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Ex 1.

$$\begin{cases} y' = -y\sqrt{y^2+1} \\ y(0) = 1 \end{cases}$$

i) Compute the sol.

Sol: The eqn is a sep. vars eqn. Stationary sols are

$$y' = 0 \iff -y\sqrt{y^2+1} = 0 \iff y \equiv 0$$

Since $y(0) = 1$ our sol is not const $\implies$ it may be determined by sep. of vars:

$$y' = -y\sqrt{y^2+1} \iff \frac{y'}{y\sqrt{y^2+1}} = -1$$

$$\iff \int \frac{y'}{y\sqrt{y^2+1}}\,dt = -\int dt + C = -t + C$$

$$\overset{u = y(t)}{\iff} \int \frac{du}{u\sqrt{u^2+1}} = -t + C$$

Let's compute $\int \frac{du}{u\sqrt{u^2+1}}$. Setting $u = \mathrm{Sh}\,v$ and recalling that $\mathrm{Ch}^2 - \mathrm{Sh}^2 \equiv 1$, that is $\mathrm{Ch}^2 = \mathrm{Sh}^2 + 1$, we have $du = \mathrm{Ch}\,v\,dv$

$$\int \frac{du}{u\sqrt{u^2+1}} = \int \frac{\mathrm{Ch}\,v}{\mathrm{Sh}\,v \underset{\sqrt{\mathrm{Ch}^2 v} \,=\, |\mathrm{Ch}\,v| \underset{\mathrm{Ch}>0}{=} \mathrm{Ch}\,v}{\underset{\parallel}{\sqrt{\mathrm{Sh}^2 v + 1}}}}\,dv$$

$$= \int \frac{1}{\mathrm{Sh}\,v}\,dv$$

$$= 2 \int \frac{e^v}{\underset{(e^v-1)(e^v+1)}{e^{2v}-1}}\, dv$$

$$= 2 \int \frac{e^v}{e^v-1} - \frac{e^v}{e^v+1}\, dv$$

$$= 2 \left(\log|e^v-1| - \log|e^v+1| \right)$$

$$= 2 \log\left| \frac{e^v-1}{e^v+1} \right|$$

$$\Rightarrow \quad 2 \log\left| \frac{e^v-1}{e^v+1} \right| = -t + C$$

$$v = \mathrm{Sh}^{-1} y(t) \quad \underset{t=0}{\Rightarrow} \quad C = 2 \log\left| \frac{e^{\mathrm{Sh}^{-1} 1} - 1}{e^{\mathrm{Sh}^{-1} 1} + 1} \right|$$

$$\Rightarrow \quad \left| \frac{e^v-1}{e^v+1} \right| = e^{-t/2 + C/2} = \frac{e^{\mathrm{Sh}^{-1} 1} - 1}{e^{\mathrm{Sh}^{-1} 1} + 1} e^{-t/2}$$

$$=: k e^{-t/2}$$

$$\Rightarrow \quad \frac{e^v-1}{e^v+1} = \underset{?}{\textcircled{\pm}} k e^{-t/2}$$

$$t = 0 \Rightarrow \quad \frac{e^{\mathrm{Sh}^{-1} 1} - 1}{e^{\mathrm{Sh}^{-1} 1} + 1} = \pm k \quad \Rightarrow \quad \textcircled{+}$$

$$\Rightarrow \quad \frac{e^v-1}{e^v+1} = k e^{-t/2} \iff e^v - 1 = k e^{-t/2} (e^v + 1)$$

$$\iff e^v \left(1 - k e^{-t/2} \right) = 1 + k e^{-t/2}$$

$$\Longleftrightarrow \quad e^{v} = \frac{1+ke^{-t/2}}{1-ke^{t/2}}$$

$$\Longleftrightarrow \quad v = \log\left(\frac{1+ke^{-t/2}}{1-ke^{-t/2}}\right)$$

