2$.
(2) $f(x,y) = x\left((\log x)^2 + y^2\right)$ on $D =]0,+\infty[\times\mathbb{R}$.
(3) $f(x,y) = xy(x + y)$, on $D = \mathbb{R}^2$.
(4) $f(x,y,z) = x^2 + 3y^2 + 2z^2 - 2xy + 2xz$ on $D = \mathbb{R}^3$.
(5) $f(x,y,z) = x^4 + y^4 + z^4 - xyz$, on $D = \mathbb{R}^3$.

Exercise 3.4.6. Let

$$f(x,y) = (x^2 + y^2)^2 - 3x^2y, \ (x,y) \in \mathbb{R}^2.$$

i) Compute (if it exists) $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find stationary points of f. iii) Find eventual global min/max of f on $\mathbb{R}^2$ and find $f(\mathbb{R}^2)$.

EXERCISE 3.4.7. Let

$$f(x,y,z) := (x^2+y^2+z^2)^2 - xyz,\ (x,y,z) \in \mathbb{R}^3.$$

i) Show that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. ii) Find stationary points of f. iii) Show that f has global minimum on $\mathbb{R}^3$ and find $f(\mathbb{R}^3)$.

EXERCISE 3.4.8. Let $f(x,y) := x^2(1-y)$ on $D := \{(x,y) \in \mathbb{R}^2 \ :\ x^2 + |y| \leqslant 4\}$. Study the sign of f, determine its eventual stationary points on D and find min/max of f on D.

EXERCISE 3.4.9. Find the extrema of $f(x,y) := xye^{-xy}$ on $D = \{(x,y) \in \mathbb{R}^2 \ :\ 1 \leqslant x \leqslant 4,\ y \geqslant 0,\ |xy| \leqslant 1\}$.

EXERCISE 3.4.10. Find the value of the parameter $\lambda \in \mathbb{R}$ such that the function $f(x,y) := x^2 + \lambda y^2 - 4x + 2y$ has a stationary point in $(2,-1)$. What kind of point is this?

EXERCISE 3.4.11. Let $f(x,y) := x^2\left(y^2 - (x-1)^2\right)$, $(x,y) \in \mathbb{R}^2$. i) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? If yes, compute it. ii) Find and classify the stationary points of f on $\mathbb{R}^2$. iii) What about extrema of f on $\mathbb{R}^2$? Determine $f(\mathbb{R}^2)$. iv) Show that f has min/max on $D := \{(x,y) \in \mathbb{R}^2 \ :\ y \leqslant 0,\ 0 \leqslant x \leqslant y+1\}$ and find them. What is $f(D)$?

EXERCISE 3.4.12. Consider the function $f(x,y) := x^4 + y^4 - 8(x^2+y^2)$ on $\mathbb{R}^2$. i) Compute $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find and classify the stationary points of f. What can you say about global extreme points of f? What about $f(\mathbb{R}^2)$? iii) Find the extreme points of f on the domain $D := \{(x,y) \in \mathbb{R}^2 \ :\ x^2 + y^2 \leqslant 9\}$.

EXERCISE 3.4.13. Consider the function

$$f(x,y) := \begin{cases} \frac{x^5y^2}{(x^4+y^2)^2}, & (x,y) \neq 0_2, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

i) Say if f is continuos, differentiable in 0_2 (and in this case compute $\nabla f(0_2)$). ii) Find the eventual stationary points of f on $\mathbb{R}^2$ and discuss their nature. Does f has extreme points on $\mathbb{R}^2$? iii) Show that f has min/max on the domain $D = \{(x,y) \in \mathbb{R}^2 \ :\ |x| \leqslant 1,\ |y| \leqslant 1\}$ and find them.

EXERCISE 3.4.14. Let f be the function defined as

$$f(x,y) := \begin{cases} xye^{\frac{xy}{x^2+y^2}}, & (x,y) \in \mathbb{R}^2\backslash\{0_2\}, \\ \\ 0, & (x,y) = 0_2. \end{cases}$$

i) Say if f is continuous and differentiable at 0_2. ii) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? (in the case affirmative, what is the value?). Is f bounded on $\mathbb{R}^2$? iii) Show that f has min/max on $\{(x,y) \in \mathbb{R}^2 \ :\ x^2 + y^2 \leqslant 1\}$ and find them.

CHAPTER 4

Constrained Optimization

Many applied questions can be formalized as the maximization/minimization of a certain quantity (function) f of several variables over certain *constraints* on the variables. For instance: consider the problem to find the parallelepiped with maximum volume among those with fixed surface S. This means to determine

$$\max_{x,y,z>0 \,:\, 2(xy+yz+xz)=S} xyz.$$

A general form for this problem is

$$\text{find max/min } f(x_1,\dots,x_d) \text{ subject } g_1(x_1,\dots,x_d)=0,\dots,g_k(x_1,\dots,x_d)=0.$$

The method developed in the previous Chapter, based on the individuation of the stationary points, doesn't work in this context. Indeed, the set of points

$$\mathscr{M} := \{(x_1,\dots,x_d)\in\mathbb{R}^d \;:\; g_1(x_1,\dots,x_d)=0,\dots,g_k(x_1,\dots,x_d)=0\},$$

has no interior points in general. This means that conditions of Fermat's Thm are never fulfilled here, hence min/max are not necessarily stationary points for f. To tackle this problem, let's consider a simplified version of the initial isoperimetric problem, that is let's consider the problem to find, among all the rectangles with fixed perimeter S, those with maximum area. Formally, we want to determine

$$\max_{(x,y)\in]0,+\infty[\,:\, 2(x+y)=S} xy.$$

In this case, of course, we can reduce the problem to a well known one just by noticing that, by the constraint $2(x+y)=S$ we have, in particular, $y=\frac{S}{2}-x$. Hence we have to maximize

$$xy = x\left(\frac{S}{2}-x\right).$$

This is now a function of one variable that we've to maximize for $0 \leqslant x \leqslant \frac{S}{2}$. To do this we can use the tools of one variable calculus. Trying to catch the moral, we could think that if we have to solve

$$\max_{g(x,y)=0} f(x,y),$$

by the equation $g(x,y)=0$ we can "extract" for instance $y=\varphi(x)$. Then the problem can be reduced to

$$\max_x f(x,\varphi(x)).$$

This last, differently from the former, looks like an "unconstrained maximization" to which we can apply the usual tools of calculus. Of course, to make real this idea, there're two main points to solve:

- first, is it always possible say that $g(x,y)=0$ iff $y=\varphi(x)$?

- second, admitting φ exists, how this can be used to maximize $f(x, \varphi(x))$?

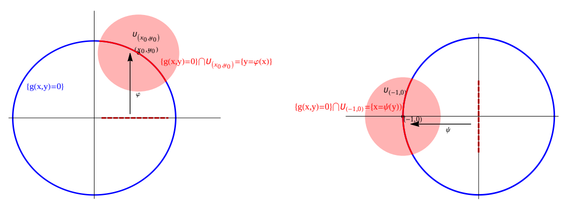
The answer to the first question is given by the *Dini's theorem*, the second is the so called *Lagrange's multipliers theorem*. For pedagogical reasons, we will first see these results in the simplest (but non trivial) case when $f = f(x, y)$ and $g = g(x, y)$, then we will extend to the general case.

4.1. Implicit functions: scalar case

Let's consider a set of the form

$$\mathscr{M} := \{(x, y) \in \mathbb{R}^2 \ : \ g(x, y) = 0\}$$

where $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Excluding degenerate cases, we expect that $\mathscr{M}$ be a curve in $\mathbb{R}^2$. This curve should be actually a graph of a function because, at least intuitively, through the equation we should able to express one of the two coordinates as function of the other. For instance, if the equation is $x^2 + y^2 - 1 = 0$ (that is $x^2 + y^2 = 1$) we can see that $x = \pm\sqrt{1 - y^2}$ as well as $y = \pm\sqrt{1 - x^2}$.



If we fix a point $(x_0, y_0) \in \mathscr{M}$, then *in a neighborhood of* (x_0, y_0) *the set* $\mathscr{M}$ *is precisely the graph of just one function of type* $y = \varphi(x)$ *or* $x = \psi(y)$. Notice that for all points except $(\pm 1, 0)$ this graph could be in the form $y = \varphi(x)$ but for points $(\pm 1, 0)$ it is impossible to represents the set of the solution (namely the circumference) as a graph of this type. It is, however, possible to express the set as a graph of type $x = \psi(y)$.

Suppose that *the set* $\mathscr{M} = \{g(x, y) = 0\}$ *be, locally, the graph of a function of type* $y = \varphi(x)$, that is

$$\{g(x, y) = 0\} \cap U_{(x_0, y_0)} = \{y = \varphi(x)\}.$$

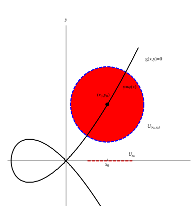
Assuming that φ be also a regular function,

$$g(x, \varphi(x)) \equiv 0, \implies 0 \equiv \frac{d}{dx} g(x, \varphi(x)) = \partial_x g(x, \varphi(x)) + \partial_y g(x, \varphi(x)) \varphi'(x). \tag{4.1.1}$$

If $\partial_y g(x, \varphi(x)) \neq 0$ to get

$$\varphi'(x) = -\frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}.$$

In particular, setting $x = x_0$, being $y_0 = \varphi(x_0)$ we would have $\varphi'(x_0) = -\frac{\partial_x g(x_0, y_0)}{\partial_y g(x_0, y_0)}$.



The condition $\partial_y g(x_0, y_0) \neq 0$ turns out to be the right condition for the existence of the function φ:

Theorem 4.1.1 (Dini). *Let $g : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, D open and $g \in \mathscr{C}^1(D)$. Let $(x_0, y_0) \in D$ such that $g(x_0, y_0) = 0$ and suppose that*

$$\partial_y g(x_0, y_0) \neq 0. \tag{4.1.2}$$

There exists then $U_{(x_0,y_0)}$ neighborhood of (x_0, y_0) in D, U_{x_0} neighborhood of x_0 in $\mathbb{R}$, $\varphi : U_{x_0} \longrightarrow \mathbb{R}$ such that

$$\{g = 0\} \cap U_{(x_0,y_0)} = \left\{(x, \varphi(x)) \ : \ x \in U_{x_0}\right\}.$$

Moreover $\varphi \in C^1$ and

$$\varphi'(x) = -\frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}, \ \forall x \in U_{x_0}. \tag{4.1.3}$$

The function φ is called **implicit function** *defined by g.*

Remark 4.1.2. *A similar statement holds to explicit $x = \psi(y)$. The key hypothesis that replaces (4.1.2) is*

$$\partial_x g(x_0, y_0) \neq 0. \tag{4.1.4}$$

Then exists $\psi : U_{y_0} \subset \mathbb{R} \longrightarrow \mathbb{R}$, such that $\{g = 0\} \cap U_{(x_0,y_0)} = \{(\psi(y), y) \ : \ y \in U_{y_0}\}$.

Remark 4.1.3. *In particular:* **if $\nabla g(x_0, y_0) \neq (0, 0)$ we have that $\{g = 0\}$ is locally the graph of some function $y = \varphi(x)$ or $x = \psi(y)$ in a neighborhood of (x_0, y_0).** ■

Remark 4.1.4 (Warning!). **Dini Thm gives a sufficient condition to explicit one variable in term of the other. A frequent error is to think that the hypothesis (4.1.2) (or (4.1.4)) is sufficient: this is false!** In other words, a common error is to believe that *"if one of (4.1.2) or (4.1.4) is not fulfilled, then is not possible to explicit one variable in term of the other"*. Look at the following "stupid" example: let

$$g(x, y) = (x - y)^2.$$

Of course $\{g(x, y) = 0\} = \{y = x\}$ is a global graph of the function $y = x$ or $x = y$. But

$$\partial_y g(x, y) = -2(x - y) \equiv 0, \ \forall (x, y) \in \{g = 0\},$$

as well $\partial_x g(x, y) \equiv 0$ for every $(x, y) \in \{g = 0\}$. Therefore, hypotheses (4.1.2) and (4.1.4) are **never** fulfilled! ■

Example 4.1.5. *Consider the equation*

$$x^3 + y^3 - 3xy - 3 = 0.$$

Show that if (x, y) is a solution of this equation is always possible to explicit at least one between x and y as function of the other variable.

