EX 1

mercoledì 17 giugno 2020 16:25

Solve

$$\begin{cases} y' = y^2 - y - 1 \\ y(0) = 1/2 \end{cases}$$

Sol: The diff eqn is a first order (non linear) sep. vars eqn, general form

$$y' = a(t)\, f(y)$$

Here $a(t) \equiv 1$, $f(y) = y^2 - y - 1$.

Because $a \in \mathcal{C}$ and $f \in \mathcal{C}^1$ ($f, f' \in \mathcal{C}$), sol of the CP is either constant ($\Leftrightarrow f(y(0)) = 0$) or non constant (and it may be determined by method of sep of vars)

Now $f(y(0)) = f(1/2) = \frac{1}{4} - \frac{1}{2} - 1 \neq 0$, thus our sol has to be found by sep of vars:

$$y' = \frac{dy}{dt} = y^2 - y - 1 \quad \Leftrightarrow$$

$$\frac{dy}{y^2 - y - 1} = dt \quad \Leftrightarrow$$

$$\int \frac{dy}{y^2 - y - 1} = t + C \qquad C \in \mathbb{R} \qquad (*)$$

$$\int \frac{dy}{y^2-y-1} = t + C \qquad C \in \mathbb{R} \qquad (*)$$

elementary primitive: $\Delta = (-1)^2 - 4 \cdot 1 \cdot (-1)$

$$= 5 > 0$$

$$\Rightarrow (y^2 - y - 1) = (y - y_1)(y - y_2)$$

where $\quad y_{1,2} = \dfrac{1 \pm \sqrt{5}}{2}$

$$\Rightarrow \frac{1}{y^2-y-1} = \frac{1}{(y-y_1)(y-y_2)}$$

$$= \left(\frac{1}{y-y_1} - \frac{1}{y-y_2} \right) \frac{1}{y_1 - y_2}$$

$$\Rightarrow \int \frac{dy}{y^2-y-1} = \frac{1}{y_1-y_2} \left(\int \frac{dy}{y-y_1} - \int \frac{dy}{y-y_2} \right)$$

$$= \frac{1}{y_1-y_2} \log \left| \frac{y-y_1}{y-y_2} \right|$$

$$= \frac{1}{\sqrt{5}} \log \left| \frac{y - \frac{1+\sqrt{5}}{2}}{y - \frac{1-\sqrt{5}}{2}} \right| = \frac{1}{\sqrt{5}} \log \left| \frac{2y - 1 - \sqrt{5}}{2y - 1 + \sqrt{5}} \right|$$

$\Rightarrow$ (returning to $(*)$)

$$\frac{1}{\sqrt{5}} \log \left| \frac{2y-1-\sqrt{5}}{2y-1+\sqrt{5}} \right| = t + C$$

We determine C by imposing the initial cond: at $t=0$

$$C = \frac{1}{\sqrt{5}} \log \left| \frac{\cancel{2}\,\cancel{\tfrac{1}{2}} - 1 - \sqrt{5}}{\cancel{2}\,\cancel{\tfrac{1}{2}} - 1 + \sqrt{5}} \right| = \frac{1}{\sqrt{5}} \log 1 = 0$$

Therefore, the solution in implicit form is

$$\frac{1}{\sqrt{5}} \log \left| \frac{2y-1-\sqrt{5}}{2y-1+\sqrt{5}} \right| = t$$

$$\Leftrightarrow \quad \left| \frac{2y-1-\sqrt{5}}{2y-1+\sqrt{5}} \right| = e^{\sqrt{5}\,t}$$

$$\Leftrightarrow \quad \frac{2y-1-\sqrt{5}}{2y-1+\sqrt{5}} = \pm\, e^{\sqrt{5}\,t}$$

?