$$v = Sh^{-1}y \quad \Longleftrightarrow \quad y(t) = Sh\left(\log\left(\frac{1+ke^{-t/2}}{1-ke^{-t/2}}\right)\right)$$

ii) Interval of def of y and graph of y

Sol: Domain $= \{ t ; \ 1 - ke^{-t/2} > 0 \}$

$$\Rightarrow \quad e^{-t/2} < 1/k \qquad (k > 0)$$

$$-t/\cancel{2} < \log 1/k = -2\log k$$

$$t > 2\log k \Longleftrightarrow I =]2\log k, +\infty[\,.$$

Recalling $\quad k = \dfrac{e^{Sh^{-1}1} - 1}{e^{Sh^{-1}1} + 1} < 1 \Rightarrow \log k < 0$

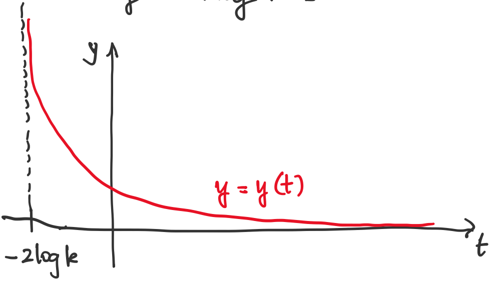
$$t \to 2\log k \quad \Rightarrow \quad y(t) \to Sh \log \frac{2}{0+} = Sh +\infty = +\infty$$

$$t \to +\infty \quad \underset{e^{-t/2} \to 0}{\Rightarrow} \quad y(t) \to Sh \log \frac{1}{1} = Sh\, 0 = 0$$

Furthermore, since $y' = -\underbrace{y}_{+}\underbrace{\sqrt{y^2+1}}_{+}$

and $y > \text{Sh} \log 1 = \text{sh}\, 0 = 0$ $\Rightarrow y' < 0$

$\Rightarrow y \downarrow$

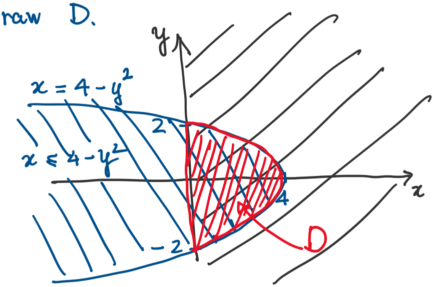


Ex 2. $f(x,y) = 2x(y-1) - y^2$

$D = \{(x,y) : 0 \leqslant x \leqslant 4 - y^2\}$

i) Draw D.

Sol:



$4 - y^2 \geqslant 0$

$\Updownarrow$

$y^2 \leqslant 4 \iff \boxed{|y| \leqslant 2}$

$\underbrace{0 \leqslant x \leqslant 4 - y^2}$ $\boxed{\leqslant 4}$

$\Downarrow$

$\boxed{0 \leqslant x \leqslant 4}$

ii) min/max f on D.

Sol: D is clearly closed & bdd

(defined through large ineq.)

$\Rightarrow$ D is compact and since $f \in \mathcal{C}(D)$ $\Rightarrow$

$\exists \min_D f, \max_D f$ (Weierstrass).

Now, let $(x,y)\in D$ be a min/max pt.

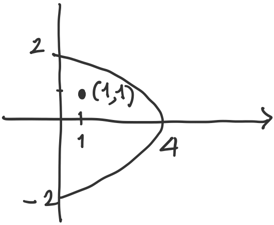
We've the alternative

- $\boxed{(x,y)\in \operatorname{Int} D}$ $\Rightarrow$ (Fermat) $\nabla f = \vec{0}$