Sol. — Define $g(x, y) := x^3 + y^3 - 3xy - 3$. If we prove that when (x, y) is a solution then **at least one** between $\partial_x g(x, y)$ or $\partial_y g(x, y)$ is different from 0 then at least one of (4.1.2) and (4.1.4) is fulfilled: therefore, Dini Thm applies at least in one of the two cases and we are done. To this aim let's see if there are points on $\{g = 0\}$ where both (4.1.2) and (4.1.4) fail: we have to find solution of

$$\begin{cases} (x,y) \in \{g=0\}, \\ \partial_x g(x,y) = 0, \\ \partial_y g(x,y) = 0. \end{cases} \iff \begin{cases} x^3 + y^3 - 3xy - 3 = 0 \\ 3x^2 - 3y = 0, \\ 3y^2 - 3x = 0. \end{cases} \iff \begin{cases} x^3 + y^3 - 3xy - 3 = 0, \\ x^2 = y, \\ y^2 = x. \end{cases}$$

By two last equations $y^4 = y$, that is $y(y^3 - 1) = 0$ whose solutions are $y = 0, 1$. As $y = 0$ we have $x = 0$, and for $y = 1$ we have $x = 1$, that is the points $(0,0)$ and $(1,1)$. Now the question is: do they satisfy also the first condition? It is easy to check that the answer is no! ■

4.2. Lagrange multipliers: scalar case

Let's now consider the problem

$$\min_{\mathscr{M}}/\max_{\mathscr{M}} f(x,y), \quad \text{where } \mathscr{M} := \{g(x,y) = 0\}.$$

Our goal here is to find an analogous of condition $\nabla f = 0$ for constrained min/max points. Assume that (x_0, y_0) be a min/max for f on $\mathscr{M}$ and assume that, in a neighborhood of (x_0, y_0), $\mathscr{M}$ be the graph of some regular function. According to Dini's Theorem and, in particular, to the Remark 4.1.3, a sufficient condition for this is

$$\nabla g(x_0, y_0) := (\partial_x g(x_0, y_0), \partial_y g(x_0, y_0)) \neq 0_2.$$

Definition 4.2.1. *We say that g is* **submersive** *at (x_0, y_0) if $\nabla g(x_0, y_0) \neq 0_2$.*

In this setting we have

Theorem 4.2.2 (Lagrange). *Assume that $f \in \mathscr{C}^1(D)$, $D \supset \mathscr{M} := \{g = 0\}$, $g \in \mathscr{C}^1(\mathbb{R}^2)$, g submersive at (x_0, y_0). If $(x_0, y_0) \in \mathscr{M}$ is a local min/max for f on $\mathscr{M}$ then*

$$\exists \lambda \in \mathbb{R} \ : \ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0). \tag{4.2.1}$$

Points (x_0, y_0) where (4.2.1) holds are called **constrained stationary points***.*

Proof — Suppose that, for instance, $\partial_y g(x_0, y_0) \neq 0$ (the other case is treated similarly). Then, by Dini's theorem, there exists a neighborhood U of (x_0, y_0) and a function $y = \varphi(x)$ such that

$$\mathscr{M} \cap U = \{(x, \varphi(x)) \ : \ x \in I_{x_0}\}.$$

Now, because (x_0, y_0) is a local minimum for f on $\mathscr{M}$ (same argument in the case of a maximum),

$$f(x, \varphi(x)) \geqslant f(x_0, \varphi(x_0)), \ \forall x \in I_{x_0},$$

that is the auxiliary function $h(x) := f(x, \varphi(x))$ has a minimum at x_0. Therefore, by the one variable Fermat theorem,

$$h'(x_0) = 0.$$

But

$$h'(x) = \frac{d}{dx} f(x, \varphi(x)) = \partial_x f(x, \varphi(x)) + \partial_y f(x, \varphi(x)) \varphi'(x),$$

hence

$$0 = \partial_x f(x_0, y_0) + \partial_y f(x_0, y_0) \varphi'(x_0).$$

Now, recalling that for the implicit function φ we have the (4.1.3) we deduce

$$0 = \partial_x f(x_0, y_0) - \partial_y f(x_0, y_0)\frac{\partial_x g(x_0, y_0)}{\partial_y g(x_0, y_0)},$$

that is

$$\partial_x f(x_0, y_0)\partial_y g(x_0, y_0) - \partial_y f(x_0, y_0)\partial_x g(x_0, y_0) = 0, \iff \nabla f(x_0, y_0) \perp \left(\partial_y g(x_0, y_0), -\partial_x g(x_0, y_0)\right).$$

But then

$$\nabla f(x_0, y_0) \propto (\partial_x g(x_0, y_0), \partial_y g(x_0, y_0)) = \nabla g(x_0, y_0). \quad \blacksquare$$

In practice, to determine constrained min/max we can proceed as follows. First we make sure if they exists (by a Weierstrass like argument). Then we look for constrained stationary points of f on $\mathscr{M}$. The min/max is among them. Finally, we could just evaluate f on these points: those where f is minimum are the mins, those where f is maximum are the maxs.

EXAMPLE 4.2.3. *Find points of the ellipse $x^2 + 2y^2 - xy = 9$ at min/max distance to the origin.*

SOL. — Let $\mathscr{M} := \{x^2 + 2y^2 - xy = 9\} = \{x^2 + 2y^2 - xy - 9 = 0\} =: \{g = 0\}$. We have to minimize/maximize the distance to the origin, that is the function

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Because of the properties of the root, to minimize this function or just $x^2 + y^2$ is the same (it produces the same points but of course not the same values!) being $\sqrt{x^2 + y^2}$ min/max iff $x^2 + y^2$ it is, we replace the previous f with

$$f(x, y) = x^2 + y^2,$$

which is easier to be managed.

Existence: $f \in \mathscr{C}(\mathbb{R}^2)$ and $\mathscr{M}$ is clearly closed. If we don't recognize an ellipse immediately (hence we can conclude that $\mathscr{M}$ is also bounded) we can easily show easily this: recalling that

$$xy \leqslant \frac{1}{2}(x^2 + y^2),$$

if $(x, y) \in \mathscr{M}$ then

$$x^2 + 2y^2 = 9 + xy \leqslant 9 + \frac{1}{2}(x^2 + y^2), \implies \frac{1}{2}x^3 + \frac{3}{2}y^2 \leqslant 9, \implies \frac{1}{2}x^2, \frac{3}{2}y^2 \leqslant 9,$$

by which $x^2 \leqslant 18$ (hence $|x| \leq \sqrt{18}$) and $y^2 \leqslant 6$ (that is $|y| \leqslant \sqrt{6}$). In any case both x, y are bounded hence $\mathscr{M}$ is bounded. The conclusion is that, according to Weierstrass theorem, $\mathscr{M}$ is compact, hence f admits both min/max.

Determination: by the previous argument we know that min/max points for f exist. Let's see if we can apply the previous theorem. We need to check if g is submersive on $\mathscr{M}$. To this aim let's see where g is not submersive. This happens iff

$$\nabla g = 0, \iff (2x - y, 4y - x) = (0, 0), \iff \begin{cases} 2x - y = 0, \\ 4y - x = 0, \end{cases} \iff x = y = 0.$$

Therefore g is not submersive at point $(0, 0) \notin \mathscr{M}$, hence g is submersive on $\mathscr{M}$.

According to the previous theorem in a min/max point we must have $\nabla f \propto \nabla g$. Being $\nabla f = (2x, 2y)$ this means

$$(2x, 2y) = \lambda(2x-y, 4y-x), \iff \begin{cases} 2x = \lambda(2x-y), \\ \\ 2y = \lambda(4y-x) \end{cases}$$

Of course $(0,0)$ is a solution of the system, but because $(0,0) \notin \mathscr{M}$ it cannot be considered as candidate to be en extrema for f. Are there non trivial solutions? Notice that if $x = 0$ (or $y = 0$) then necessarily $y = 0$ ($x = 0$). Hence we can assume $x, y \neq 0$: in such case by dividing the two equations we get

$$\frac{2x}{2y} = \frac{2x-y}{4y-x}, \iff x(4y-x) - (2x-y)y = 0, \iff 2xy + y^2 - x^2 = 0.$$

This can be rewritten as

$$(x+y)^2 - 2x^2 = 0, \iff (x+y)^2 = 2x^2, \iff x + y = \pm\sqrt{2}x, \iff y = (\pm\sqrt{2}-1)x.$$

Therefore we have points $(x, (\pm\sqrt{2}-1)x)$. Of course we have to look at those of them that belongs to $\mathscr{M}$:

$$(x, (\sqrt{2}-1)x) \in \mathscr{M}, \iff x^2 + 2(\sqrt{2}-1)^2x^2 - (\sqrt{2}-1)x^2 = 9, \iff (8-5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8-5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right)$ (same sign, 2 points). Similarly

$$(x, (-\sqrt{2}-1)x) \in \mathscr{M}, \iff x^2 + 2(-\sqrt{2}-1)^2x^2 - (-\sqrt{2}-1)x^2 = 9, \iff (8+5\sqrt{2})x^2 = 9, \iff x = \pm\frac{3}{\sqrt{8+5\sqrt{2}}}.$$

This produces points $\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right)$ (same sign, two points). Now, being

$$f\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right) = \frac{36-18\sqrt{2}}{8-5\sqrt{2}} > f\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right) = \frac{36+18\sqrt{2}}{8+5\sqrt{2}}$$

we have that the first points are max for f (hence point of $\mathscr{M}$ at max distance from 0_2), the latter are min. ∎

4.3. Dini Theorem: case of systems

A natural extension of the classical Dini's theorem is the following. Consider a set defined as

$$\mathscr{M} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \ : \ g_1(x_1, \ldots, x_d) = 0, \ldots, g_n(x_1, \ldots, x_d) = 0\}.$$

The intuitive idea is that, under good conditions, it should be possible to represent points $(x_1, \ldots, x_d) \in \mathscr{M}$ in a neighborhood of a certain $(\xi_1, \ldots, \xi_d) \in \mathscr{M}$ by expressing n coordinates in function of the remaining $d-n$, as for instance

$$(x_1, \ldots, x_d) \in \mathscr{M}, \iff \begin{cases} x_1 = \psi_1(x_{n+1}, \ldots, x_d), \\ \vdots \\ x_n = \psi_n(x_{n+1}, \ldots, x_d). \end{cases}$$

We can see this under the same form of the Dini's Thm. Call $g = (g_1, \ldots, g_n) : \mathbb{R}^d \longrightarrow \mathbb{R}^n$, and write

$$x = (x_1, \ldots, x_n), \ \ y = (x_{n+1}, \ldots, x_d),$$

in such a way that $(x_1,\dots,x_d)\in\mathscr{M}$ iff $g(x,y)=0$. We set also $x_0:=(\xi_1,\dots,\xi_n)$ and $y_0:=(\xi_{n+1},\dots,\xi_d)$. We look then for a function $x=\psi(y)$ such that

$$\{g=0\}\cap U_{(x_0,y_0)}=\{(\psi(y),y)\}.$$

Such ψ would be a function of $y\in\mathbb{R}^{d-n}$ into $\mathbb{R}^n$ and should fulfill

$$g(\psi(y),y)\equiv 0.$$

Deriving this identity with the chain rule

$$\partial_x g(\psi(y),y)\psi'(y)+\partial_y g(\psi(y),y)=0.$$

Here $\partial_x g$ denotes the Jacobian matrix of g respect to $x=(x_1,\dots,x_n)$ while $\partial_y g$ is the same respect to $y=(x_{n+1},\dots,x_d)$,

$$\partial_x g=\begin{bmatrix}\partial_1 g_1 & \partial_2 g_1 & \dots & \partial_n g_1\\ \partial_1 g_2 & \partial_2 g_2 & \dots & \partial_n g_2\\ \vdots & & & \vdots\\ \partial_1 g_n & \partial_2 g_n & \dots & \partial_n g_n\end{bmatrix},\quad \partial_y g=\begin{bmatrix}\partial_{n+1} g_1 & \partial_{n+2} g_1 & \dots & \partial_d g_1\\ \partial_{n+1} g_2 & \partial_{n+2} g_2 & \dots & \partial_d g_2\\ \vdots & & & \vdots\\ \partial_{n+1} g_n & \partial_{n+2} g_n & \dots & \partial_d g_n\end{bmatrix}.$$

Therefore

$$\psi'(y)=-[\partial_x g(\psi(y),y)]^{-1}\partial_y g(\psi(y),y),$$

provided $\partial_x g(\psi(y),y)$ **be invertible**. As $y=y_0$, $\psi(y_0)=x_0$ and we have $\partial_x g(x_0,y_0)$ needs to be invertible. As for the Dini's Theorem this turns out to be the appropriate condition to ask:

Theorem 4.3.1. *Let* $g=(g_1,\dots,g_n):D\subset\mathbb{R}^d\longrightarrow\mathbb{R}^n$ *be a* $\mathscr{C}^1$ *function and* $(x_0,y_0)\in\mathbb{R}^d$ *be such that* $g(x_0,y_0)=0$. *Suppose that*

$$\det\partial_x g(x_0,y_0)\neq 0,$$

There exists then an implicit function $\psi\in\mathscr{C}^1$ *and a neighborhood* $U_{(x_0,y_0)}$ *of* (x_0,y_0) *such that*

$$\{g=0\}\cap U_{(x_0,y_0)}=\{(\psi(y),y)\}.$$

Moreover

$$\psi'(y)=-\left[\partial_x g(\psi(y),y)\right]^{-1}\partial_y g(\psi(y),y). \tag{4.3.1}$$

Example 4.3.2. *Show that the system*

$$\begin{cases} x^3-3xy^2+z^3+1=0,\\ \\ x-2y^2-3z^2+4=0,\end{cases}$$

is a graph of (y,z) *as function of* x *in a neighborhood of the point* $(x,y,z)=(1,1,1)$. *Compute* $y'(1)$.

Sol. — Easily we see that $(x,y,z)=(1,1,1)$ is a solution. The problem asks to express (y,z) as function of x in a neighborhood of $(1,1,1)$: this is possible, according to Dini's thm, if the jacobian $\partial_{(y,z)}g(1,1,1)$ is invertible, where of course $g(x,y,z)=(x^3-3xy^2+z^3+1,x-2y^2-3z^2+4)$. We have

$$\partial_{(y,z)}g(x,y,z)=\begin{bmatrix}-6xy & 3z^2\\ \\ -4y & -6z\end{bmatrix},\quad\Longrightarrow\quad \partial_{(y,z)}g(1,1,1)=\begin{bmatrix}-6 & 3\\ \\ -4 & -6\end{bmatrix},$$

clearly invertible (its determinant is $36 + 12 = 48$). So the first requirement is fulfilled. For the second, instead to use the (4.3.1) we proceed directly: *let's derive the two equations considering $y = y(x)$ and $z = z(x)$*. We get

$$\begin{cases} (x^3 - 3xy^2 + z^3 + 1)' = 0, \\ \\ (x - 2y^2 - 3z^2 + 4)' = 0, \end{cases} t, \iff \begin{cases} 3x^2 - 3(y^2 + 2xyy') + 3z^2 z' = 0, \\ \\ 1 - 4yy' - 6zz' = 0. \end{cases}$$

Now, replacing $(x, y, z) = (1, 1, 1)$ we obtain

$$\begin{cases} 3 - 3(1 + 2y'(1)) + 3z'(1) = 0, \\ \\ 1 - 4y'(1) - 6z'(1) = 0, \end{cases} \iff \begin{cases} -2y'(1) + z'(1) = 0, \\ \\ 4y'(1) + 6z'(1) = 1, \end{cases} \iff y'(1) = \frac{1}{16},\ z'(1) = \frac{1}{8}. \blacksquare$$

4.4. Lagrange Multipliers: general case

Let's now consider the problem

$$\min_{\mathscr{M}}/\max_{\mathscr{M}} f(x_1, \dots, x_d), \text{ on } \mathscr{M} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \ :\ g_1(x_1, \dots, x_d) = 0, \dots, g_n(x_1, \dots, x_d) = 0\}.$$

The general argument is similar but technically much more involved than that one presented as special case above. We will limit to sketch it.