by initial cond $\Rightarrow$ $-$

$$\Rightarrow \quad \frac{2y-1-\sqrt{5}}{} = e^{\sqrt{5}\,t}$$

$$\Rightarrow \quad \frac{2y-1-\sqrt{5}}{2y-1+\sqrt{5}} = e^{\sqrt{5}\,t}$$

$$\Rightarrow \quad 1 - \frac{2\sqrt{5}}{2y-1+\sqrt{5}} = e^{\sqrt{5}\,t}$$

$$\Leftrightarrow \quad 1 - e^{\sqrt{5}\,t} = \frac{2\sqrt{5}}{2y-1+\sqrt{5}}$$

$$\Leftrightarrow \quad \left(2y-1+\sqrt{5}\right)\left(1-e^{\sqrt{5}\,t}\right) = 2\sqrt{5}$$

$$\Leftrightarrow \quad y = \frac{\left(e^{\sqrt{5}\,t}-1\right)\left(\sqrt{5}-1\right)+2\sqrt{5}}{2\left(1-e^{\sqrt{5}\,t}\right)}. \qquad \blacksquare$$

$$f(x,y) = 3(x^2+y^2)^2 - 4xy^3$$

1) $\lim\limits_{(x,y)\to\infty} f.$ Clearly $f(x,0) = 3x^4 \xrightarrow[+\infty]{}$ if $|x| = \|(x,0)\| \to +\infty$

Thus, IF the limit exists, THEN it equals $+\infty$.

To prove that this is actually true, we see f on polar coords

$$f = 3(\rho^2)^2 - 4\rho\cos\theta\,(\rho\sin\theta)^3$$

$$= \rho^4\left(3 - 4\cos\theta\,(\sin\theta)^3\right)$$

$$4\cos\theta\,(\sin\theta)^3 = 2\cdot \underbrace{2\sin\theta\cos\theta}_{=\sin(2\theta)}\cdot(\sin\theta)^2$$

$$= \rho^4\left(3 - 2\underbrace{\sin(2\theta)\,(\sin\theta)^2}\right)$$

Now, because $-1 \leq \quad \leq 1$

$1 = 3-2 \leq \qquad \leq 3+2 = 5$

$$\Rightarrow f(x,y) \geq \rho^4 = \|(x,y)\|^4 \xrightarrow[\|(x,y)\| \to +\infty]{} +\infty$$

Conclusion: $\lim\limits_{(x,y)\to\infty_2} f = +\infty$

2. min/max f on $\mathbb{R}^2$

Existence: $\mathbb{R}^2$ is closed but unbounded ($\Rightarrow$ not compact)

However, by 1. $\lim\limits_{\infty_2} f = +\infty$

$\Rightarrow \nexists \max\limits_{\mathbb{R}^2} f$ while $\exists \min\limits_{\mathbb{R}^2} f$.

Determination (of $\min\limits_{\mathbb{R}^2} f$): Let $(x,y)\in\mathbb{R}^2$ be a global min for f. Since f is clearly differentiable and $\mathbb{R}^2$ is open (every pt. lies in the $\operatorname{Int}\mathbb{R}^2$) $\Rightarrow$ (Fermat)

$$\nabla f(x,y) = \vec{0}$$

Now

$$\nabla f = \vec{0} \Leftrightarrow \begin{cases} \partial_x f = 0 \\ \partial_y f = 0 \end{cases} \Leftrightarrow \begin{cases} 6(x^2+y^2)2x - 4y^3 = 0 \\ 6(x^2+y^2)2y - 12xy^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 3(x^2+y^2)x - y^3 = 0 \\ \boxed{y\left[x^2+y^2 - xy\right] = 0} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = 0 \\ 3x^3 = 0 \end{cases} \quad \vee \quad \begin{cases} x^2+y^2 = xy \\ 3x^2y - y^3 = 0 \end{cases}$$

$$\Updownarrow$$

$$(0,0) \quad \vee \quad \begin{cases} x^2+y^2 = xy \\ \boxed{y(3x^2 - y^2) = 0} \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = 0 \\ x^2 = 0 \end{cases} \quad \vee \quad \begin{cases} 3x^2 = y^2 \\ x^2+y^2 = xy \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$(0,0) \qquad \begin{cases} y = \pm\sqrt{3}\,x \\ \_\_\_\_ \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} y = \sqrt{3}\,x \\ 4x^2 = \sqrt{3}\,x^2 \end{cases} \quad \vee \quad \begin{cases} y = -\sqrt{3}\,x \\ 4x^2 = -\sqrt{3}\,x^2 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