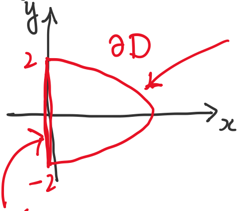
Now since $\nabla f = (2(y-1),\ 2x-2y)$ we've

$$\nabla f = \vec{0} \iff \begin{cases} 2(y-1) = 0 \\ 2(x-y) = 0 \end{cases} \iff \begin{cases} y = 1 \\ x = y = 1 \end{cases}$$

Thus the unique st pt for f in Int D is $(1,1)$:



- $(x,y)\notin \operatorname{Int} D$ $\Rightarrow$ $(x,y)\in \partial D$



$\{(4-y^2,\ y)\ \ -2\leq y\leq 2\} =: B.$

$\{(0,y):\ -2\leq y\leq 2\} =: A$

On A: $f(0,y) = -y^2 \qquad y\in[2,2]$

Clearly $y=0$, $\underline{\underline{(0,0)}}$ is max for f on A

$y=\pm 2$ $\underline{\underline{(0,\pm 2)}}$ are min pts for f on A.

On B: $f(4-y^2, y) = 2(4-y^2)(y-1) - y^2$

$$= 2(4y - 4 - y^3 + y^2) - y^2$$

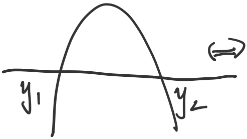
$$= -2y^3 + y^2 + 8y - 8 =: g(y)$$

We've to discuss g on $[-2,2]$.

$g' = -6y^2 + 2y + 8 \geqslant 0 \iff$

$\iff -3y^2 + y + 4 \geqslant 0 \iff \Delta = 1 + 4\cdot 3\cdot 4$

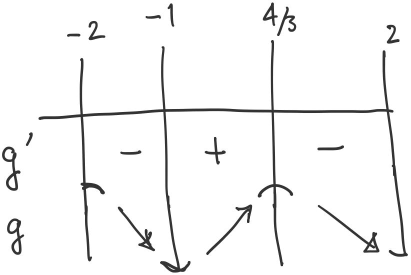
$= 49 > 0$



$\iff \frac{-1+\sqrt{49}}{-6} \leqslant y \leqslant \frac{-1-\sqrt{49}}{-6}$

$\Updownarrow$

$-2 < -1 \leqslant y \leqslant \frac{8}{6} = \frac{4}{3} < 2$



$y = -2 \Rightarrow (0,-2)$

$y = 2 \Rightarrow (0,2)$

$y=-1 \Rightarrow \quad (3,-1)$

$y = 4/3 \Rightarrow \quad \left(4-\frac{16}{9}, 4/3\right) = \left(\frac{20}{9}, 4/3\right)$

Conclusion:

Pt	f	
$(1,1)$	-1	
$(0,\pm 2)$	-4	
$(3,-1)$	-13	MIN.
$\left(\frac{20}{9}, 4/3\right)$	$\frac{40}{9}\cdot\frac{1}{3}-\frac{16}{9} = \frac{40-48}{9\cdot 3} = -\frac{8}{27}$	MAX.

Ex 3:

$$\mathcal{M} = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+(x+y)z=1\}$$

i) $\mathcal{M}\neq\emptyset$, $\mathcal{M}=\{g=0\}$ g submersion on $\mathcal{M}$.

Sol: First $(\pm 1,0,0)\in\mathcal{M} \Rightarrow \mathcal{M}\neq\emptyset$.

Let $g = x^2+y^2+(x+y)z-1$ in such a way that $\mathcal{M}=\{g=0\}$.

Recall: g submersive on $\mathcal{M} \Leftrightarrow \nabla g \neq \vec{0}$ on $\mathcal{M}$

Let's check

$$\nabla g = \vec{0} \Leftrightarrow (2x+z, 2y+z, x+y) = \vec{0}$$

$$\Leftrightarrow \begin{cases} 2x+z=0 \\ 2y+z=0 \\ x+y=0 \end{cases} \Leftrightarrow \begin{cases} 2x+z=0 \\ -2x+z=0 \\ y=-x \end{cases} \left.\begin{matrix} \\ \\ \end{matrix}\right] + \Rightarrow z=0$$

$$\Leftrightarrow \begin{cases} z = 0 \\ x = 0 \\ y = 0 . \end{cases}$$

$$\Rightarrow \nabla g = \vec{0} \Leftrightarrow (x,y,z) = \vec{0} \notin M \Rightarrow \nabla g \neq \vec{0} \text{ on } M.$$

ii) Is M compact?