The first step is to discuss when $\mathscr{M} = \{g = 0\}$ is, locally (that is in a neighborhood of each of its points), a graph of some function. According to Dini's Thm 4.3.1, to express locally $\mathscr{M}$ as graph where $(x_{i_1}, x_{i_2}, \dots, x_{i_n})$ are functions of the remaining $d - n$ we need that

$$\det[\partial_{(x_{i_1}, x_{i_2}, \dots, x_{i_n})} g] = \det \begin{bmatrix} \partial_{x_{i_1}} g_1 & \partial_{x_{i_2}} g_1 & \dots & \partial_{x_{i_n}} g_1 \\ \partial_{x_{i_1}} g_2 & \partial_{x_{i_2}} g_2 & \dots & \partial_{x_{i_n}} g_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_{i_1}} g_n & \partial_{x_{i_2}} g_n & \dots & \partial_{x_{i_n}} g_n \end{bmatrix} \neq 0.$$

The matrix $\partial_{(x_{i_1}, x_{i_2}, \dots, x_{i_n})} g$ is the sub-matrix of the Jacobian matrix of g by which we select columns $i_1, i_2, \dots, i_n$. Now, because it is indifferent which are the $i_1, \dots, i_n$, we wish just

$$\exists 1 \leqslant i_1 < i_2 < \dots < i_n \leqslant d \ :\ \det[\partial_{(x_{i_1}, x_{i_2}, \dots, x_{i_n})} g] \neq 0,$$

that is *at least one of the $n \times n$ sub-determinants of the Jacobian of g be $\neq 0$*. It is well known that this is equivalent to say that

$$\text{rank}[\partial g] = n.$$

Definition 4.4.1. *We say that $g : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ is* **submersive at** *x if* $\text{rank}[\partial g(x)] = n$. *If g is submersive at every point of a set S, we say that g is a* **submersion on** *S. In particular, if g is submersion on $\mathscr{M} := \{x \in \mathbb{R}^d \ :\ g(x) = 0\}$ we say that $\mathscr{M}$ is a* **differential manifold**.

We notice that ∂g is a $n \times d$ matrix (with $d > n$ in our setting), hence to say that $\text{rank}[\partial g] = n$ means also that the rank is maximum. With a proof similar to that one seen above it is possible now to prove

Theorem 4.4.2 (Lagrange multipliers theorem). *Assume that $f \in \mathscr{C}^1(D;\mathbb{R})$, $D \supset \mathscr{M} := \{g = 0\}$, be a differential manifold. Then, if $\xi \in \mathscr{M}$ is a local min/max for f on $\mathscr{M}$ we necessarily have*

$$\exists \lambda_1, \dots, \lambda_n \in \mathbb{R} \ : \nabla f(\xi) = \sum_{i=1}^{n} \lambda_i \nabla g_i(\xi). \tag{4.4.1}$$

Points ξ where (4.4.1) holds are called **constrained stationary points**.

The (4.4.1) says that $\nabla f(\xi)$ is linearly dependent by $\nabla g_1(\xi), \dots, \nabla g_n(\xi)$ or, equivalently,

$$\operatorname{rank}[\nabla f(\xi)\ \nabla g_1(\xi)\ \dots \nabla g_n(\xi)] = n. \tag{4.4.2}$$

Because g is assumed to be submersive on $\mathscr{M}$, $\operatorname{rank}[\nabla g_1(\xi)\ \dots \nabla g_n(\xi)] = \operatorname{rank}[\partial g] = n$, Therefore, to check the (4.4.2) it is sufficient to check that *all the $(n+1)\times(n+1)$ sub-determinants of the matrix $[\nabla f\ \nabla g_1\ \cdots\ \nabla g_n]$ vanish.*

Example 4.4.3. *Let $\mathscr{M} := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ xy + z^2 = 1,\ x^2 + y^2 = 1\right\}$. i) Show that $\mathscr{M}$ is a non empty differential manifold. ii) Say if $\mathscr{M}$ is compact or less. iii) Find points of $\mathscr{M}$ at minimum/maximum distance to the origin.*

Sol. — **i)** Let's check that $\mathscr{M} \neq \emptyset$. To this aim let's look to points of type $(x,x,z) \in \mathscr{M}$. Imposing this we get $2x^2 = 1$, that is $x = \pm\frac{1}{\sqrt{2}}$. By the first, then, $x^2 + z^2 = 1$, that is $z^2 = 1 - x^2 = 1 - \frac{1}{2} = \frac{1}{2}$, i.e. $z = \pm\frac{1}{\sqrt{2}}$. Therefore $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) \in \mathscr{M}$ (all combinations of sign provided sign of the first two coordinates are equal). Let $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $g(x,y,z) := (xy + z^2 - 1, x^2 + y^2 - 1)$. Clearly $g \in \mathscr{C}^1$ and $\mathscr{M} = Z(g)$. Let's find points where g **is not** submersive. This means

$$\operatorname{rank} g'(x,y,z) < 2, \iff \operatorname{rank}\begin{bmatrix} y & x & 2z \\ 2x & 2y & 0 \end{bmatrix} < 2, \iff \begin{cases} 2(y^2 - x^2) = 0, \\ -4xz = 0, \\ -4yz = 0. \end{cases}$$

Now, this produces the two cases

$$\begin{cases} x = 0, \\ y^2 = 0, \iff y = 0, \\ z \in \mathbb{R} \end{cases} \quad \text{or} \quad \begin{cases} z = 0, \\ x^2 - y^2 = 0, \iff y = x, \ \vee\ y = -x. \end{cases}$$

Therefore, g is not submersive at points $(0,0,z)$, $z \in \mathbb{R}$ and $(x,x,0), (x,-x,0)$, $x \in \mathbb{R}$. Which of them belongs to $\mathscr{M}$? Clearly $(0,0,z) \notin \mathscr{M}$ for any $z \in \mathbb{R}$; moreover,

$$(x,x,0) \in \mathscr{M}, \iff \begin{cases} x^2 = 1, \\ 2x^2 = 1, \end{cases} \text{(impossible)}, \quad (x,-x,0) \in \mathscr{M}, \iff \begin{cases} -x^2 = 1, \\ 2x^2 = 1, \end{cases} \text{(impossible)}.$$

Therefore g is submersive on $\mathscr{M}$ and consequently $\mathscr{M}$ is a differential manifold of dimension 1 in $\mathbb{R}^3$.
ii) Because $\mathscr{M} = Z(g)$ and $g \in \mathscr{C}$ it follows that $\mathscr{M}$ is closed. It is also bounded because, by second constraint, $x^2 + y^2 = 1$ we deduce $|x|, |y| \leqslant 1$, and by the first

$$z^2 = 1 - xy \leqslant 2, \implies |z| \leqslant \sqrt{2}.$$

iii) We should minimize/maximize the function $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$. Because this is min/max exactly when the same happens for $f(x,y,z) = x^2 + y^2 + z^2$, we use this last function to find extrema. In the previous point we have seen that $\mathscr{M}$ is compact, hence min/max exist by Weierstrass Thm. Because $f \in \mathscr{C}^1(\mathbb{R}^3)$ we have that extrema points are stationary points. To find them we use the Lagrange Thm (that can be used by point i)): we have

$$(x,y,z) \in \mathscr{M} \text{ stationary point } \iff \text{rank} \begin{bmatrix} F'(x,y,z) \\ \\ \nabla f(x,y,z) \end{bmatrix} = 2, \iff \det \begin{bmatrix} y & x & 2z \\ 2x & 2y & 0 \\ 2x & 2y & 2z \end{bmatrix} = 0.$$

Computing the determinant by third column,

$$2z(2y^2 - 2x^2) = 0, \iff z(y-x)(y+x) = 0.$$

Candidates are therefore the points $(x,y,0)$, $x,y \in \mathbb{R}$, $(x,x,z),(x,-x,z)$, with $x,z \in \mathbb{R}$. Now

$$(x,y,0) \in \mathscr{M}, \iff \begin{cases} x^2 = 1, \\ \\ x^2 + y^2 = 1, \end{cases} \iff (x,y,0) = (\pm 1, 0, 0).$$

Similarly

$$(x,x,z) \in \mathscr{M}, \iff \begin{cases} x^2 + z^2 = 1, \\ \\ 2x^2 = 1, \end{cases} \iff \left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}} \right);$$

$$(x,-x,z) \in \mathscr{M}, \iff \begin{cases} -x^2 + z^2 = 1, \\ \\ 2x^2 = 1, \end{cases} \iff \left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{\sqrt{2}} \right);$$

It is easy to conclude: $(\pm 1, 0, 0)$ are the points at min distance, $\left(\pm \frac{1}{\sqrt{2}}, \mp \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{\sqrt{2}} \right)$ are at max distance. ■

Example 4.4.4. *Let* $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 = 1,\ x^2 - yz = 0\}$. *i) Prove that* $\mathscr{M}$ *is a differential manifold. ii) Prove that* $\mathscr{M}$ *is compact. iii) Find points of* $\mathscr{M}$ *with maximum quote.*

Sol. — **i)** $\mathscr{M}$ is defined by constraints $g_1(x,y,z) := x^2 + y^2 + z^4 - 1$, $g_2(x,y,z) := x^2 - yz$, clearly $\mathscr{C}^1(\mathbb{R}^3)$. Setting $g(x,y,z) := (g_1(x,y,z), g_2(x,y,z))$, we have to check that g is submersive on $\mathscr{M}$. Now

$$g \text{ is not submersive on } (x,y,z) \iff \text{rank } g'(x,y,z) = \text{rank} \begin{bmatrix} 2x & 2y & 4z^3 \\ \\ 2x & -z & -y \end{bmatrix} < 2,$$

and this happens iff all the 2×2 sub-determinants of $g'(x,y,z)$ vanish. This means

$$\begin{cases} -2xz - 4xy = 0, \\ -2xy - 8xz^3 = 0, \\ -2y^2 + 4z^4 = 0 \end{cases} \iff \begin{cases} x(z+2y) = 0, \\ x(y + 4z^3) = 0, \\ y^2 = 2z^4, \end{cases}$$

which produces the alternatives

$$(1)\ \begin{cases} x = 0, \\ y^2 = 2z^4, \end{cases} \implies (x,y,z) = (0, \pm\sqrt{2}z^2, z) \in \mathscr{M} \iff \begin{cases} 2z^2 + z^4 = 1, \\ \\ \pm\sqrt{2}z^3 = 0, \end{cases} \implies \text{impossible}$$

$$(2)\ \begin{cases} z = -2y, \\ y = -4z^3, \\ y^2 = 2z^4. \end{cases} \iff \begin{cases} z = -2y, \\ y = 8y^3, \\ y^2 = 32y^4. \end{cases} \implies \begin{array}{l} (2a)\ \begin{cases} y = 0, \\ z = 0, \end{cases} \implies (x,0,0) \notin \mathscr{M} \\ \\ (2b)\ \begin{cases} z = -2y, \\ 1 = 8y^2, \\ 1 = 32y^2, \end{cases} \implies \text{impossible.} \end{array}$$

All this means that there aren't points on $\mathscr{M}$ on which g is not submersive.
ii) Notice that

$$\mathscr{M} \subset \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^4 = 1\},$$

(which isn't the unit ball!), and because

$$x^2 + y^2 + z^4 = 1, \implies x^2 \leqslant 1,\ y^2 \leqslant 1, z^4 \leqslant 1, \implies |x| \leqslant 1,\ |y| \leqslant 1,\ |z| \leqslant 1.$$

Therefore $\mathscr{M}$ is bounded. Moreover $\mathscr{M}$ is closed because is the zero set of g continuous, and by this follows that $\mathscr{M}$ is compact.
iii) We have to maximize the function $f(x,y,z) := z$ on $\mathscr{M}$. Because f is continuous and $\mathscr{M}$ compact, the existence of global max is assured by Weierstrass Thm. Moreover $f \in \mathscr{C}^1$ and $\mathscr{M}$ is a differential manifold, therefore maximum points have to be stationary for f on $\mathscr{M}$. By Lagrange multipliers Thm, these are necessarily such that

$$(x,y,z) \in \mathscr{M},\ \text{s.t. rank} \begin{bmatrix} \nabla f(x,y,z) \\ \nabla g_1(x,y,z) \\ \nabla g_2(x,y,z) \end{bmatrix} = 2,$$

and this happens iff

$$0 = \det \begin{bmatrix} \nabla f(x,y,z) \\ \nabla g_1(x,y,z) \\ \nabla g_2(x,y,z) \end{bmatrix} = \det \begin{bmatrix} 0 & 0 & 1 \\ 2x & 2y & 4z^3 \\ 2x & -z & -y \end{bmatrix} = -2xz - 4yz = -2z(x+2y).$$

This produces the following alternatives:

$$z = 0, \iff (x,y,z) = (x,y,0),\ \text{or}\ x + 2y = 0, \iff x = -2y, \iff (x,y,z) = (-2y,y,z).$$

We have to check now which of these points belong to $\mathscr{M}$:

$$(x,y,0) \in \mathscr{M}, \iff \begin{cases} x^2 + y^2 = 1, \\ x^2 = 0, \end{cases} \iff (x,y,z) = (0,\pm 1, 0).$$

In the second case

$$(-2y,y,z) \in \mathscr{M} \iff \begin{cases} 4y^2 + y^2 + z^4 = 1, \\ 4y^2 - yz = 0, \end{cases} \iff \begin{cases} 5y^2 + z^4 = 1, \\ \\ y(4y - z) = 0. \end{cases}$$

By this we have the alternatives $y = 0$ (hence $z^4 = 1$ and we find points $(0,0,\pm 1)$) or $z = 4y$ which produces

$$\begin{cases} 5y^2 + 4^4 y^4 = 1, \\ z = 4y. \end{cases}$$

Solving the first equation we get

$$y^2 = \frac{-5 \pm \sqrt{25+4^5}}{2\cdot 4^4}, \iff y = \sqrt{\frac{-5+\sqrt{25+4^5}}{2\cdot 4^4}} =: \hat{y}.$$

Therefore we have a further candidate, the point $(-2\hat{y}, \hat{y}, 4\hat{y})$. The maximum point is therefore between $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ and $(-2\hat{y}, \hat{y}, 4\hat{y})$, and is simply that one with the maximum z. Because $4\hat{y} < 1$ we have easily that $(0, 0, 1)$ is the maximum. ■