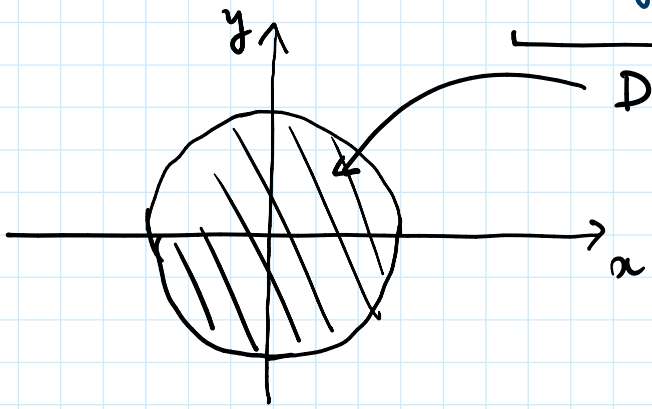
$$x = 0 \qquad\qquad (0,0)$$

$$\Updownarrow$$

$\Updownarrow$ $(0,0)$ $\quad (0,0)$

Conclusion: there's a unique stationary point for f, it is $(0,0)$.
Therefore, there's a unique possible candidate to be min pt for f on $\mathbb{R}^2$ $\Rightarrow$ $(0,0)$ is the (unique) global min pt for f on $\mathbb{R}^2$.

3. min / max f on $\underbrace{\{x^2+y^2 \leq 1\}}_{D}$



Existence: D is closed & bded, $f \in \mathcal{C}$

$\Downarrow$ (Weierstrass)

$\exists$ min / max f on D.

Determination: let $(x,y) \in D$ be a min/max pt for f. We've the following alternative

- IF $(x,y) \in \operatorname{Int} D$ $\Rightarrow$ (Fermat) $\nabla f = 0$. By 2.
$\Rightarrow$ $\boxed{(x,y) = (0,0)}$ Here $f \geq 0$.

• IF $(x,y) \in D \setminus \text{Int}\, D = \partial D$ then not necessarily $\nabla f = \vec{0}$. We have to investigate on f on ∂D. On $x^2+y^2=1$

$$f(x,y) = 3\cdot 1^2 - 4xy^3$$

$$= 3 - 4\cos\theta(\sin\theta)^3 =: g(\theta)$$

$$(x,y) = (\cos\theta, \sin\theta)$$

Let's study $g(\theta)$ $\theta \in [0, 2\pi]$

$$g'(\theta) = -4\left[-\sin\theta\,(\sin\theta)^3 + (\cos\theta)^2 3(\sin\theta)^2\right]$$

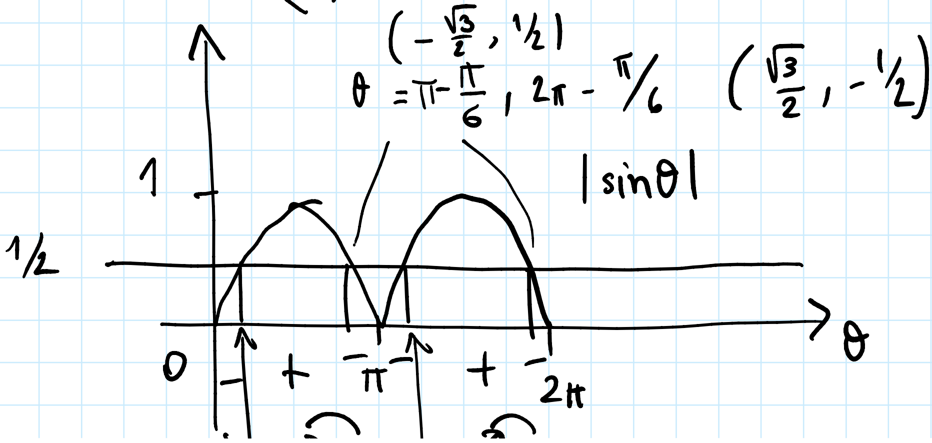
$$= -4(\sin\theta)^2\left[-(\sin\theta)^2 + 3\underbrace{(\cos\theta)^2}_{1-(\sin\theta)^2}\right]$$

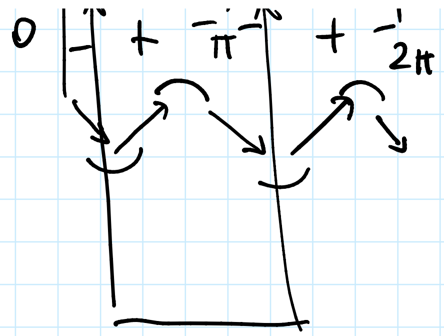
$$= -4(\sin\theta)^2\left[1 - 4(\sin\theta)^2\right]$$