M is closed (standard: $M = \{g=0\}$, $g \in \mathcal{C}$)

Is M bded? Take pts $(x, 0, z) \in M \Leftrightarrow$

$$x^2 + xz = 1 \Leftrightarrow z = \frac{1-x^2}{x} \quad x \neq 0$$

$$\Rightarrow \left(x, 0, \frac{1-x^2}{x}\right) \in M \quad \forall\ x \neq 0$$

But if $x \to \infty$ $\quad \|(\qquad)\| \to +\infty$

$$\left(\|\quad\|^2 = x^2 + \left(\frac{1-x^2}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2x + x^2\right.$$

$$\left. = 2x^2 + 1/x^2 - 2x \xrightarrow[x \to +\infty.]{} +\infty \right)$$

$\Rightarrow M$ is unbded $\Rightarrow M$ is not compact.

iii) Pts of M at min/max dist to $\vec{0}$

We have to det $\underset{M}{\min/\max} \sqrt{x^2+y^2+z^2}$

or, equivalently,

$$\underset{M}{\min/\max} \underbrace{(x^2+y^2+z^2)}_{f}$$

First: Since M is closed & unbded (ii)

and $f \in \mathcal{C}$ plus (clearly)

$$\lim_{(x,y,z)\to\infty_3} f = +\infty$$

$$\Rightarrow \quad \nexists \max_{M} f \quad \text{but} \quad \exists \min_{M} f .$$

Second: To det min pts we apply Lagrange multipliers thm (we can after i). At $(x,y,z) \in M$ min pt for f

$$\nabla f = \lambda \nabla g \iff \text{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 1.$$

Now

$$\text{rk}\begin{bmatrix} 2x & 2y & 2z \\ 2x+z & 2y+z & x+y \end{bmatrix} = 1 \iff$$

$$\left\{ \begin{array}{l} \begin{vmatrix} 2x & 2y \\ 2x+z & 2y+z \end{vmatrix} = 0 \iff \cancel{2}\cancel{x(2y}+z) - \cancel{2}\cancel{y(2x}+z) = 0 \\[2ex] \begin{vmatrix} 2x & 2z \\ 2x+z & x+y \end{vmatrix} = 0 \iff \cancel{2}x(x+y) - \cancel{2}z(2x+z) = 0 \\[2ex] \begin{vmatrix} 2y & 2z \\ 2y+z & x+y \end{vmatrix} = 0 \iff \cancel{2}y(x+y) - \cancel{2}z(2y+z) = 0 \end{array} \right.$$

$$\Leftrightarrow \begin{cases} (x-y)z = 0 \\ x(x+y) - z(2x+z) = 0 \\ y(x+y) - z(2y+z) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ 2x^2 - z(2x+z) = 0 \end{cases} \vee \begin{cases} z = 0 \\ x(x+y) = 0 \\ y(x+y) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ 2x^2 - 2xz - z^2 = 0 \end{cases} \vee \begin{cases} z = 0 \\ x = 0 \\ y^2 = 0 \end{cases} \vee \begin{cases} z = 0 \\ y = -x \end{cases}$$

$\in M \Leftrightarrow$

$$2x^2 + 2xz = 1$$

$$\Updownarrow$$

$$\begin{cases} y = x \\ 2x^2 - 2xz - z^2 = 0 \end{cases}$$

?