Example 4.4.5. *A segment of length L is divided into n parts $x_1, \ldots, x_n$. Find the maximum of $x_1 \cdots x_n$. Deduce by this the classical inequality*

$$\sqrt[n]{x_1 \cdots x_n} \leqslant \frac{x_1 + \ldots + x_n}{n}, \ \forall x_1, \ldots, x_n \geqslant 0.$$

Sol. — We have to find

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1 \cdots x_n.$$

First: let's prove that the maximum exists. Indeed, let

$$\mathscr{M} := \{x_1 + \ldots + x_n = L \ :\ x_1, \ldots, x_n > 0\}.$$

Clearly $\mathscr{M}$ is an $n-1$ differential manifold defined by a unique constraint $g(x_1, \ldots, x_n) = x_1 + \ldots + x_n - L$, clearly submersive on all $\mathbb{R}^n$ ($\nabla g \equiv (1, 1, \ldots, 1)$). In particular $\mathscr{M}$ is closed as zero set of a continuous function (g). Moreover is bounded. Indeed, because

$$x_1, \ldots, x_n > 0,\ x_1 + x_2 + \ldots + x_n = L, \implies 0 < x_j < L,\ \forall j = 1, \ldots, n.$$

Therefore $\mathscr{M}$ is compact and because $f(x_1, \ldots, x_n) = x_1 \cdots x_n$ is of course continuous we have the existence by Weierstrass Thm. Now, let's find the stationary points of f on $\mathscr{M}$. We have $(x_1, \ldots, x_n) \in \mathscr{M}$ is stationary iff

$$1 = \operatorname{rank}\begin{bmatrix} \nabla f(x_1, \ldots, x_n) \\ \\ \nabla g(x_1, \ldots, x_n) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} x_2 \cdots x_n & x_1 x_3 \cdots x_n & \cdots & x_1 \cdots x_{n-1} \\ \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

that is iff all the 2×2 sub determinants vanish. Choosing column i and j respectively we have

$$\det\begin{bmatrix} x_1 \cdots x_{i-1} x_{i+1} \cdots x_n & x_1 \cdots x_{j-1} x_{j+1} \cdots x_n \\ \\ 1 & 1 \end{bmatrix} = x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n (x_j - x_i).$$

Therefore, $(x_1, \ldots, x_n) \in \mathscr{M}$ is critic for f on $\mathscr{M}$ iff

$$x_1 \cdots x_{i-1} x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n (x_j - x_i) = 0,\ \forall i \neq j = 1, \ldots, n.$$

This produces points where a coordinate is null (hence $f = 0$) and, if $x_j > 0$ for any j, $x_i - x_j = 0$ for all i, j, and this means that $(x_1, \ldots, x_n) = (\alpha, \alpha, \ldots, \alpha)$. Imposing that this belongs to $\mathscr{M}$ we find the point $(\frac{L}{n}, \ldots, \frac{L}{n})$ where $f > 0$: therefore this is the maximum! The moral is

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1 \cdots x_n = \left(\frac{L}{n}\right)^n.$$

In particular, recalling that $x_1 + \ldots + x_n = L$, this can be rewritten as

$$x_1 \cdots x_n \leqslant \left(\frac{x_1 + \ldots + x_n}{n}\right)^n, \iff \sqrt[n]{x_1 \cdots x_n} \leqslant \frac{x_1 + \ldots + x_n}{n},$$

that is just the classical inequality between arithmetic and geometric means. ■

Example 4.4.6. *Between all the convex polygons inscribed into a circumference, find those of maximum perimeter.*

Sol. — Let $r > 0$ be the radius of the circumference, $\theta_1, \dots, \theta_n$ the subsequent angles formed by the vertexes of the polygon. Then

$$\text{perimeter} \;=\; P(\theta_1, \dots, \theta_n) = \sum_{j=1}^{n} 2r \sin \frac{\theta_j}{2}.$$

Of course $0 < \theta_j < 2\pi$ and $\theta_1 + \cdots + \theta_n = 2\pi$. So, we have to find

$$\max_{\theta_1+\ldots+\theta_n=2\pi,\; 0<\theta_j<2\pi,\; j=1,\ldots,n} \sum_{j=1}^{n} 2r \sin \frac{\theta_j}{2}.$$

Let

$$\mathscr{M} := \{(\theta_1, \dots, \theta_n) \in]0, 2\pi[^n \;:\; \theta_1 + \cdots + \theta_n = 2\pi\}.$$

Clearly $\mathscr{M}$ is and $n-1$ dimensional differential manifold in $\mathbb{R}^n$. An argument similar to that one of the previous example, shows that the maximum exists. Let's find stationary points of P on $\mathscr{M}$. These fulfills

$$\operatorname{rank} \begin{bmatrix} r\cos\frac{\theta_1}{2} & \cdots & r\cos\frac{\theta_n}{2} \\ \\ 1 & \cdots & 1 \end{bmatrix} = 1, \iff r\cos\frac{\theta_i}{2} = r\cos\frac{\theta_j}{2},\ \forall i,j, \iff \theta_i = \theta_j,\ \forall i,j.$$

Therefore, the polygon with maximum perimeter has $\theta_1 = \theta_2 = \ldots = \theta_n = \frac{2\pi}{n}$, so it is a regular polygon. ■

4.5. Exercises

Exercise 4.5.1. Determine min/max of f on the set D in the following cases:

i $f = x + y$, $D = \{(x, y) \;:\; x^2 + y^2 = 1\}$;
ii) $f = 2x^2 + y^2 - x$, $D = \{(x, y) \;:\; x^2 + y^2 = 1\}$;
iii) $f = xy$, $D = \{(x, y) \;:\; x^2 + y^2 + xy - 1 = 0\}$;
iv) $f = x^2 + 5y^2 - \frac{1}{2}xy$, $D = \{(x, y) \;:\; x^2 + 4y^2 = 4\}$;
v) $f = x - 2y + 2z$, $D = \{(x, y, z) \;:\; x^2 + y^2 + z^2 = 9\}$;
vi) $f = z^2 e^{xy}$, $D = \{(x, y, z) \;:\; x^2 + y^2 + z^2 = 1\}$.

Exercise 4.5.2. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \;:\; z^2 = x^2 + y^2 + 1,\ z = 2x^2 + y^2\}$. Show that i) $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) $\mathscr{M}$ is compact. iii) $\mathscr{M}$ has points of maximum quote: find them.

Exercise 4.5.3. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \;:\; z^2 = xy + 1\}$. Show that i) $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) $\mathscr{M}$ is not compact. iii) Show that there exists points of $\mathscr{M}$ at minimum distance to the origin and find them.

Exercise 4.5.4. Let $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3 \;:\; (x^2 + y^2 + z^2)^2 - xyz = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) Say if $\mathscr{M}$ is compact or not. iii) Determine, if they exists, points on $\mathscr{M}$ at maximum distance to the origin.

Exercise 4.5.5. Let $\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 - z^2 = 0,\ x^2 - y^2 = 1\right\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) Say if $\mathscr{M}$ is compact or less. iii) Noticed that 0_3 is not on $\mathscr{M}$, show that exists points of $\mathscr{M}$ at minimum distance from 0_3 and find them.

Exercise 4.5.6. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 - xy + y^2 - z^2 = 1, \ x^2 + y^2 = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) Show that $\mathscr{M}$ is compact. iii) Find stationary points of $f(x,y,z) = xyz$ on $\mathscr{M}$. What can you say about the problem to find extrema of f on $\mathscr{M}$?

Exercise 4.5.7. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1\}$. i) Show that $\mathscr{M} \neq \emptyset$ is a differential manifold of dimension ... ii) Is $\mathscr{M}$ compact? iii) Find points of $\mathscr{M}$ at minimum distance from the origin 0_3.

Exercise 4.5.8. Find the stationary points of $f(x,y,z) := xyz$, $(x,y,z) \in \mathbb{R}^3$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (here $a,b,c > 0$). Deduce min/max of f on the ellipsoid.

Exercise 4.5.9. Compute the eventual min/max of $f(x,y,z) = xy + yz + zx$ on the plane $x + y + z = 3$.

Exercise 4.5.10. Compute the min/max distance of the point $(0,1,0)$ to the following subset of $\mathbb{R}^3$:

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ \\ x^2 + y^2 = x. \end{cases}$$

Exercise 4.5.11. Consider the set $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ z = x^2 + y^2, \ x + y + z = 0\}$. Show that $\mathscr{M}$ is not empty and is a ... Find points of $\mathscr{M}$ with min/max quote.

Exercise 4.5.12. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1, \ 2z - 3x = 0\}$ and $f(x,y,z) := xz$. i) Show that $\mathscr{M}$ is non empty and say if it is a differential manifold and what is its dimension. ii) Show that $\mathscr{M}$ is compact. iii) Find extrema of f on $\mathscr{M}$.

Exercise 4.5.13. Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ 2x^2 + 2y^2 - z^2 = 1, \ (x-y)^2 + z = 2\}$. i) Show that $\mathscr{M}$ is a differential manifold of dimension... ii) Show that $\mathscr{M}$ is not compact. iii) Find stationary points of $f(x,y,z) := z$ on $\mathscr{M}$.

Exercise 4.5.14. Let

$$f(x,y,z) := \frac{\sqrt{x^2 + \frac{y^2}{4}} - 3}{4} + z^2, \ (x,y,z) \in \mathbb{R}^3.$$

i) Compute $\lim_{(x,y,z) \to \infty_3} f(x,y,z)$: what can you deduce by this about min/max f? ii) Find and classify all the stationary points of f on $\mathbb{R}^3$. Find, if there exist, min/max of f on $\mathbb{R}^3$. What is $f(\mathbb{R}^3)$? iii) Let $\mathscr{M} := \{(x,y,z) \in \mathbb{R}^3 \ : \ f(x,y,z) = 1\}$. Prove that $\mathscr{M}$ is a non empty differential manifold of dimension... Is $\mathscr{M}$ compact? iv) Show that there exists points of $\mathscr{M}$ at min/max distance to the origin. Find them.

Exercise 4.5.15. Among all the parallelepipedes of sides $x,y,z > 0$ with fixed total surface find those with maximum volume.

Exercise 4.5.16. Find

$$\max\{xy^2z^3 \ : \ x,y,z > 0, \ x + y + z = 6\}.$$

Exercise 4.5.17 (★). Let $a_1, \dots, a_d \in \mathbb{R}$ such that $a_1^2 + \dots + a_d^2 > 0$. Find

$$\max\{x_1^2 + \dots + x_d^2 \ : \ a_1x_1 + \dots + a_dx_d = 1\}.$$

CHAPTER 5

Integration

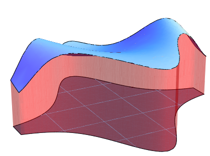
In the first Mathematical Analysis course the concept of *integral* for functions depending on one real variable has been introduced. This is a central notion in Analysis with several applications: from the calculus of areas of plane figures to a fundamental tool in Probability, Physics, Engineering. We recall that if $f = f(x) : [a,b] \longrightarrow [0,+\infty[$, the integral was defined with the following geometrical interpretation:

$$\int_{[a,b]} f(x)\,dx = \text{Area}\,(\text{Trap}(f)), \quad \text{where } \text{Trap}(f) := \{(x,y) \in \mathbb{R}^2 \;:\; x \in [a,b],\; 0 \leqslant y \leqslant f(x)\}.$$

The definition of Area passes through a formally complex *exhaustion method* based on filling the trapezoid Trap$(()f)$ with rectangles.

Relevance of integration both in Mathematics and applied disciplines, pushes for the extension of this operation to functions of several variables. In this case, however, we don't have a preferred type of domains as intervals in case of functions of one real variable. For instance, if $f = f(x,y) : E \subset \mathbb{R}^2 \longrightarrow [0,+\infty[$ we might expect

$$\int_E f(x,y)\,dxdy = \text{Volume}\,(\text{Trap}(f)), \text{ where } \text{Trap}(f) := \{(x,y,z) \in \mathbb{R}^3 \;:\; (x,y) \in E,\; 0 \leqslant z \leqslant f(x,y)\}.$$



Extending this idea, if now $f : E \subset \mathbb{R}^3 \longrightarrow [0,+\infty[$, we might expect

$$\int_E f(x,y,z)\,dxdydz = \text{Hyper-Volume}\,(\text{Trap}(f)),$$

where of course

$$\text{Trap}(f) := \{(x,y,z,w) \in \mathbb{R}^4 \;:\; (x,y,z) \in E,\; 0 \leqslant w \leqslant f(x,y,z)\}.$$

Area, Volume and Hyper-Volume are different forms of what we could name *measure* of a set. As it is complicate to construct a solid concept of Area, harder is to define a concept of volume and, more in general, a concept of *$n-$dimensional measure of a subset of* $\mathbb{R}^n$. This construction being totally out of our scope, we will take for granted focussing on the properties of multidimensional integrals. Yet, these are too technical to be proved here, thus we will limit to heuristic arguments for their justification, providing however proper statements.

5.1. Integral

In the introduction, we mentioned the concept of $n-$*dimensional measure*. This is in fact a function λ_n that assigns to $S \subset \mathbb{R}^n$ a number in $[0,+\infty]$ (we improperly will treat $+\infty$ as a number). In general, it is not possible to assign a measure to every set. However it is possible to do this on a large class of sets of $\mathbb{R}^n$, called *measurable sets*.