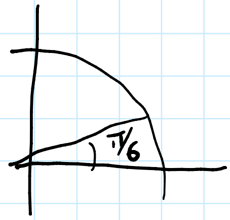
thus $g' \geq 0 \Leftrightarrow 1 - 4(\sin\theta)^2 \leq 0$

$$\Leftrightarrow 4(\sin\theta)^2 \geq 1$$

$$\Leftrightarrow |\sin\theta| \geq 1/2$$







$\theta = \pi/6 \qquad \pi/6 + \pi$

$\downarrow \qquad\qquad \downarrow$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \qquad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Conclusion: candidates for min/max are:

$(0,0) \longrightarrow f = 0$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \longrightarrow f = 3\cdot 1^2 - 4\,\frac{\sqrt{3}}{2}\cdot\frac{1}{2^3}$

$$= 3 - \frac{\sqrt{3}}{4} > 0$$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \longrightarrow f = 3\cdot 1^2 + 4\,\frac{\sqrt{3}}{2}\cdot\frac{1}{2^3}$

$$= 3 + \frac{\sqrt{3}}{4}$$

$\Rightarrow$ $(0,0)$ is global min

$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ are global max.

$\boxtimes$

$f(x,y,z) = xyz$

$D = \{ (x,y,z) \in \mathbb{R}^3 : \ 2x^2 + y^2 + z^2 = 1 \}$

$\min_D / \max f$

Existence: $f \in \mathscr{C}$, D is closed & bdd (by

$$2x^2 + y^2 + z^2 = 1 \Longrightarrow \begin{cases} 2x^2 \le 1 \\ y^2 \le 1 \\ z^2 \le 1 \end{cases} \Longleftrightarrow \begin{cases} |x| \le 1/\sqrt{2} \\ |y| \le 1 \\ |z| \le 1 \end{cases})$$

$\Rightarrow$ (Weierstrass) $\exists \min_D / \max f$

Determination: Since D has not interior points and $D = \{ g(x,y,z) = 0 \}$ where $g(x,y,z) = 2x^2 + y^2 + z^2$, we have a constrained optimization pb. Thus we apply the Lagrange multipliers thm. This says that IF g is a submersion on D (that is $\nabla g \neq \vec{0}$ on D) THEN any (x,y,z) min/max pt for f on D is such that

$$\nabla f = \lambda \nabla g .$$

Thus, first we check g is submersion on D
To this aim let's see where is not a submersion:

$$\nabla g = \vec{0} \iff (4x, 2y, 2z) = \vec{0}$$
$$\iff (x,y,z) = \vec{0} \notin D$$
$$\Rightarrow \nabla g \neq \vec{0} \text{ on } D.$$

Second: let $(x,y,z) \in D$ be any min/max pt for f. Then (Lagrange)

$$\exists \lambda : \nabla f = \lambda \nabla g \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2$$

$\iff$ all 2×2 sub determinants vanish.

$$\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ 4x & 2y & 2z \end{bmatrix}$$

thus

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff \begin{cases} 2y^2z - 4x^2z = 0 \\ 2yz^2 - 4x^2y = 0 \\ 2xz^2 - 2xy^2 = 0 \end{cases}$$

$$\iff \begin{cases} z\,(y^2-2x^2)=0 \\ y\,(z^2-2x^2)=0 \\ \boxed{x\,(z^2-y^2)=0} \end{cases}$$

$$\Updownarrow$$

$$x=0 \quad \vee \quad z=y \quad \vee \quad z=-y$$

$$\iff \begin{cases} x=0 \\ zy^2=0 \\ \cancel{yz^2=0} \end{cases} \quad \vee \quad \begin{cases} z=y \\ y(y^2-2x^2)=0 \\ \cancel{y(y^2-2x^2)=0} \end{cases} \quad \vee \quad \begin{cases} z=-y \\ -y(y^2-2x^2)=0 \\ \cancel{y(y^2-2x^2)=0} \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} x=0 \\ z=0 \end{cases} \vee \begin{cases} x=0 \\ y=0 \end{cases} \qquad \begin{cases} z=y \\ y=0 \end{cases} \vee \begin{cases} z=y \\ y^2=2x^2 \end{cases} \vee \begin{cases} z=-y \\ y=0 \end{cases} \vee \begin{cases} z=-y \\ y^2=2x^2 \end{cases}$$