Imposing $\in M$, $(x,x,z) \in M$

$$\Downarrow$$

$$\begin{cases} 2x^2 + 2xz = 1 \\ 2x^2 - 2xz - z^2 = 0 \end{cases}$$

$$4x^2 - z^2 = 1$$

$$\Downarrow$$

$$(0,0,0)$$

$$\notin M$$

$$(x,-x,0)$$

$$\in M$$

$$\Updownarrow$$

$$2x^2 = 1$$

$$\Updownarrow$$

$$x = \pm \frac{1}{\sqrt{2}}$$

(I) $(\pm 1/\sqrt{2}, \pm 1/2, 0)$

(same sign, two pts)

$$\boxed{\text{dist}^2 = \frac{1}{2} + \frac{1}{2} + 0 = 1}$$

$$\Downarrow$$

$$4x^2 = 1+z^2$$

$$\Downarrow$$

$$x = \pm \frac{1}{2}\sqrt{1+z^2}$$

thus we've pts $\left(\pm \frac{1}{2}\sqrt{1+z^2}, \pm \frac{1}{2}\sqrt{1+z^2}, z\right)$ same sign. Let's check for which z they $\in M$:

$$\cancel{2}\cdot \frac{1}{\cancel{4}2}(1+z^2) \pm 2\sqrt{1+z^2}\, z = \cancel{1}2$$

$$(x^2+y^2) \qquad\qquad (x+y)z$$

$$\Leftrightarrow \boxed{z^2 \pm 2\sqrt{1+z^2}\, z = 1}$$

$$\Leftrightarrow z^2 - 1 = \pm 2z\sqrt{1+z^2}$$

Case ⊕ $z^2 - 1 = 2z\sqrt{1+z^2}$ (notice $z^2 > 1 \Rightarrow z > 0$, $z^2 < 1 \Rightarrow z < 0$)

$$\Updownarrow$$

$$(z^2-1)^2 = 4z^2(1+z^2)$$

$$\Updownarrow$$

$$z^4 + 1 - 2z^2 = 4z^2 + 4z^4$$

$$\Updownarrow$$

$$3z^4 + 6z^2 - 1 = 0 \Leftrightarrow z^2 = \frac{-6 \pm \sqrt{48}}{6}$$

$$= -1 \pm \frac{\sqrt{48}}{6}$$

⊖ no sols.

⊕ $z^2 = -1 + \frac{\sqrt{48}}{6} < 1$

$$z = -\sqrt{\frac{\sqrt{48}}{6} - 1}$$

we finally have

(II) $$\left(\pm\frac{1}{2}\sqrt{\frac{\sqrt{48}}{6}},\ \pm\frac{1}{2}\sqrt{\frac{\sqrt{48}}{6}},\ -\sqrt{\frac{\sqrt{48}}{6}-1}\right)$$

$$\text{dist}^2 = \frac{1}{\cancel{4}}\,\frac{1}{2}\,\frac{\sqrt{48}}{6}\cdot\cancel{2} + \frac{\sqrt{48}}{6} - 1$$

$$= \frac{\cancel{3}}{2}\,\frac{\sqrt{48}}{\cancel{6}\,3} - 1 = \frac{\sqrt{48}}{6} - 1 < 1$$

Case ⊖ $$\underbrace{z^2 - 1} = -2z\sqrt{1+z^2}$$

$+$ if $-$

$-$ if $+$

$z^2 > 1 \Leftrightarrow z < 0$

$z^2 < 1 \Leftrightarrow z > 0$

Squaring we obtain the same eqn as before

$$z^2 = \frac{\sqrt{48}}{6} - 1 < 1 \Rightarrow z = +\sqrt{\frac{\sqrt{48}}{6} - 1}$$

and pts

(III) $$\left(\pm\frac{1}{2}\sqrt{\frac{\sqrt{48}}{6}},\ \pm\frac{1}{2}\sqrt{\frac{\sqrt{48}}{6}},\ +\sqrt{\frac{\sqrt{48}}{6}-1}\right)$$

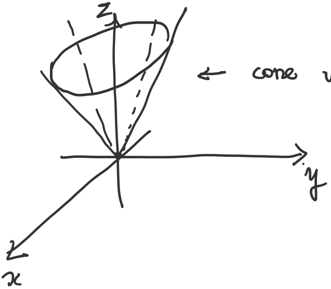
same dist^2

Conclusion: pts (II) & (III) are min pts.