Theorem 5.1.1. *There exists a function*

$$\lambda_n \ : \ \mathscr{M}_n \longrightarrow [0,+\infty],$$

where $\mathscr{M}_n \subset \mathscr{P}(\mathbb{R}^n)$ *is called* **class ($n-$dimensional) measurable sets** *fulfilling the following properties:*

i) *Open and closed sets of* $\mathbb{R}^n$ *are measurable (that is are elements of* $\mathscr{M}_n$*), in particular* $\emptyset$ *and* $\mathbb{R}^n$ *are measurable and*

$$\lambda_n(\emptyset) = 0, \quad \lambda_n(\mathbb{R}^n) = +\infty.$$

Moreover, if a set S differs by an open (closed) set for a measure 0 set, S is measurable. Precisely,

$$\text{if } \exists E \subset \mathbb{R}^n \text{ open (or closed)} \ : \ \lambda_n\left((S\backslash E) \cup (E\backslash S)\right) = 0, \implies E \in \mathscr{M}_n.$$

ii) λ_n *factorizes in the sense that*

$$\text{if } S = A \times B \in \mathscr{M}_n,\ A \in \mathscr{M}_k,\ B \in \mathscr{M}_{n-k}, \implies \lambda_n(S) = \lambda_k(A)\lambda_{n-k}(B).$$

In particular, λ_n *is coherent with elementary geometry, in the sense that*

$$\lambda_n\left([a_1,b_1] \times \cdots \times [a_n,b_n]\right) = (b_1 - a_1)\cdots(b_n - a_n).$$

iii) λ_n *is invariant by translations, rotations, reflections and, in general, the following holds true:*

$$\lambda_n(L(S) + v) = |\det L|\lambda_n(S),\ \forall L \in M_{n\times n} \text{ invertible},\ \forall v \in \mathbb{R}^n. \tag{5.1.1}$$

(notice that $L = \mathbb{I}_n$ *is translation invariance;* L *=orthogonal matrix* $LL^t = \mathbb{I}_n$ *and* $v = 0$ *is rotation invariance).*

iv) λ_n *is countably additive, that is if* $(S_j)_{j\in\mathbb{N}} \subset \mathscr{M}_n$ *are disjoint, that is* $S_i \cap S_j = \emptyset$ *if* $i \neq j$*, then*

$$\lambda_n\left(\bigcup_{j=1}^{\infty} S_j\right) = \sum_{j=1}^{\infty} \lambda_n(S_j).$$

Some remarks may be useful to understand previous statement. i) says that large classes of common sets (like open or closed sets) are measurable. This is a good news, because most of sets we consider are open or closed. Just think to the case of sets like

$$O = \left\{(x_1,\dots,x_n) \in \mathbb{R}^n \ : \ g_j(x_1,\dots,x_n) > 0,\ j = 1,\dots,k\right\} \text{ (open if } g_j \in \mathscr{C}(\mathbb{R}^n),\ j = 1,\dots,k),$$

$$C = \left\{(x_1,\dots,x_n) \in \mathbb{R}^n \ : \ g_j(x_1,\dots,x_n) \geqslant 0,\ j = 1,\dots,k\right\} \text{ (closed if } g_j \in \mathscr{C}(\mathbb{R}^n),\ j = 1,\dots,k)$$

Furthermore, if our set S differs by an open/closed set by a measure 0 set, then S is measurable. Measure zero (or null) sets are particularly important. How can we imagine these sets? Here some examples:

- singletons $S = \{x^*\}$ are measure 0 sets: indeed, if $x^* = (x_1^*, \dots, x_n^*)$ we may see
$$\{x^*\} = [x_1^*, x_1^*] \times \cdots \times [x_n^*, x_n^*], \implies \lambda_n(\{x^*\}) = (x_1^* - x_1^*) \cdots (x_n^* - x_n^*) = 0.$$
- countable sets $S = \{x_j^* : j \in \mathbb{N}\}$ (like naturals or rationals in reals) are null sets: indeed, by countable additivity
$$\lambda_n(S) = \sum_j \lambda_n(\{x_j^*\}) = \sum_j 0 = 0.$$
- "lower dimensional" sets are in general measure zero sets: imagine a segment in the plane, say for simplicity $S = \{(x, c) : x \in [a, b]\}$. In $\mathbb{R}^2$ this is a null set: indeed
$$\lambda_2(S) = \lambda_2([a, b] \times [c, c]) = (b - a)(c - c) = 0.$$
To give a more precise and general statement we have
$$\text{if } g \in \mathscr{C}(\mathbb{R}^n) \text{ is non constant}, \implies \{(x_1, \dots, x_n) : g(x_1, \dots, x_n) = 0\} \text{ is a null set.}$$

Let $f : E \subset \mathbb{R}^n \longrightarrow [0, +\infty[$. We call *trapezoid* delimited by f the set
$$\mathrm{Trap}(f) := \left\{(x, y) \in \mathbb{R}^{n+1} : x \in E,\ 0 \leqslant y \leqslant f(x)\right\}.$$

Definition 5.1.2. *Let $f : E \longrightarrow [0, +\infty[$. If* $\mathrm{Trap}(f) \in \mathscr{M}_{n+1}$ *we pose*
$$\int_E f := \lambda_{n+1}\left(\mathrm{Trap}(f)\right). \tag{5.1.2}$$

First question: under which conditions is $\mathrm{Trap}(f)$ is measurable? Here some important cases:

Proposition 5.1.3. *Let $f : E \longrightarrow [0, +\infty[$ be continuous on E closed or open. Then* $\mathrm{Trap}(f)$ *is measurable. Thus, in particular,*
$$\int_E f$$
is well defined.

Proof — For E closed it is easy to prove that $\mathrm{Trap}(f)$ is closed too. For E open, $\mathrm{Trap}(f)$ is not open in general. However, one can prove that
$$S := \{(x, y) \in \mathbb{R}^{n+1} : x \in E,\ 0 < y < f(x)\} \text{ is open,}$$
thus $S \in \mathscr{M}_n$. Clearly $S \subset \mathrm{Trap}(f)$ (thus $S \backslash \mathrm{Trap}(f) = \emptyset$) while
$$\mathrm{Trap}(f) \backslash S = \left\{(x, y) \in \mathbb{R}^{n+1} : y = 0 \ \vee\ y = f(x)\right\}.$$
Now, because
$$\{(x, y) \in \mathbb{R}^{n+1} : y = 0\},\ \{(x, y) \in \mathbb{R}^{n+1} : y = f(x)\} = \{(x, y) \in \mathbb{R}^{n+1} : y - f(x) = 0\}$$
are both defined by equations involving continuous functions ($g(x, y) = y$ in the first case, $g(x, y) = y - f(x)$ in the second), they have $\lambda_{n+1} = 0$. We conclude that $\lambda_{n+1}(\mathrm{Trap}(f) \backslash S) = 0$ because $\mathrm{Trap}(f) \backslash S$ is made of two measure zero sets and λ_{n+1} is additive. The conclusion is that $\mathrm{Trap}(f)$ differs by S open for a measure zero set, thus it is measurable. ■

Of course, we might need to consider discontinuous functions, but for purposes of this course $f \in \mathscr{C}(E)$ on E open or closed is more than sufficient. Throughout this Chapter we will develop a number of methods to compute integrals. These can be used to compute measures:

Proposition 5.1.4. *If E is open or closed,*

$$\lambda_n(E) = \int_E 1. \tag{5.1.3}$$

Proof — Just notice that if $f \equiv 1$,

$$\mathrm{Trap}(f) = \left\{(x, y) \in \mathbb{R}^{n+1} \ : \ x \in E,\ 0 \leqslant y \leqslant 1\right\} \equiv E \times [0, 1].$$

Thus,

$$\int_E 1\ dx = \lambda_{n+1}\left(E \times [0, 1]\right) \overset{factor.}{=} \lambda_n(E)\lambda_1([0, 1]) = \lambda_n(E). \quad \blacksquare$$

So far, we defined the integral of a positive function. Let's now consider $f : E \longrightarrow \mathbb{R}$ and define

$$f_+(x) := \begin{cases} f(x), & \text{if } f(x) \geqslant 0, \\ 0, & \text{if } f(x) < 0, \end{cases} \qquad f_-(x) := \begin{cases} -f(x), & \text{if } f(x) \leqslant 0, \\ 0, & \text{if } f(x) > 0. \end{cases}$$

Functions $f_\pm$ are called, respectively, **positive part** and **negative part** of f. Easily it turns out that

$$f \in \mathscr{C}(E), \implies f_\pm \in \mathscr{C}(E).$$

Moreover, both $f_\pm$ are positive and, finally,

$$f = f_+ - f_-, \ \ |f| = f_+ + f_-.$$

Notice that, if $f \in \mathscr{C}(E)$, E open/closed,

$$\int_E |f| < +\infty, \iff \int_E f_+ < +\infty, \int_E f_- < +\infty.$$

This justifies the following

Definition 5.1.5. *Let $f \in \mathscr{C}(E)$, E open or closed in $\mathbb{R}^n$. We say that f is* **integrable** *if*

$$\int_E |f| < +\infty.$$

We pose

$$\int_E f := \int_E f_+ - \int_E f_-.$$

and we denote the set of integrable functions with $L^1(E)$.

The properties of the integral are very similar to those of one dimensional integral:

Proposition 5.1.6. *The following properties holds:*

i) *(linearity) if $f, g \in L^1(E)$ then $\alpha f + \beta g \in L^1(E)$ for any $\alpha, \beta \in \mathbb{R}$ and*

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g;$$

ii) *(isotonicity) if $f \leqslant g$ on E with $f, g \in L^1(E)$ then $\int_E f \leqslant \int_E g$;*

iii) *(triangular inequality) if $f \in L^1(E)$ then $\left|\int_E f\right| \leqslant \int_E |f|$;*

iv) *(decomposition) if* $f \in L^1(E)$ *and* $E = A \cup B$ *with* $A, B \in \mathscr{M}_n$, $\int_E f = \int_A f + \int_B f$.

It remains now to develop an efficient method of calculus for integrals. This is based on two fundamental tools: *reduction formula* and *change of variables formula*.

5.2. Reduction formula

In this Section we introduce the technique based on the *reduction formula* that allows to reduce the calculus of a multiple variables integral to iterated one variable integrals. For pedagogical reasons we present first the case of double integrals, then we will extend to the general case.

5.2.1. Double Integrals. To understand the idea, let's consider the problem to compute

$$\int_E f(x, y)\, dxdy.$$

As we know, integrals are continuous versions of discrete sums. Thinking to these we could write

$$\sum_{(x,y)\in E} f(x, y) = \sum_{x\in\mathbb{R}} \left(\sum_{y\in\mathbb{R}\,:\,(x,y)\in E} f(x, y) \right) = \sum_{y\in\mathbb{R}} \left(\sum_{x\in\mathbb{R}\,:\,(x,y)\in E} f(x, y) \right).$$

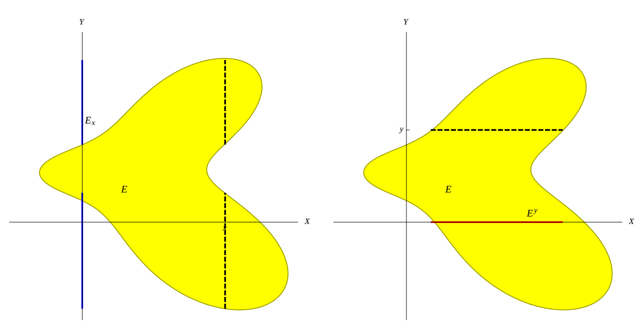
If the sums were finite, there wouldn't be any problem in reordering terms and summing as we prefer, so the previous formula would be a consequence of associativity and commutativity. However, when sum contains infinitely many terms the story is much more complicate. Leaving aside for a moment this problem, we might expect that the analogous for integrals of the previous formula is

$$\int_E f(x, y)\, dxdy = \int_{\mathbb{R}} \left(\int_{E^y} f(x, y)\, dy \right) dx = \int_{\mathbb{R}} \left(\int_{E_x} f(x, y)\, dx \right) dy.$$

The two sets E_x and E^y are defined as

$$E_x := \{y \in \mathbb{R} \,:\, (x, y) \in E\}, \ (x - \textbf{section}), \quad E^y := \{x \in \mathbb{R} \,:\, (x, y) \in E\}, \ (y - \textbf{section}).$$

Notice that, fixed $x \in \mathbb{R}$, E_x is the set of ordinates y of points of E with abscissa x, that is y such that $(x, y) \in E$. In other words, E_x is the projection on the $y-$axis of the "slice" of E along the vertical straight line at abscissa x. It turns out that if $f \in L^1$ the reduction formula holds true.



Proposition 5.2.1. *Let $f \in L^1(E)$, then*

$$\int_E f(x,y)\,dxdy = \int_{\mathbb{R}} \left(\int_{E_x} f(x,y)\,dy \right) dx = \int_{\mathbb{R}} \left(\int_{E^y} f(x,y)\,dx \right) dy. \tag{5.2.1}$$

Remark 5.2.2. *Notice that E_x (and E^y) may be empty for certain values of x (resp y). For such x (y), clearly $\int_{E_x} f = 0$ ($\int_{E^y} f = 0$). Therefore*

$$\int_{\mathbb{R}} \left(\int_{E_x} f(x,y)\,dy \right) dx = \int_{x\in\mathbb{R}\,:\,E_x\neq\emptyset} \int_{E_x} f(x,y)\,dy\,dx.$$

However, for future use we prefer to keep a lighter notation as in (5.2.1). ■

The (5.2.1) requires $f \in L^1(E)$, that is

$$\int_E |f(x,y)|\,dxdy < +\infty.$$

To check this, in principle one should compute a double integral. Notice that if we know $f \in L^1(E)$ then, by the reduction formula,

$$\int_E |f(x,y)|\,dxdy = \int_{\mathbb{R}} \left(\int_{E_x} |f(x,y)|\,dy \right) dx = \int_{\mathbb{R}} \left(\int_{E^y} |f(x,y)|\,dx \right) dy,$$

so in particular

$$\int_{\mathbb{R}} \left(\int_{E_x} |f(x,y)|\,dy \right) dx,\ \int_{\mathbb{R}} \left(\int_{E^y} |f(x,y)|\,dx \right) dy \ < +\infty.$$

It turns out that also the vice versa holds true:

Proposition 5.2.3. *Let $f \in \mathscr{C}(E)$, $E \subset \mathbb{R}^2$ open/closed set. If one of the following iterated integrals*

$$\int_{\mathbb{R}} \left(\int_{E_x} |f(x,y)|\,dy \right) dx,\ \int_{\mathbb{R}} \left(\int_{E^y} |f(x,y)|\,dx \right) dy \tag{5.2.2}$$

is finite, then $f \in L^1(E)$ and reduction formula (5.2.1) holds.