$$\Updownarrow$$

$(0,y,0)$, $y\in\mathbb{R}$ $\qquad$ $(0,0,z)$, $z\in\mathbb{R}$ $\qquad$ $(x,0,0)$, $x\in\mathbb{R}$ $\qquad$ $(x,0,0)$, $x\in\mathbb{R}$

$$\begin{cases} z=y \\ y=\pm\sqrt{2}\,x \end{cases} \qquad\qquad \begin{cases} z=-y \\ y=\pm\sqrt{2}\,x \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$(x, \pm\sqrt{2}x, \pm\sqrt{2}x)$ $\qquad$ $(x, \pm\sqrt{2}x, \mp\sqrt{2}x)$

same sign $\qquad$ opposite sign

$x\in\mathbb{R}$ $\qquad$ $x\in\mathbb{R}$

These pts are those where $\nabla f = \lambda \nabla g$. We have

These pts are those where $\nabla f = \lambda \nabla g$. We have to check now which among these belong to D:

$$(x,0,0) \in D \iff 2x^2 = 1 \iff x = \pm 1/\sqrt{2}$$

$$f\left(\pm\frac{1}{\sqrt{2}},0,0\right) = 0.$$

$$(0,y,0) \in D \iff y^2 = 1 \iff y = \pm 1$$

$$f(0,\pm 1,0) = 0$$

$$(0,0,z) \in D \iff z^2 = 1 \iff z = \pm 1$$

$$f(0,0,\pm 1) = 0$$

$$(x,\pm\sqrt{2}x,\pm\sqrt{2}x) \in D \iff 2x^2 + 2x^2 + 2x^2 = 1 \iff 6x^2 = 1$$

$$\iff x = \pm\frac{1}{\sqrt{6}}$$

$$f\left(\frac{1}{\sqrt{6}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right) = +\frac{1}{3\sqrt{6}}$$

$$f\left(-\frac{1}{\sqrt{6}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{6}}$$

$$(x,\pm\sqrt{2}x,\mp\sqrt{2}x) \in D \iff 6x^2 = 1 \iff x = \pm\frac{1}{\sqrt{6}}$$

$$f\left(\frac{1}{\sqrt{6}},\pm\frac{1}{\sqrt{3}},\mp\frac{1}{\sqrt{3}}\right) = -\frac{1}{3\sqrt{6}}$$

$$f\left(-\frac{1}{\sqrt{6}},\pm\frac{1}{\sqrt{3}},\mp\frac{1}{\sqrt{3}}\right) = +\frac{1}{3\sqrt{6}}$$

Conclusion: $\left(\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right)$

are max pts

$\left(-\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{3}}, \mp\frac{1}{\sqrt{3}}\right)$