Ex 4: $D = \{ z \geqslant 0, \ x^2 + 4y^2 \leqslant z^2 \leqslant 8 - x^2 - 4y^2 \}$

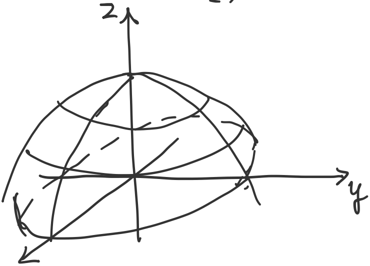
i) Draw D.

$$z^2 \geqslant x^2 + 4y^2 \overset{z \geqslant 0}{\Longleftrightarrow} z \geqslant \sqrt{x^2 + 4y^2}$$



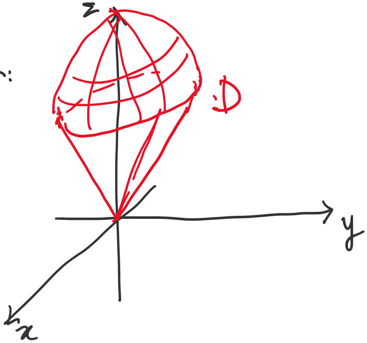
← cone with elliptic sect.

$$z^2 \leqslant 8 - (x^2 + 4y^2) \underset{z \geqslant 0}{\Longleftrightarrow} 0 \leqslant z \leqslant \sqrt{8 - (x^2 + 4y^2)}$$



x

Putting together:



iia) Vol D $= \displaystyle\int_D dx\,dy\,dz$

$$\underbrace{x^2+4y^2}_{\text{must be } \leqslant \text{ than}} \leqslant z^2 \leqslant 8-(x^2+4y^2)$$

$$x^2+4y^2 \leqslant 8-(x^2+4y^2) \Longleftrightarrow \cancel{2}(x^2+4y^2) \leqslant \cancel{8}\,4$$

$$\Rightarrow \int_D dx\,dy\,dz = \int_{x^2+4y^2\leqslant 4} \left(\int_{x^2+4y^2 \leqslant z^2 \leqslant 8-(x^2+4y^2)} dz \right) dx\,dy$$

$$x^2+4y^2 \leqslant z^2 \leqslant 8-(x^2+4y^2) \overset{z\geqslant 0}{\Updownarrow} \sqrt{x^2+4y^2} \leqslant z \leqslant \sqrt{8-(x^2+4y^2)}$$

$$= \int_{x^2+4y^2\leqslant 4} \left(\int_{\sqrt{x^2+4y^2}}^{\sqrt{8-(x^2+4y^2)}} dz \right) dx\,dy$$

$$= \int_{x^2+4y^2\le 4} \left(\sqrt{8-(x^2+4y^2)} - \sqrt{x^2+4y^2}\right) dx\,dy$$

Use adapted polar coords:

$$\begin{cases} x = \rho\cos\theta \\ 2y = \rho\sin\theta \end{cases} \iff \phi^{-1} \begin{cases} x = \rho\cos\theta \\ y = \frac{1}{2}\rho\sin\theta \end{cases}$$