Combining the previous Propositions we have an algorithm to check if $f \in L^1(E)$ and to compute its integral by using the reduction formula: *first, one check if one of the (5.2.2) is finite (which one of the two is indifferent and the choice could be done in terms of computational ease); second, one uses (5.2.1) to compute the integral*. Notice that, in particular, if $f \geqslant 0$ the check (5.2.2) leads at same time to the calculation of the integral by (5.2.1).

Example 5.2.4. *Discuss if* $f(x,y) := x^3 e^{-yx^2} \in L^1([0,+\infty[\times[1,2])$ *and compute its integral.*

Sol. — Clearly $f \in \mathscr{C}(E)$ where $E = [0,+\infty[\times[1,2]$ is closed. Applying (5.2.2), notice that if $E = [0,+\infty[\times[1,2]$, $E_x = \emptyset$ if $x < 0$, $E_x = [1,2]$ if $x \geqslant 0$, therefore

$$\begin{aligned}\int_{\mathbb{R}} \int_{E_x} |f|\,dy\,dx &= \int_0^{+\infty} \left(\int_1^2 x^3 e^{-yx^2}\,dy \right) dx = \int_0^{+\infty} x \left[-e^{-yx^2} \right]_{y=1}^{y=2} dx \\ &= \int_0^{+\infty} xe^{-x^2} - xe^{-2x^2}\,dx = \left[\frac{-e^{-x^2}}{2} \right]_{x=0}^{x=+\infty} - \left[\frac{-e^{-2x^2}}{4} \right]_{x=0}^{x=+\infty} = \frac{1}{4}.\end{aligned}$$

We deduce $f \in L^1$ and because $f \geqslant 0$, thus $|f| = f$, the same calculation and (5.2.1) gives $\int_{[0,+\infty[\times[1,2]} f = \frac{1}{4}$. ■

EXAMPLE 5.2.5. *Discuss if* $f(x,y) := e^{-x} \in L^1(E)$ *where* $E = \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant 0,\ 0 \leqslant y \leqslant x^2\}$. *In such case compute the integral of* f *on* E.

SOL. — Clearly $f \in \mathscr{C}(E)$ where E is closed (defined by large inequalities on continuous functions). Applying (5.2.2), notice that $E_x = \emptyset$ if $x < 0$, $E_x = [0, x^2]$ if $x \geqslant 0$, therefore

$$\begin{aligned}\int_{\mathbb{R}}\int_{E_x} |f|\,dy\,dx &= \int_0^{+\infty}\left(\int_0^{x^2} e^{-x}\,dy\right)\,dx = \int_0^{+\infty} x^2 e^{-x}\,dx = \int_0^{+\infty} x^2(-e^{-x})'\,dx \\ &= \left[-x^2e^{-x}\right]_{x=0}^{x=+\infty} + \int_0^{+\infty} 2xe^{-x}\,dx = 2\int_0^{+\infty} x(-e^{-x})'\,dx \\ &= 2\left[[-xe^{-x}]_{x=0}^{x=+\infty} + \int_0^{+\infty} e^{-x}\,dx\right] = 2\,[-e^{-x}]_{x=0}^{x=+\infty} = 2.\end{aligned}$$

Therefore $f \in L^1(E)$ and because $f \geqslant 0$ the same calculation and (5.2.1) gives $\int_E f = 2$. ■

A remarkable example of (5.2.1) is obtained by taking $f \equiv 1$. Recalling that $\int_E 1 = \lambda_2(E)$ we obtain

(5.2.3) $$\lambda_2(E) = \int_{\mathbb{R}}\left(\int_{E_x} 1\,dy\right)\,dx = \int_{\mathbb{R}} \lambda_1(E_x)\,dx = \int_{\mathbb{R}} \lambda_1(E^y)\,dy.$$

EXAMPLE 5.2.6. *Compute the area of a disk of radius* r.

SOL. — Because the measure is translations–invariant, we can center the disk into the origin. So, let's consider

$$E = \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant r^2\right\}.$$

This set E is closed, hence measurable. Applying the integration by slices we have

$$\lambda_2(E) = \int_{\mathbb{R}} \lambda_1(E_x)\,dx.$$

Let's determine an $x-$section. We have

$$y \in E_x, \iff (x,y) \in E, \iff x^2 + y^2 \leqslant r^2, \iff y^2 \leqslant r^2 - x^2, \iff y \in \left[-\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2}\right].$$

Of course we need $r^2 - x^2 \geqslant 0$, that is $x^2 \leqslant r^2$, $|x| \leqslant r$, otherwise $E_x = \emptyset$. Therefore

$$\lambda_2(E) = \int_{\mathbb{R}} \lambda_1(E_x)\,dx = \int_{|x|\leqslant r} \lambda_1\left(\left[-\sqrt{r^2 - x^2}, \sqrt{r^2 - x^2}\right]\right)\,dx = \int_{|x|\leqslant r} 2\sqrt{r^2 - x^2}\,dx.$$

Being $x \longmapsto \sqrt{r^2 - x^2}$ continuous on $[-r, r]$, the last integral is equal to a Riemann one, so

$$\lambda_2(E) = \int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx = 4\int_0^r \sqrt{r^2 - x^2}\,dx = 4r\int_0^r \sqrt{1 - \frac{x^2}{r^2}}\,dx.$$

Therefore, setting $\frac{x}{r} = \sin\theta$, $\theta \in [0, \frac{\pi}{2}]$,

$$\lambda_2(E) = 4r\int_0^{\pi/2} \sqrt{1 - (\sin\theta)^2}\, r\cos\theta\,d\theta = 4r^2\int_0^{\pi/2} (\cos\theta)^2\,d\theta.$$

Now $\int(\cos\theta)^2 = \int \cos\theta(\sin\theta)' = \cos\theta\sin\theta + \int(\sin\theta)^2 = \frac{1}{2}\sin(2\theta) + \theta - \int(\cos\theta)^2$ hence

$$\lambda_2(E) = 4r^2 \left[\frac{1}{4}\sin(2\theta) + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \pi r^2. \quad \blacksquare$$

Warning! If $f \notin L^1$ the reduction formula might be false even if the iterated integral are finite.

Example 5.2.7. *Let*

$$f(x,y) = \frac{x-y}{(x+y)^3},\ (x,y) \in E := [0,1]^2.$$

Then $\int_{\mathbb{R}} \left(\int_{E_x} f\, dy\right)\, dx \neq \int_{\mathbb{R}} \left(\int_{E^y} f\, dx\right)\, dy$. *Hence, in particular,* $f \notin L^1([0,1]^2)$.

Sol. — Notice first that

$$E_x = \{y \in \mathbb{R} \ :\ (x,y) \in [0,1]^2\} = \begin{cases} \emptyset, & x \notin [0,1], \\ \\ [0,1] & x \in [0,1] \end{cases}$$

and similarly for E^y. Therefore

$$\int_{E^y} f(x,y)\, dx = \begin{cases} 0, & y \notin [0,1], \\ \\ \displaystyle\int_0^1 \frac{x-y}{(x+y)^3}\, dx = \int_0^1 \frac{1}{(x+y)^2}\, dx - 2y\int_0^1 \frac{1}{(x+y)^3}\, dx. & y \in [0,1]. \end{cases}$$

Except for $y = 0$ (therefore for a measure 0 set) both integrals are finite and their value is

$$\left[\frac{(x+y)^{-1}}{-1}\right]_{x=0}^{x=1} - 2y\left[\frac{(x+y)^{-2}}{-2}\right]_{x=0}^{x=1} = \frac{1}{y} - \frac{1}{y+1} + y\left(\frac{1}{(y+1)^2} - \frac{1}{y^2}\right) = -\frac{1}{(y+1)^2}.$$

Hence

$$\int_{\mathbb{R}} \left(\int_{E^y} f(x,y)\, dx\right)\, dy = \int_0^1 \left(-\frac{1}{(y+1)^2}\right)\, dy = \left[(y+1)^{-1}\right]_{y=0}^{y=1} = \frac{1}{2} - 1 = -\frac{1}{2}.$$

Exchanging x with y we obtain the same result except for the sign: $\int_{\mathbb{R}} \left(\int_{E_x} f(x,y)dy\right)\, dx = \frac{1}{2}$. $\blacksquare$

5.2.2. General Multiple Integrals. The previous mechanism can be extended to functions f of n variables. Let $f = f(z_1, \dots, z_n)$ and imagine we group $(z_1, \dots, z_n)$ into two blocks, one of k variables and the remaining of $n-k$ variables. For simplicity with notations we write

$$f = f(x,y), \text{ where } x = (x_1, \dots, x_k) \in \mathbb{R}^k,\ y = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}.$$

As above, we will denote by E_x (resp. E^y) the $x-$section (resp. $y-$section) of E defined as

$$E_x := \{y \in \mathbb{R}^n \ :\ (x,y) \in E\}, \quad E^y := \{x \in \mathbb{R}^m \ :\ (x,y) \in E\}.$$

Be careful because now $E_x \subset \mathbb{R}^n$ while $E^y \subset \mathbb{R}^m$. With these notations we have the

THEOREM 5.2.8 (FUBINI–TONELLI). *Let $f \in L^1(E)$, $E \subset \mathbb{R}^{m+n}$. Then the* **reduction formula** *holds*

$$\int_E f = \int_{\mathbb{R}^m} \left(\int_{E_x} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^n} \left(\int_{E^y} f(x,y)\, dx \right) dy. \tag{5.2.4}$$

Moreover, if $f \in \mathscr{C}(E)$ and one among

$$\int_{\mathbb{R}^m} \left(\int_{E_x} |f(x,y)|\, dy \right) dx, \quad \int_{\mathbb{R}^n} \left(\int_{E^y} |f(x,y)|\, dx \right) dy,$$

is finite, then $f \in L^1(E)$ (and the reduction formula (5.2.4) holds). In particular, by taking $f = 1$ we have the **slicing formula**

$$\lambda_{m+n}(E) = \int_{\mathbb{R}^m} \lambda_n(E_x)\, dx = \int_{\mathbb{R}^n} \lambda_m(E^y)\, dy. \tag{5.2.5}$$

Fubini–Tonelli theorem is a versatile tool to integrate functions of several variables. For instance: consider a function of three variables $f = f(x,y,z) \in \mathscr{C}(E)$, $E \subset \mathbb{R}^3$ open/closed. In this common case, the three variables may be grouped is six different ways, this leading to six different possible applications of reduction formula:

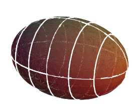
$$x \text{ and } (y,z), \quad \int_E f = \int_{\mathbb{R}} \left(\int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x \in E_{(y,z)}} f\, dx \right) dydz,$$

$$y \text{ and } (x,z), \quad \int_E f = \int_{\mathbb{R}} \left(\int_{(x,z)\in E_y} f\, dxdz \right) dy = \int_{\mathbb{R}^2} \left(\int_{y \in E_{(x,z)}} f\, dy \right) dxdz,$$

$$z \text{ and } (x,y), \quad \int_E f = \int_{\mathbb{R}} \left(\int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x \in E_{(y,z)}} f\, dx \right) dydz,$$

Which choice is the best one depends by the complexity of calculus. ■

EXAMPLE 5.2.9. *Compute the volume of a rugby ball $E = \left\{ (x,y,z) \in \mathbb{R}^3 \; : \; \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1 \right\}$, $(a,b > 0)$.*



SOL. — Clearly E is closed in $\mathbb{R}^3$, hence measurable. Slicing E along the z-axis,

$$m_3(E) = \int_{\mathbb{R}} m_2(E_z)\, dz.$$

Now,

$$(x,y,z) \in E, \iff \frac{x^2+y^2}{a^2} \leqslant 1 - \frac{z^2}{b^2}, \iff (x,y) \in B\left(0_2, \sqrt{1 - \frac{z^2}{b^2}}\right] =: E_z$$

This of course if $1-\frac{z^2}{b^2}\geqslant 0$, that is $z^2\leqslant b^2$, namely $|z|\leqslant b$, otherwise $E_z=\emptyset$. It follows that

$$\begin{aligned} m_3(E) &= \int_{|z|\leqslant b} m_2\left(B\left(0_2, a\sqrt{1-\frac{z^2}{b^2}}\right]\right)\,dz = \int_{|z|\leqslant b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz \overset{R=L}{=} \int_{-b}^{b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz \\ &= \pi a^2\left([z]_{-b}^{b} - \left[\frac{z^3}{3b^2}\right]_{-b}^{b}\right) = \pi a^2\left(2b-\frac{2}{3}b\right) = \pi\frac{4}{3}a^2 b. \end{aligned}$$

Taking $a=b=r$ we obtain the volume of a sphere of radius r, the well known $\frac{4}{3}\pi r^3$. ■

5.3. Change of variables

Let $T:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ a transformation. If T is a linear bijection we know that

$$\lambda_n(T(E)) = |\det T|\lambda_n(E).$$

What happens if T is a general (non linear) bijection?

