venerdì 28 febbraio 2020 08:36

Ex. 3.4.1 Compute

#1 $\partial_{(\sqrt{3},1)} \underbrace{\log(1+x^2y^2)}_{f(x,y)}$ at $(x,y)=(1,1)$

Remind that

$$\partial_{\vec v} f(\vec x) = \lim_{t\to 0} \frac{f(\vec x + t\vec v) - f(\vec x)}{t}$$

$$\Rightarrow \quad \partial_{(\sqrt{3},1)} f(1,1) = \lim_{t\to 0} \frac{f\big((1,1)+t(\sqrt{3},1)\big) - f(1,1)}{t}$$

$$= \lim_{t\to 0} \frac{f(1+t\sqrt{3},\, 1+t) - f(1,1)}{t}$$

$$= \lim_{t\to 0} \frac{\log\big(1 + (1+t\sqrt{3})^2(1+t)^2\big) - \log 2}{t} \to \frac{0}{0}$$

$$\overset{H}{=} \lim_{t\to 0} \frac{\dfrac{1}{1+(1+t\sqrt{3})^2(1+t)^2}\Big[2(1+t\sqrt{3})\sqrt{3}(1+t)^2 + (1+\sqrt{3}t)^2\, 2(1+t)\Big]}{1}$$

$$= \frac{2\sqrt{3}+2}{2} = \sqrt{3}+1. \quad \blacksquare$$

#4 $\partial_{(-1,1)} \dfrac{xy}{x^2+y^4}$ at $(0,0)$

$$f = \begin{cases} \dfrac{xy}{x^2+y^4} & (x,y) \neq (0,0) \end{cases}$$

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Here

$$\partial_{(-1,1)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(-1,1)) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{f(-t,t) - \cancel{0}}{t}$$

$$= \lim_{t\to 0} \frac{\dfrac{-t.t}{t^2+t^4}}{t} = \lim_{t\to 0} \frac{-t^2}{t^2+t^4}\,\frac{1}{t}$$

$$= \lim_{t\to 0} \frac{-1}{1+t^2} \cdot \frac{1}{t} \quad \nexists \quad \Rightarrow \partial_{(-1,1)} f(0,0)$$

doesn't $\exists$ ■

<u>Rmk:</u> The directional deriv is not a good concept of derivative: it may happens that

$$\exists\ \partial_{\vec{v}} f(\vec{x}) \qquad \forall\, \vec{v} \in \mathbb{R}^d$$

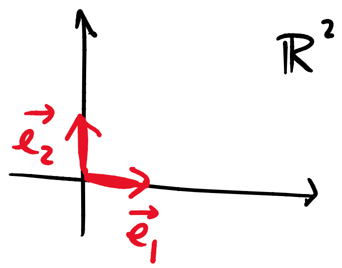
but f is not even continuous at $\vec{x}$. ■

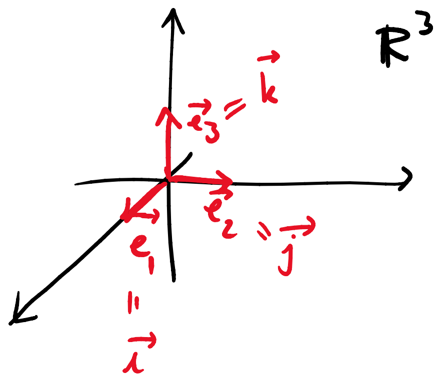
Nevertheless, some part dir. derivatives play an important role: recall that the canonical base for $\mathbb{R}^d$

for $\mathbb{R}^d$

$$\vec{e}_j = (0,0,\dots,0,\overset{j}{1},0\dots0) \qquad j=1,2,\dots,d$$

$d=2$





If $\vec{x} \in \mathbb{R}^d$

$$\vec{x} = (x_1, x_2, \dots, x_d) = \sum_{j=1}^{d} x_j \vec{e}_j$$

Def: (partial derivative)

$$\partial_j f(\vec{x}) := \partial_{\vec{e}_j} f(\vec{x})$$

(other notations $\partial_j \equiv \partial_{x_j} \equiv \frac{\partial}{\partial x_j}$)