are min pts. $\blacksquare$

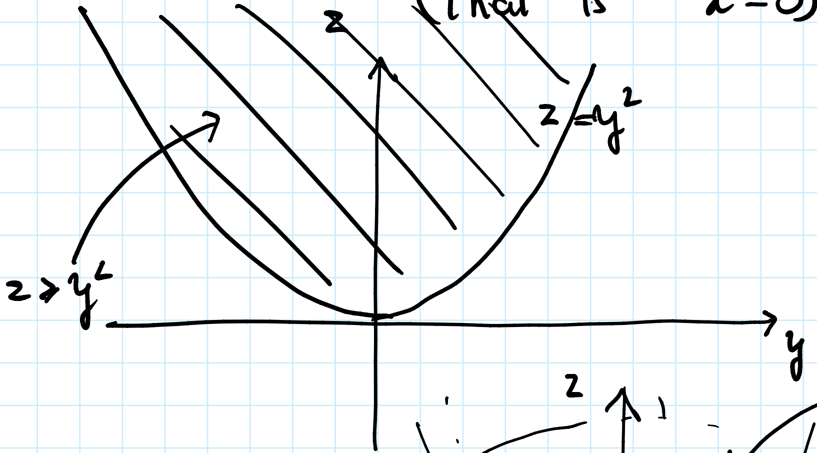
EX 4

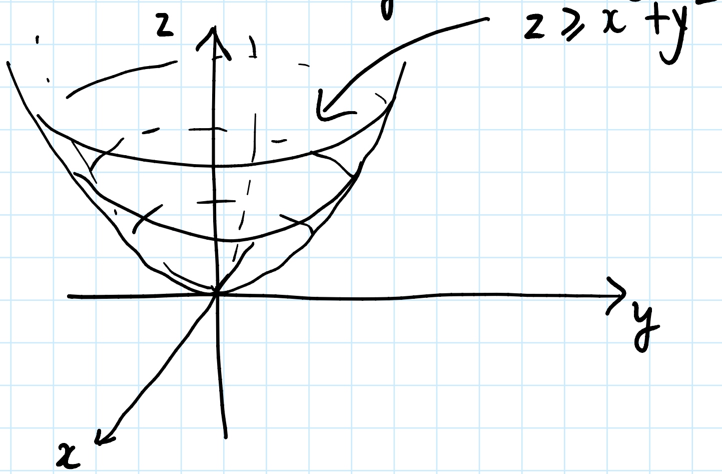
mercoledì 17 giugno 2020 16:27

$$D = \{ (x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z \leq 1-(x+y) \}$$

1. Draw D.

$z \geq x^2+y^2$ is invariant by rotations around z-axis, (because x^2+y^2 is constant along these rotations). We plot first a section on the plane yz (that is $x=0$) : $z \geq y^2$