Theorem 5.3.1. *Let $T:E\subset\mathbb{R}^n\longrightarrow T(E)$ be a diffeomorphism (that is $T,T^{-1}\in\mathscr{C}^1$) on E open/closed set. Then*

$$\lambda_n(T(E)) = \int_E |\det T'(\xi)|\,d\xi. \tag{5.3.1}$$

Proof — (sketch): We decompose E as disjoint union of neighborhoods of some of its points, let's say

$$E=\bigcup_j U_{x_j}.$$

Then, by countable additivity,

$$\lambda_n(T(E)) = \sum_j \lambda_n(T(U_{x_j})).$$

If T is regular (differentiable) then $T(x)=T(x_0)+T'(x_0)(x-x_0)+o(x-x_0)\sim_{x_0} T(x_0)+T'(x_0)(x-x_0)$. The sense of $\sim_{x_0}$ is that $T(x)$ can be replaced by $T(x_0)+T'(x_0)(x-x_0)$ in a neighborhood of x_0 and the approximation is more precise smaller is this neighborhood. Therefore, we may expect that

$$\begin{aligned} \lambda_n(T(U_{x_j})) &\approx \lambda_n(T(x_j)+T'(x_j)(U_{x_j}-x_j)) = \lambda_n(T'(x_j)(U_{x_j}-x_j)) = |\det T'(x_j)|\lambda_n(U_{x_j}-x_j) \\ &= |\det T'(x_j)|\lambda_n(U_{x_j}). \end{aligned}$$

Hence

$$\lambda_n(T(E)) \approx \sum_j |\det T'(x_j)|\lambda_n(U_{x_j}) \approx \int_E |\det T'(\xi)|\,d\xi.$$

Of course this is not yet a proof and it needs a lot of technical work to make it a rigorous argument, but this is the main idea on which is based. ■

We may review (5.3.1) as

$$\int_{T(E)} dx = \int_E |\det T'(\xi)|\, d\xi, \quad \overset{\Phi:=T^{-1}}{\Longleftrightarrow} \quad \int_F dx = \int_{\Phi(F)} |\det(\Phi^{-1})'(\xi)|\, d\xi.$$

This last is a special case of change of variable formula:

Theorem 5.3.2 (change of variables formula). *Let $f : F \longrightarrow \mathbb{R}$, $f \in \mathscr{C}(F)$, F open/closed, and $\Phi : F \longrightarrow \Phi(F)$ be a diffeomorphism, that is $\Phi, \Phi^{-1} \in \mathscr{C}^1$. Then*

$$\boxed{\int_F f(x)\, dx \overset{y=\Phi(x),\ x=\Phi^{-1}(y)}{=} \int_{\Phi(F)} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)|\, d\xi.} \tag{5.3.2}$$

Proof — (sketch) As in the previous proof, divide F in a disjoint union of small neighbourhoods,

$$F = \bigcup_j U_{x_j},$$

in such a way that

$$\int_F f = \sum_j \int_{U_{x_j}} f.$$

We may choose U_{x_j} in such a way that $f(x) \approx f(x_j)$ for all $x \in U_{x_j}$ (by continuity). Thus

$$\int_{U_{x_j}} f \approx \int_{U_{x_j}} f(x_j) = f(x_j) \int_{U_{x_j}} = f(x_j) \int_{\Phi(U_{x_j})} |\det(\Phi^{-1})'|.$$

Now, because Φ is a diffeomorphism, we may imagine that $\Phi(U_{x_j}) = V_{\Phi(x_j)}$ is a neighbourhood of $\Phi(x_j)$. Thus, if $\xi \in V_{\Phi(x_j)}$, by continuity $f(\Phi^{-1}(\xi)) \approx f(\Phi^{-1}(\Phi(x_j))) = f(x_j)$ for all $\xi \in V_{\Phi(x_j)}$. Furthermore,

$$\bigcup_j V_{\Phi(x_j)} = \bigcup_j \Phi(U_{x_j}) = \Phi\left(\bigcup_j U_{x_j}\right) = \Phi(F).$$

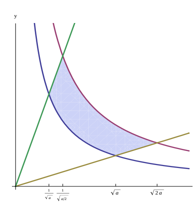
Summing up all previous remarks we would have

$$\int_F f \approx \sum_j f(x_j) \int_{\Phi(U_{x_j})} |\det(\Phi^{-1})'| = \sum_j \int_{V_{\Phi(x_j)}} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)| = \int_{\Phi(F)} f(\Phi^{-1}(\xi))|\det(\Phi^{-1})'(\xi)|\, d\xi. \quad \blacksquare$$

Example 5.3.3. *Compute*

$$\int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy, \quad 0 < a < 1.$$

Sol. — The domain is closed in $\mathbb{R}^2$, hence measurable and $f \in \mathscr{C}$.



Notice that

$$\frac{y^4 \arctan(xy)}{(x^2+y^2)^2} = \left(\frac{y}{x}\right)^4 \frac{\arctan(xy)}{\left(1+\left(\frac{y}{x}\right)^2\right)^2}.$$

It seems therefore natural to introduce the new variables

$$\xi = xy, \quad \eta = \frac{y}{x}, \quad (\xi,\eta) := \Psi(x,y),$$

where $\Psi :]0,+\infty[^2 \longrightarrow]0,+\infty[^2$, $\Psi(x,y) = (xy, \frac{y}{x})$ is clearly $\mathscr{C}^1$. We need Ψ^{-1}. If $(\xi,\eta) \in]0,+\infty[^2$ then

$$\begin{cases} \xi = xy, \\ \eta = \frac{y}{x}, \end{cases} \iff \begin{cases} \xi = \eta x^2, \\ y = \eta x, \end{cases} \iff \begin{cases} x = \sqrt{\frac{\xi}{\eta}}, \\ y = \sqrt{\xi\eta}, \end{cases} \iff \Psi^{-1}(\xi,\eta) = \left(\sqrt{\frac{\xi}{\eta}}, \sqrt{\xi\eta}\right).$$

Therefore

$$I(a) := \int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy = \int_{1\leqslant \xi\leqslant 2,\ a\leqslant\eta\leqslant\frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi\, |\det(\Psi^{-1})'(\xi,\eta)|\, d\xi d\eta.$$

and because

$$|\det(\Psi^{-1})'(\xi,\eta)| = \frac{1}{|\det\Psi'(\Psi^{-1}(\xi,\eta))|},$$

with

$$\Psi'(x,y) = \begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{bmatrix}, \implies \det\Psi'(x,y) = \frac{y}{x} + x\frac{y}{x^2} = 2\frac{y}{x} = 2\eta.$$

we have

$$I(a) = \int_{1\leqslant\xi\leqslant 2,\ a\leqslant\eta\leqslant\frac{1}{a}} \frac{\eta^4}{(1+\eta^2)^2} \arctan\xi \frac{1}{2\eta}\, d\xi d\eta = \frac{1}{2}\left(\int_1^2 \arctan\xi\, d\xi\right)\left(\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta\right).$$

Now

$$\int_1^2 \arctan\xi\, d\xi = [\xi\arctan\xi]_1^2 - \int_1^2 \frac{\xi}{1+\xi^2}\, d\xi = 2\arctan 2 - \frac{\pi}{4} - \frac{1}{2}\log\frac{5}{2},$$

while

$$\int_a^{\frac{1}{a}} \frac{\eta^3}{(1+\eta^2)^2}\, d\eta = \int_a^{\frac{1}{a}} \frac{\eta}{1+\eta^2}\, d\eta - \int_a^{\frac{1}{a}} \frac{\eta}{(1+\eta^2)^2}\, d\eta = -\log a + \frac{1}{2}\frac{1-a^2}{1+a^2}. \quad \blacksquare$$

5.3.1. Polar coordinates in $\mathbb{R}^2$. A very important change of variable in plane integration is

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \end{cases} \iff (x,y) = \Psi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta).$$

Here we may notice that change of variable is defined in the form $(x,y) = \Psi(\rho,\theta)$. This means that, referring to notations of (5.3.2), present Ψ is just Φ^{-1}. Thus

$$\det(\Phi^{-1})' = \det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{bmatrix} = \rho(\cos^2\theta + \sin^2\theta) = \rho,$$

and (5.3.2) becomes

$$\int_E f(x,y)\,dxdy = \int_{E_{pol}} f(\rho\cos\theta, \rho\sin\theta)\rho\, d\rho d\theta, \tag{5.3.3}$$

where E_{pol} is E in polar coordinates.

EXAMPLE 5.3.4. *Compute*

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dxdy.$$

SOL. — We have

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}}\, dxdy &= \int_{\rho\geqslant 0, \theta\in[0,2\pi]} e^{-\rho}\rho\, d\rho d\theta = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\rho}\rho\, d\theta\right) d\rho = 2\pi\int_0^{+\infty} \rho e^{-\rho}\, d\rho \\ &= 2\pi\left([-\rho e^{-\rho}]_{\rho=0}^{\rho=+\infty} + \int_0^{+\infty} e^{-\rho}\, d\rho\right) = 2\pi. \ \blacksquare\end{aligned}$$

EXAMPLE 5.3.5 (GAUSSIAN INTEGRAL). *A very beautiful (and relevant) application of the (5.3.3) is the formula*

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}.$$

More in general: if C is a $d\times d$ positive symmetric matrix,

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}C^{-1}x\cdot x}\, dx = \sqrt{(2\pi)^d \det C}. \tag{5.3.4}$$

SOL. — Let's start by the integral

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{x^2+y^2}{2}}\, dx\right) dy = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dx\right) dy = \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx\right)^2.$$

On the other hand, by (5.3.3)

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\frac{\rho^2}{2}}\rho\, d\theta\right) d\rho = 2\pi\int_0^{+\infty} e^{-\frac{\rho^2}{2}}\rho\, d\rho = 2\pi\left[e^{-\frac{\rho^2}{2}}\right]_{\rho=0}^{\rho=+\infty} = 2\pi,$$

and by this the conclusion follows.

To compute (5.3.4) notice first that, being C symmetric, it is diagonalizable: this means that there exists T invertible such that $T^{-1}CT = \mathrm{diag}(\sigma_1, \dots, \sigma_d)$. Furthermore, because C is symmetric, T is also orthogonal, that is $T^{-1} = T^t$ (transposed matrix). Therefore $C = TDT^{-1}$, hence

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}C^{-1}x\cdot x}\, dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}(TDT^{1-})^{-1}x\cdot x}\, dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}(TD^{-1}T^{1-})x\cdot x}\, dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}T^{-1}x\cdot T^{-1}x}\, dx.$$

Now, set $y = T^{-1}x$, in such a way that $x = Ty$ and

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}T^t x\cdot T^t x}\, dx = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\, |\det T|\, dy = \int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\, dy.$$

Last = is justified because, being T orthogonal, $TT^t = \mathbb{I}$, hence $1 = \det(TT^t) = \det T \det T^t = (\det T)^2$ by which $|\det T| = 1$. Moreover,

$$D^{1-} y \cdot y = \sum_j \frac{1}{\sigma_j} y_j^2,$$

therefore

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}D^{-1}y\cdot y}\, dy = \int_{\mathbb{R}^d} \prod_{j=1}^d e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j = \prod_{j=1}^d \int_{\mathbb{R}} e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j \overset{x_j = \frac{y_j}{\sqrt{\sigma_j}}}{=} \prod_{j=1}^d \sqrt{\sigma_j} \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{(2\pi)^d \sigma_1 \cdots \sigma_d}.$$

To conclude just notice that

$$\sigma_1 \cdots \sigma_d = \det D = \det(T^{-1}CT) = \det T^{-1} \det C \det T = \det C. \quad \blacksquare$$

5.3.2. Spherical and cylindrical coordinates. The analogous of polar coordinates for functions of three variables are *spherical coordinates*:

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \qquad (\rho,\theta,\varphi) \in [0,+\infty[\times[0,2\pi]\times[0,\pi]. \\ z = \rho\cos\varphi. \end{cases}$$

Also in this case the change of variable is defined in the form

$$(x, y, z) = \Psi(\rho, \theta, \varphi),$$

thus, referring to (5.3.2), $\Psi = \Phi^{-1}$. Hence,

$$\det(\Phi^{-1})' = \det \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix} = \rho^2\sin\varphi.$$

Therefore, (5.3.2) reads as

$$\int_E f(x, y, z)\, dxdydz = \int_{E_{spher}} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\varphi\, d\theta\, d\rho.$$

Here E_{spher} is E in spherical coordinates. This type of change of variable is often useful when f has some spherical symmetry, that is it depends on $x^2 + y^2 + z^2$.

Example 5.3.6. *Using spherical coordinates, compute the volume of a sphere of radius r.*

Sol. — We have

$$\begin{aligned} \lambda_3\left(\{x^2+y^2+z^2 \leqslant r^2\}\right) &= \int_{x^2+y^2+z^2\leqslant r^2} dxdydz = \int_{0\leqslant\rho\leqslant r,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} \rho^2\sin\varphi\, d\rho d\theta d\varphi \\ &= 2\pi\left(\int_0^\pi \sin\varphi\, d\varphi\right)\left(\int_0^r \rho^2\, d\rho\right) = \frac{4}{3}\pi r^3. \quad \blacksquare \end{aligned}$$

When f has not a central symmetry but it is symmetric respect to some of the axes, a further variant of polar coordinates may be useful. Let first introduce this system of coordinates defined as

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \qquad (\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R}. \\ z = z. \end{cases}$$

Also in this case the change of variables is defined in the form

$$(x, y, z) = \Psi(\rho, \theta, z), \text{ where } \Psi = \Phi^{-1}.$$

Being,

$$\det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho,$$

according to (5.3.2) we have

$$\int_E f(x,y,z)\, dxdydz = \int_{E_{cil}} f(\rho\cos\theta, \rho\sin\theta, z)\rho\, d\rho d\theta dz.$$

This change of variables is particularly useful in the case of functions symmetric respect to the z axis (that is depending on $x^2 + y^2$ that becomes ρ^2 in new coords).

Example 5.3.7. *Compute the volume of the rugby ball* $E = \{(x,y,z) \in \mathbb{R}^3 : \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1\}$ *by adapting cylindrical coordinates.*

Proof — Adapting the cylindrical coords $(x, y, z) = \Psi^{-1}(\rho, \theta, z) := (a\rho\cos\theta, a\rho\sin\theta, bz)$ we have

$$\det(\Psi^{-1})' = \det\begin{bmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ a\sin\theta & a\rho\cos\theta & 0 \\ 0 & 0 & b \end{bmatrix} = ba^2\rho,$$

therefore

$$m_3(E) = \int_{\rho^2+\tilde{z}^2\leqslant 1,\ \rho\geqslant 0,\ \theta\in[0,2\pi],\ \tilde{z}\in\mathbb{R}} ba^2\rho\, d\rho d\theta dz = 2\pi a^2 b \int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho\, d\rho dz.$$

To compute the last integral we may use polar coords for $(\rho, z) = (r\cos\alpha, r\sin\alpha)$. Then

$$\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho\, d\rho dz = \int_{-\frac{\pi}{2}\leqslant\alpha\leqslant\frac{\pi}{2},\ 0\leqslant r\leqslant 1} (r\cos\alpha) r\, dr d\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha\, d\alpha \int_0^1 r^2\, dr = \frac{2}{3}.$$

Moral: $m_3(E) = \frac{4\pi}{3}a^2 b$. ■

5.4. Barycenter, center of mass, inertia moments

Through multiple integrals we can define several quantities relevant in Geometry and Physics. To fix ideas consider a set $E \subset \mathbb{R}^3$. We call **barycenter** of E the point $(\overline{x}, \overline{y}, \overline{z})$ defined as

$$\overline{x} = \frac{1}{\lambda_3(E)} \int_E x\, dx\, dy\, dz, \quad \overline{y} = \frac{1}{\lambda_3(E)} \int_E y\, dx\, dy\, dz, \quad \overline{z} = \frac{1}{\lambda_3(E)} \int_E z\, dx\, dy\, dz.$$

In other words, the barycenter is the point whose coords are the mean values of the coords of E. With special symmetries some of the coords of the barycenter may vanish. For instance, if E is symmetric with respect to the plane yz, that is $(x, y, z) \in E$ iff $(-x, y, z) \in E$), then $\overline{x} = 0$. Indeed, if $\Phi(x, y, z) = (-x, y, z)$ we have $\Phi(E) = E$ therefore, by change of variables,

$$\int_E x\, dxdydz = \int_{\Phi(E)} x\, dxdydz = \int_E (-x)|\det \Phi'(x, y, z)|\, dxdydz = -\int_E x\, dxdydz$$

by which $\int_E x\, dxdydz = 0$.

