Principles of Math An 2

- Introduction to Diff Eqns (1 week)
- Diff Calc for functs of several vars. (2 weeks)
- Integration for functs of several vars. (1 week)

1. Introd. to Diff. Eqns

What is a Differential Equation

First: it is an equation
↓
identity with unknowns

$$x + 2 = 0$$

$$x^5 + x + 1 = 0 \qquad x^2 + 2x - 5 = 0$$

$$xy + x + y = 0$$

algebraic eqns.

$$e^x y + e^y x = 0$$

Here a or x, y are numbers.

$\sin x = 0$

$X = X$

We may notice that equations

- may have / may have not sols.
- if there's a sol this can be unique / non unique (ex. we may have infinitely many sols)
- sometimes they have explicit sols (sol = formula) / numerical sols

Diff Eqns are equations where the unknown is a function, This function is a funct of one real var

$$\boxed{y = y(t)} \equiv y(x)$$

that enters into the eqn with some of its derivatives

(1) $y(t) + y'(t) = 0 \quad \forall t \in [a,b]$

(2) $y''(t) = -k\, y(t)$

(3) $y'(t) = y(t)\,(1 - y(t))$

$$\boxed{F(y(t), y'(t), y''(t), t) = 0}$$

Motivating Examples

Ex 1: (Malthus Model)

We consider the evolution (time) of a population

Pb: We know initial population, we want to det. the future pop.

the future pop.

Modelling:

1. $P = P(t)$ (number of individuals at time t)

2. $P(0)$ is assumed to be known

3. $P(t)$, $P(t+h)$ $h>0$

$P(t+h) - P(t)$ = absolute var of population

$$\frac{P(t+h) - P(t)}{h\,P(t)} = \text{growth rate of the pop. between } t \text{ and } t+h.$$

$$\frac{P(t+h) - P(t)}{h} \cdot \frac{1}{P(t)} \overset{h \to 0}{\approx} \frac{P'(t)}{P(t)}$$

instantaneous growth rate

4. (Malthus model) Hypothesis: instantaneous growth rate = constant

$$\boxed{\frac{P'(t)}{P(t)} = k \qquad (k \text{ constant})}$$

$$\Updownarrow$$

$$\begin{cases} P'(t) = k\,P(t) \\ P(0) = P_0 \end{cases} \qquad k,\ P_0 \text{ known.}$$