This way $\boxed{x^2+4y^2=\rho^2}$

Set $\{(x,y) : x^2+4y^2 \le 4\}$ becomes

$$\{(\rho,\theta) : \rho^2 \le 4\} = \{0 \le \rho \le 2,\ 0 \le \theta \le 2\pi\}$$

$$|\det (\phi^{-1})'(\rho,\theta)| = \frac{1}{2}\rho$$

$$\Rightarrow \text{Vol } D = \int_{\substack{0\le\rho\le 2\\ 0\le\theta\le 2\pi}} \left(\sqrt{8-\rho^2} - \overset{|\rho| = \rho \ (\rho>0)}{\overset{\parallel}{\sqrt{\rho^2}}}\right)\frac{1}{2}\rho \, d\rho\, d\theta$$

$$\overset{RF}{=} \frac{1}{2}\int_0^{2\pi}\left(\int_0^2 \left(\rho(8-\rho^2)^{1/2} - \rho^2\right)d\rho\right)d\theta$$

$$= \frac{\cancel{2}\pi}{\cancel{2}} \cdot \left\{ \left[-\frac{2}{3}(8-\rho^2)^{3/2}\right]_0^2 - \left[\frac{\rho^3}{3}\right]_0^2 \right\}$$

$$= \pi\left\{-\frac{2}{3}\left(4^{3/2} - 8^{3/2}\right) - \frac{8}{3}\right\}$$

$$= \frac{\pi}{3}\left\{ 2\left(8^{3/2} - 4^{3/2}\right) - 8 \right\}.$$

iib) Compute the barycentre of D

$$\bar{x} = \frac{1}{\operatorname{Vol} D} \int_D x \, dx\,dy\,dz$$

$$\int_D x\,dx\,dy\,dz = \int_{x^2+4y^2\le 4} \left(\int_{\sqrt{x^2+4y^2}}^{\sqrt{8-(x^2+4y^2)}} x \, dz \right) dx\,dy$$

$$= \int_{x^2+4y^2\le 4} x\left(\sqrt{8-(x^2+4y^2)} - \sqrt{x^2+4y^2}\right) dx\,dy$$

$$\overset{\text{pol coords}}{=} \int_{\substack{0\le\rho\le 2\\ 0\le\theta\le 2\pi}} \rho\cos\theta\left(\sqrt{8-\rho^2}-\rho\right)\rho\,d\rho\,d\theta$$

$$\overset{\text{Rid Form}}{=} \int_0^2 \rho^2\left(\sqrt{8-\rho^2}-\rho\right)\left(\underset{\substack{\shortparallel\\ 0}}{\int_0^{2\pi}\cos\theta\,d\theta}\right)d\rho$$

$$= 0$$

Similarly $\int_D y\,dx\,dy\,dz = 0$

Finally

$$\int_D z\,dx\,dy\,dz = \int_{x^2+4y^2\le 4} \int_{\sqrt{x^2+4y^2}}^{\sqrt{8-(x^2+4y^2)}} z\,dz\;dx\,dy$$

$$= \int_{x^2+4y^2\le 4} \left[\frac{z^2}{2}\right]_{\sqrt{x^2+4y^2}}^{\sqrt{8-(x^2+4y^2)}} dx\,dy$$

$$= \frac{1}{2} \int_{\substack{0\le\rho\le 2\\ 0\le\theta\le 2\pi}} \left(8-\rho^2-\rho\right)\rho \; d\rho\, d\theta$$

$$= \frac{\cancel{2}\pi}{\cancel{2}} \int_0^2 \left(8\rho-\rho^3-\rho^2\right) d\rho$$

$$= \pi\left[\, 4\rho^2\Big|_0^2 - \frac{\rho^4}{4}\Big|_0^2 - \frac{\rho^3}{3}\Big|_0^2 \,\right]$$

$$= \pi\left[\, \underset{\substack{\shortparallel\\ 12}}{16-4} - \frac{8}{3} \,\right] = \pi\,\frac{28}{3}$$

Thus $(\bar{x},\bar{y},\bar{z}) = \left(0, 0, \dfrac{\frac{28}{3}\pi}{\mathrm{Vol}\, D}\right).$

☑