If E represents a solid body with density of mass $\varrho = \varrho(x, y, z)$, the total mass is, by definition,

$$\mu(E) := \int_E \varrho(x, y, z)\, dxdydz.$$

In Physics it is then important the **center of mass**: it is the point where the sum of all the forces acting on E could be applied to get the same effect. This point has coords (x_G, y_G, z_G)

$$x_G = \frac{1}{\mu(E)} \int_E x\varrho(x, y, z)\, dxdydz, \quad y_G = \frac{1}{\mu(E)} \int_E y\varrho(x, y, z)\, dxdydz, \quad z_G = \frac{1}{\mu(E)} \int_E z\varrho(x, y, z)\, dxdydz.$$

If the body is homogeneous (that is $\varrho \equiv \varrho_0 \in \mathbb{R}$) the center of mass coincide with the barycenter as it is easy to see.

Another important quantity for Physics is the **inertia moment with respect to some axis**. For instance, if the axis is the z one, this is defined by

$$I_z := \int_E (x^2 + y^2)\varrho(x, y, z)\, dxdydz.$$

Example 5.4.1. *Determine the barycenter of a spherical cap* $E := \{(x, y, z) \,:\, x^2 + y^2 + z^2 \leqslant r^2,\ z \geqslant h\}$ *con* $0 \leqslant h < r$.

Sol. — By symmetries, it is evident that $\overline{x} = \overline{y} = 0$. Let's compute

$$\overline{z} = \frac{1}{\lambda_3(E)} \int_E z\, dxdydz.$$

It seems convenient to slice E perpendicularly to the $z-$axis:

$$\begin{aligned}\lambda_3(E) &= \int_h^r \left(\iint_{x^2+y^2 \leqslant r^2 - z^2} dxdy \right) dh = \int_h^r \pi(r^2 - z^2)\, dz = \pi r^2 (r - h) - \pi \left[\frac{z^3}{3} \right]_{z=h}^{z=r} \\ &= \pi(r - h)\left(r^2 - \frac{1}{3}(r^2 + rh + h^2) \right).\end{aligned}$$

Similarly

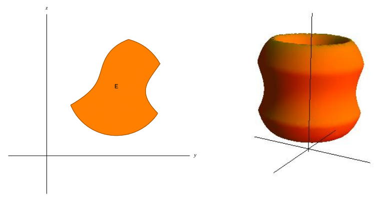
$$\begin{aligned}\int_E z\, dxdydz &= \int_h^r \left(\iint_{x^2+y^2 \leqslant r^2 - z^2} z\, dxdy \right) dz = \int_h^r z \left(\iint_{x^2+y^2 \leqslant r^2 - z^2} dxdy \right) dz = \int_h^r z\pi(r^2 - z^2)\, dz \\ &= \pi r^2 \left[\frac{z^2}{2} \right]_{z=h}^{z=r} - \pi \left[\frac{z^4}{4} \right]_{z=h}^{z=r} = \pi r^2 \frac{r^2 - h^2}{2} - \pi \frac{r^4 - h^4}{4} = \pi \frac{r^2 - h^2}{2} \left(r^2 - \frac{r^2 + h^2}{2} \right) \\ &= \pi \frac{(r^2 - h^2)^2}{4}.\end{aligned}$$

By this we get $\overline{z}$. In the case $h = 0$ (that is when E is the half-sphere) we have $\overline{z} = \frac{3}{8}r$. ■

Let $D \subset \mathbb{R}^3$ by a domain obtained by a rotation around one of the axes of a plane set E. To fix ideas, let's assume that the rotation be around the $z-$axis of a domain E in the plane yz. This domain can be identified by $\{(0, y, z) \ : \ (y, z) \in E\} \subset \mathbb{R}^3$. Therefore, D can be represented as

$$D = \{(y\cos\theta, y\sin\theta, z) \ : \ (y, z) \in E, \ \theta \in [0, 2\pi]\} = \Phi(E \times [0, 2\pi]),$$

where Φ is nothing but the cylindrical coords map.



By the formula of change of variables

$$\lambda_3(D) = \int_{E\times[0,2\pi]} J_\Phi(y, \theta, z)\, dy\, d\theta\, dz = \int_{E\times[0,2\pi]} y\, dy\, d\theta\, dz = 2\pi \int_E y\, dydz$$

that gives the **Pappo's Theorem**:

(5.4.1) $$\lambda_3(D) = 2\pi\lambda_2(E)\overline{y}.$$

Example 5.4.2. *Let's compute the volume of a thorus* $\mathbb{T}_{r,R} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (\sqrt{x^2 + y^2} - R)^2 + z^2 \leqslant r^2\right\}$ *(*$0 < r < R$*).*

Sol. — By (5.4.1)

$$\lambda_3(\mathbb{T}_{r,R}) = 2\pi m_2\left(\{(y - R)^2 + z^2 \leqslant r^2\right)\overline{y} = 2\pi 1\pi r^2\overline{y} = 4\pi^2 r^2\overline{y}.$$

Here $\overline{y}$ it's the ordinate of the barycenter of the disk $E := \{(y - R)^2 + z^2 \leqslant r^2\}$, so

$$\overline{y} = \frac{1}{\lambda_2(E)}\int_E y\, dydz = \frac{1}{\pi r^2}\int_{(y-R)^2+z^2\leqslant r^2} y\, dydz.$$

Changing to polar coord $y - R = \rho\cos\theta$, $z = \rho\sin\theta$, we have easily

$$\overline{y} = \frac{1}{\pi r^2}\int_0^{2\pi}\left(\int_0^r \rho(R + \rho\cos\theta)\, d\rho\right)\, d\theta = \frac{1}{\pi r^2} 2\pi\frac{r^2}{2}R = R,$$

(as it is natural!). Hence $\lambda_3(\mathbb{T}_{r,R}) = 4\pi^2 r^2 R$. ■

5.5. Exercises

Exercise 5.5.1. Compute

1. $\int_{0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant 1-y^2} xe^y\, dxdy$.
2. $\int_{0\leqslant y\leqslant 1-x^2} \frac{x}{2+y}\, dxdy$.
3. $\int_{|y|\leqslant 1-x^2} \frac{1}{1+y}\, dxdy$
4. $\int_{0\leqslant x,y\leqslant 1,\ 0\leqslant z\leqslant 3-x+y} x\sin(\pi y)\, dxdydz4$.
5. $\int_{x\geqslant 0,\ y\geqslant 0,\ x+y+z\leqslant 1} xyz\, dxdydz$.
6. $\int_{0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 2,\ 0\leqslant z\leqslant 6-x^2-y^2} x\log(1 + y)\, dxdydz$.

Exercise 5.5.2. Compute:

1. $\displaystyle\int_{[0,1]^2} e^{\max\{x^2,y^2\}}\,dxdy$. 2. $\displaystyle\int_{[0,1]\times[2,4]} \frac{1}{(x-y)^2}\,dxdy$. 3. $\displaystyle\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4}\,dxdy$.

4. $\displaystyle\int_{1\leqslant x\leqslant 2,\ \frac{1}{x}\leqslant y\leqslant x} \frac{x}{y}\,dxdy$. 5. $\displaystyle\int_{[0,1]^3} e^{\max\{x,y,z\}}\,dxdy$. 6. $\displaystyle\int_{[1,+\infty[^3} y^3z^8e^{-xy^2z^3}\,dxdydz$.

Exercise 5.5.3. Compute

1. $\displaystyle\int_D x\sqrt{y^2-x^2}\,dxdy,\ \ D=\{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant y\}$.

2. $\displaystyle\int_D \frac{x^2e^{-x^2}}{1+(xy)^2}\,dxdy,\ \ D=\{(x,y)\in\mathbb{R}^2\ :\ |xy|\leqslant 1\}$.

3. $\displaystyle\int_D zy^2\sqrt{x^2+zy}\,dxdydz,\ \ D=\{(x,y,z)\in\mathbb{R}^3\ :\ 0\leqslant z\leqslant x^2,\ 0\leqslant x\leqslant 1,\ 0\leqslant y\leqslant 1\}$.

Exercise 5.5.4. For which values $\alpha\in\mathbb{R}$ the function $f_\alpha(x,y):=\frac{1}{(x-y)^\alpha}$ belongs to $L^1([1,+\infty[\times[0,1])$? In this case compute the integral $\int_{[1,+\infty[\times[0,1]} f_\alpha$.

Exercise 5.5.5. Let $D:=\{(x,y)\in\mathbb{R}^2\ :\ x\geqslant 0,\ y\geqslant 0,\ x^2+y^2\leqslant r^2\}$. Draw D and describe it in polar coords. Determine its barycenter and compute the integral

$$\int_D \frac{x+y}{x^2+y^2}\,dxdy.$$

Exercise 5.5.6 (polar, spherical, cylindrical coords). Draw (if possible) and compute the volume of

1. $\left\{(x,y,z) : 9(1-\sqrt{x^2+y^2})^2+4z^2\leqslant 1\right\}$.
2. $\left\{(x,y,z)\ :\ x^2+y^2+z^2\leqslant r^2,\ \left(x-\frac{r}{2}\right)^2+y^2\leqslant\frac{r^2}{4}\right\}$.
3. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\right\}$, $(a,b,c>0)$.
4. $\left\{(x,y,z)\ :\ x^2+y^2\leqslant 1,\ x^2+z^2\leqslant 1,\ y^2+z^2\leqslant 1\right\}$.
5. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2\leqslant 4,\ 4x^2+4y^2+z^2\leqslant 64\right\}$.
6. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2+z^2\leqslant 16,\ x^2+y^2\geqslant 4\right\}$
7. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant\sqrt{x^2+y^2},\ x^2+y^2+z^2\leqslant 1\right\}$.
8. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant x^2+y^2,\ z\leqslant 18-x^2-y^2\right\}$.

Exercise 5.5.7. Discuss existence and (if possible) the value of the following integrals:

1. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\,dxdy.$
2. $\displaystyle\int_{x^2+y^2\leqslant 4} \sqrt{4-x^2-y^2}\,dxdy$
3. $\displaystyle\int_{x^2+2y^2\leqslant 1} \frac{1}{1+x^2+2y^2}\,dxdy.$
4. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+(x^2+2y^2)^2}\,dxdy.$
5. $\displaystyle\int_{x^2+y^2\leqslant 16,\ -5\leqslant z\leqslant 4} \sqrt{x^2+y^2}\,dxdydz.$
6. $\displaystyle\int_{\mathbb{R}^3} \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,dxdydz.$
7. $\displaystyle\int_{\mathbb{R}^3} \frac{1}{1+(x^2+2y^2+3z^2)^2}\,dxdydz.$
8. $\displaystyle\int_{\mathbb{R}^2} \frac{1}{1+(x^2+xy+y^2)^2}\,dxdy.$
9. $\displaystyle\int_{\mathbb{R}^3} e^{-(x^2+y^2+z^2-xy+yz-xz)}\,dxdydz.$
10. $\displaystyle\int_{\mathbb{R}^3} \frac{1}{1+x^4+y^4+z^4}\,dxdydz.$

Exercise 5.5.8. By using carefully the suggested change of variables, compute

1. $\displaystyle\int_D xy\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 3,\ x\leqslant y\leqslant 3x\},\ (u=xy,\ v=\frac{y}{x}).$
2. $\displaystyle\int_D y^2\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 2,\ 1\leqslant xy^2\leqslant 2\}.\ (u=xy,\ v=xy^2).$
3. $\displaystyle\int_D \sqrt{x^2-y^2}\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant x^2-y^2\leqslant 2,\ px\leqslant y\leqslant qx\}\ (-1<p<q<1).\ (u=x^2-y^2,\ v=\frac{y}{x}).$

Exercise 5.5.9. Let $a>1$ and

$$E_a := \left\{(x,y)\in\mathbb{R}^2\ :\ \frac{1}{ax}\leqslant y\leqslant\frac{1}{x},\ x^2\leqslant y\leqslant ax^2\right\}.$$

Draw E_a. Show that $\Phi(x,y) := \left(xy, \frac{y}{x^2}\right)$ is a diffeomorphism modulo null sets on E_a. Use this to compute

$$I(a) := \int_{E_a} \frac{x^2}{y}e^{xy}\,dxdy.$$

Compute $\lim_{a\to+\infty} I(a)$ and check if $I=\int_{x^2\leqslant y\leqslant\frac{1}{x}} \frac{x^2}{y}e^{xy}\,dxdy.$

Exercise 5.5.10 (★). Let $D := \{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant x\leqslant y\}$ and

$$f(x,y) := \frac{x^{3/2}}{\sqrt{y-x}}e^{-(xy)^{3/2}},\ \ (x,y)\in D.$$

Show that $\Phi(x,y) := (xy, x/y)$ is a diffeomorphism modulo null sets on D. Use this to say if $f\in L^1(D)$. In such case compute $\int_D f$.

Exercise 5.5.11 (★). Let $D := \{(x,y)\in[0,+\infty[^2\ :\ xy\geqslant 1\}$, and $f(x,y) := \frac{\log(xy)}{y(x+y^2)^7}$, $(x,y)\in D$. Show that $\Phi(x,y) := (xy, \frac{y^2}{x})$ is a diffeomorphism modulo null sets on E. Use this to say if $f\in L^1(D)$. In such case compute $\int_D f$.