To determine $P = P(t)$ we have to solve

$$P'(t) = k\,P(t)$$

$$\Updownarrow$$

$$(\log P(t))' = \boxed{\frac{P'(t)}{P(t)} = k}$$

$$\Updownarrow$$

$$(\underbrace{\log P(t)}_{})' = k$$

$$Q(t)' = k$$

$$\Updownarrow$$

$$\log P(t) = Q(t) = \int k\,dt = kt + c \qquad c \in \mathbb{R}$$

$$\boxed{\log P(t) = kt + c}$$

$$\Updownarrow$$

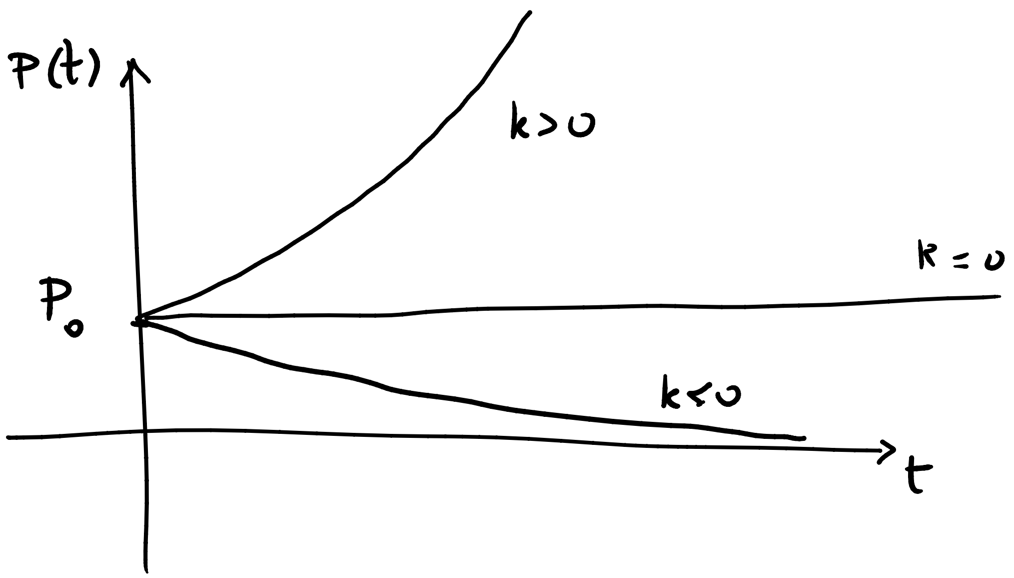
$$P(t) = e^{kt+c} = e^{c} e^{kt}$$

$$\boxed{P(t) = C e^{kt}} \qquad C > 0$$

To finish, we impose the initial cond

$$C = C e^{k0} = P(0) = P_0$$

$$\Rightarrow \qquad \boxed{P(t) = P_0 e^{kt}}$$

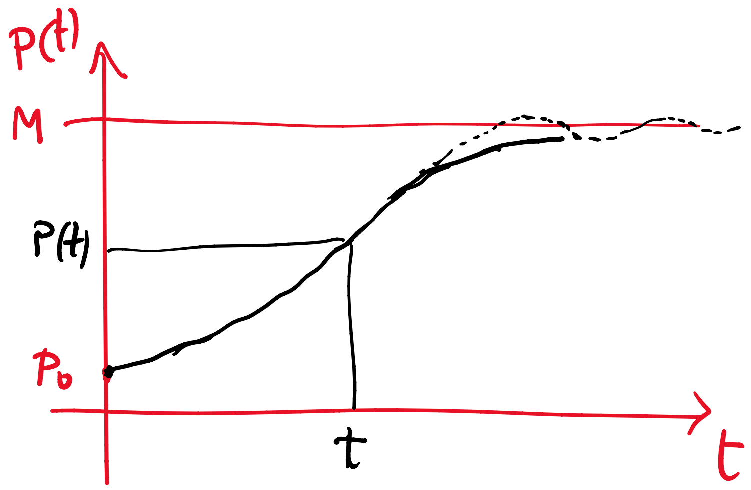


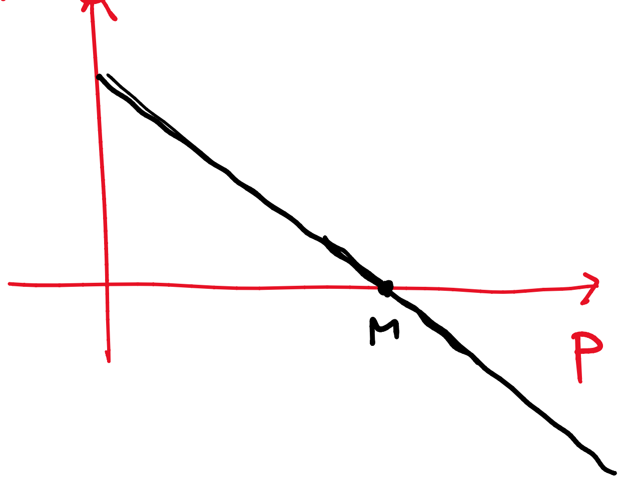
Ex 2 (Logistic Model)

We consider still a population evolving in time. We assume an instantaneous growth rate such that there can be up to a number $M > 0$ of indiv.

$$\frac{P'(t)}{P(t)} = k\,(M - P(t))$$

inst growth rate





So we're reduced to the Pb:

$$\begin{cases} P'(t) = k\,P(t)\,(M - P(t)) \\ P(0) = P_0 \end{cases}$$

k, M, P_0 known.

To det $P(t)$ we have to solve this

mathematical pb.

Notice that

$$P'(t) = k\, P(t)\,(M - P(t)) \Leftrightarrow$$

$$\frac{P'(t)}{P(t)} = k\,(M - P(t))$$

$$\Updownarrow$$

$$\boxed{\frac{P'(t)}{P(t)\,(M - P(t))} = k}$$

$$\downarrow$$

$$\frac{P'}{P\,(M-P)} =$$

$$\frac{1}{P\,(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

$M - P + P$

$$= \frac{P'}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

$$= \frac{1}{M}\left(\frac{P'}{P} - \frac{P'}{P-M}\right)$$

$$\| $$

$$(\log P)' - (\log(P-M))'$$

$f' - g'$

$= (f-g)'$

$$= (f \cdot g)'$$

$$= \frac{1}{M}\left[\log P - \log(P-M)\right]'$$

$$= \left[\frac{1}{M}\log\frac{P}{P-M}\right]'$$

So

$$\frac{P'(t)}{P(t)\,(M-P(t))} = k \iff \underbrace{\left[\frac{1}{M}\log\left|\frac{P(t)}{P(t)-M}\right|\right]'}_{Q(t)' = k} = k$$