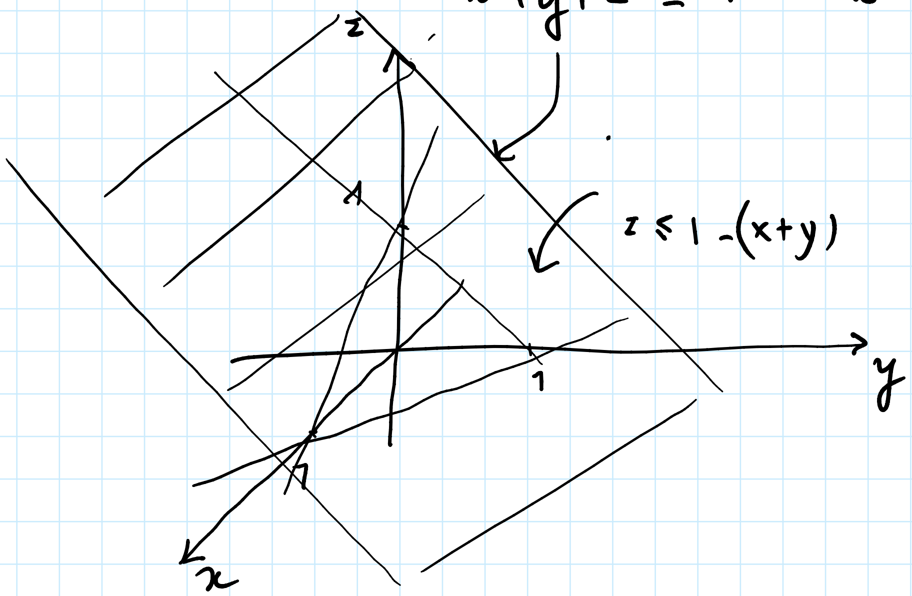
Let's pass to $z \leq 1-(x+y) \iff$

$$x+y+z \leq 1$$

Notice that

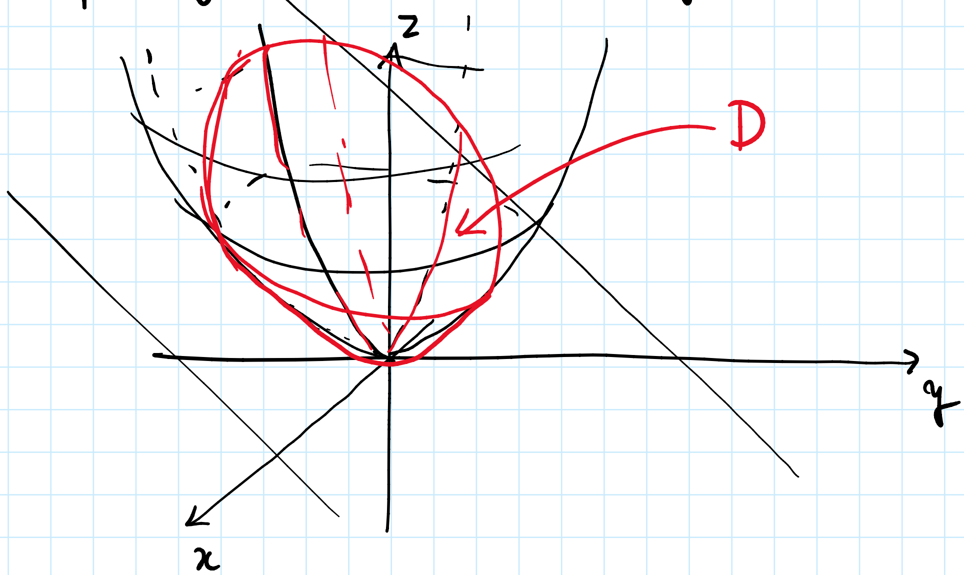
$x+y+z = 1$ is a plane

$x+y+z=1$ is a plane



$z \le 1-(x+y)$ half space under plane

Combining the two figures:



2. Vol D

$$\mathrm{Vol}\, D = \int_D 1 \, dx\,dy\,dz$$

$$= \int_{\underbrace{x^2+y^2 \le z \le 1-(x+y)}_{\downarrow}} dx\,dy\,dz$$

$$x^2+y^2 \leq 1-(x+y)$$

$$\Downarrow$$

$$x^2+y^2+(x+y) \leq 1$$

$$+\,2\cdot\frac{1}{2}x+2\,\frac{1}{2}y\;+\frac{1}{4}-\frac{1}{4}\;+\frac{1}{4}-\frac{1}{4}$$

$$\left(x+\tfrac{1}{2}\right)^2 \qquad \left(y+\tfrac{1}{2}\right)^2$$

thus

$$x^2+y^2+x+y \leq 1 \iff \left(x+\tfrac{1}{2}\right)^2+\left(y+\tfrac{1}{2}\right)^2 \leq 1+\frac{1}{2} = 3/2$$

$$\Rightarrow \operatorname{Vol} D = \int_{x^2+y^2 \leq z \leq 1-(x+y)} dx\,dy\,dz$$

$$\overset{RF}{=} \int_{(x+1/2)^2+(y+1/2)^2 \leq 3/2} \left(\int_{x^2+y^2}^{1-(x+y)} dz\right) dx\,dy$$

$$= \int_{(x+1/2)^2+(y+1/2)^2 \leq 3/2} \left(1-(x+y)-(x^2+y^2)\right) dx\,dy$$

$$= \int\limits_{(x+\frac{1}{2})^2+(y+\frac{1}{2})^2 \leq \frac{3}{2}} \left[1 - \left((x+\tfrac{1}{2})^2 + (y+\tfrac{1}{2})^2 - \tfrac{1}{2}\right)\right] dx\,dy$$

$$\underset{\begin{cases} u = x + \frac{1}{2} \\ v = y + \frac{1}{2} \end{cases}}{=}$$

easily $|\det(\Phi^{-1})'| = 1$

$$= \int\limits_{u^2+v^2 \leq 3/2} \left(\frac{3}{2} - (u^2+v^2)\right) du\,dv$$

$$\underset{\substack{u = \rho\cos\theta \\ v = \rho\sin\theta}}{=} \int\limits_{\substack{\rho^2 \leq 3/2,\ \rho \geq 0 \\ 0 \leq \theta \leq 2\pi}} \left(\frac{3}{2} - \rho^2\right)\rho \, d\rho\, d\theta$$

$$\overset{RF}{=} \int_0^{2\pi} \left(\int_0^{\sqrt{3/2}} \frac{3}{2}\rho - \rho^3 \, d\rho \right) d\theta$$

$$= 2\pi \int_0^{\sqrt{3/2}} \frac{3}{2}\rho - \rho^3 \, d\rho$$

$$= 2\pi \left[\frac{3}{2} \frac{\rho^2}{2} \Big|_0^{\sqrt{3/2}} - \frac{\rho^4}{4}\Big|_0^{\sqrt{3/2}} \right]$$

$$= 2\pi \left[\frac{3}{4}\,\frac{3}{2} - \frac{1}{4}\left(\frac{3}{2}\right)^2 \right]$$

$$= 2\pi \frac{9}{8}\left(1 - \tfrac{1}{2}\right) = \pi\, 9/8 \qquad \blacksquare$$

$$= 2\pi \frac{9}{8} \left(1 - 1/2 \right) = \pi \, 9/8 \qquad \blacksquare$$