$$\Rightarrow \quad Q(t) = kt + c \qquad c \in \mathbb{R}$$

$$\parallel$$

$$\frac{1}{\cancel{M}}\log\left|\frac{P}{P-M}\right| = Mkt + \underbrace{CM} = kMt + c$$

$$\Rightarrow \quad \left|\frac{P}{P-M}\right| = e^{kMt+c} = e^{c}\, e^{kMt}$$

$$\left|\frac{P}{P-M}\right| = C\, e^{kMt} \qquad C > 0$$

$$\Leftrightarrow \boxed{\frac{P}{P-M} = \pm C e^{kMt} \equiv C e^{kMt}} \quad C \in \mathbb{R}$$
$$(*)$$

$$\Leftrightarrow \quad P = (P-M) C e^{kMt}$$

$$\Leftrightarrow \quad P\left(1 - C e^{kMt}\right) = -MCe^{kMt}$$

$$\Leftrightarrow \quad P(t) = -\frac{MCe^{kMt}}{1 - Ce^{kMt}} \qquad C \in \mathbb{R}$$

By imposing $P(0) = P_0$ in $(*)$

$$C = \frac{P_0}{P_0 - M}$$

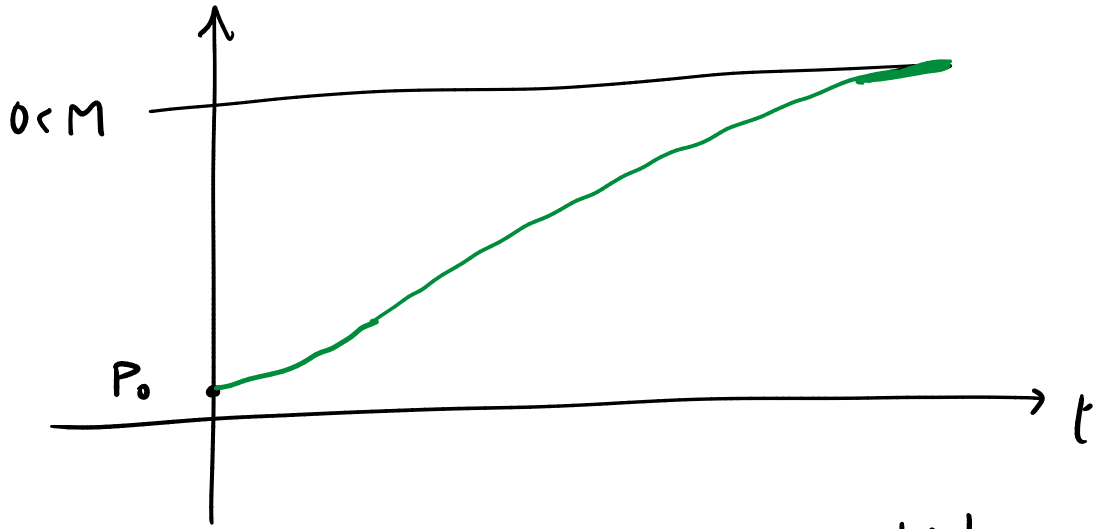
$$\Rightarrow \quad P(t) = + \frac{M \frac{P_0}{-P_0+M} e^{kMt}}{1 - \frac{P_0}{P_0 - M} e^{kMt}}$$

$$= \frac{MP_0}{\cancel{M-P_0}} \quad \frac{e^{kMt}}{P_0 - M - P_0 e^{kMt}}$$

$(-1)\nearrow \; \cancel{P_0 - M}$

$$= \frac{M P_0 \, e^{kMt}}{M + P_0\left(e^{kMt} - 1\right)}$$

$$P(t) = \frac{M P_0 \, e^{kMt}}{M + P_0\left(e^{kMt} - 1\right)} \overset{?}{<} M$$



$k > 0$

$$t \sim 0 \qquad P(t) \sim \frac{M P_0 \, e^{kMt}}{M} = P_0 \, e^{kMt}$$

$$t \sim +\infty \qquad P(t) \sim \frac{M \cancel{P_0} \, \cancel{e^{kMt}}}{\cancel{P_0} \, \cancel{e^{kMt}}} = M$$