Notice that

$$\partial_{\vec{e}_j} f(\vec{x}) \underset{\text{def}}{=} \lim_{t\to 0} \frac{f(\vec{x}+t\vec{e}_j) - f(\vec{x})}{t}$$

$\vec{x} = (x_1, \dots, x_d)$

$\vec{e}_j = (0,0,\dots,0,1,0,\dots 0)$

$\vec{x} + t\vec{e}_j = (x_1 \dots x_d) + (0,0 \dots 0, t, 0 \dots 0)$

$= (x_1, x_2, \dots, x_{j-1}, x_j + h, x_{j+1} \dots x_d)$

$$= \lim_{t \to 0} \frac{f(x_1, x_2, \dots, x_{j-1}, x_j + t, x_{j+1}, \dots, x_d) - f(x_1, \dots, x_d)}{t}$$

$$= \lim_{t \to 0} \frac{f(\_\_\_, x_j + t, \_\_\_) - f(\_, x_j, \_)}{t}$$

$$\left[\lim_{t \to 0} \frac{g(x_j + t) - g(x_j)}{t} \qquad g(x_j) = f(\_\_\_, x_j, \_\_\_) \right.$$

$$= \frac{d f}{d x_j}(\_\_\_, x_j, \_\_\_)$$

Example: $f(x,y) = x^2 + xy.$

$$\partial_x f(x,y) = 2x + y$$

$$\partial_y f(x,y) = 0 + x = x$$

$$f(x,y,z) = e^x \sin(y+z) - z^2$$

$$\partial_x f = e^x \sin(y+z)$$

$$\partial_y f = e^x \cos(y+z)$$

$$\partial_z f = e^x \cos(y+z) - 2z.$$

$\square$

Differential

Let's return on

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad (*)$$

where $f = f(x)$, $x \in \mathbb{R}$. Notice that

$$(*) \Leftrightarrow \lim \left[\frac{f(x+h) - f(x)}{h} - f'(x) \right] = 0$$

$$(*) \iff \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} - f'(x) \right] = 0$$

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)\,h}{h} = 0$$

$$\iff \boxed{f(x+h) - f(x) - f'(x)\,h = o(h)} = o(|h|)$$

$$\left[\quad \varepsilon(h) \text{ is } o(h) \quad \text{if} \quad \lim_{h\to 0} \frac{\varepsilon(h)}{h} = 0 \quad \right]$$

This property can be extended to the case

$$f = \vec{f}(\vec{x}) \qquad \vec{x} \in \mathbb{R}^d, \quad \vec{f} \in \mathbb{R}^m$$

The key idea is:

<u>Def</u>: We say that $\vec{f} = \vec{f}(\vec{x})$ is <u>differentiable</u> at point $\vec{x}$ if $\exists\ \vec{f}'(\vec{x})$ $m \times d$ matrix

such that

$$(*) \quad \boxed{\underbrace{\vec{f}(\vec{x}+\vec{h})}_{\in \mathbb{R}^m} - \underbrace{\vec{f}(\vec{x})}_{\in \mathbb{R}^m} - \underbrace{\vec{f}'(\vec{x})\,\overset{\mathbb{R}^d}{\vec{h}}}_{\in \mathbb{R}^m} = o(\|\vec{h}\|)}$$

$$\vec{f} : D \subset \mathbb{R}^d \to \mathbb{R}^m$$

$m \times d$ matrix

$$m \text{ lines.} \quad \begin{array}{c} \overbrace{\qquad\qquad\qquad}^{d \text{ columns}} \\ \left[\begin{array}{c} \hline \hline \end{array} \right] \end{array} \begin{pmatrix} h_1 \\ \vdots \\ h_d \end{pmatrix} = \begin{pmatrix} = \end{pmatrix}$$

$m \times a$ matrix

m lines. $\begin{bmatrix} — \\ — \end{bmatrix} \begin{pmatrix} \vdots \\ h_d \end{pmatrix} = \begin{pmatrix} \\ - \end{pmatrix} \in \mathbb{R}^m$

$(*)$ means

$$\frac{\vec{f}(\vec{x}+\vec{h}) - \vec{f}(\vec{x}) - \vec{f}'(\vec{x})\vec{h}}{\|\vec{h}\|} \xrightarrow[\vec{h}\to 0]{} 0$$

The matrix $\vec{f}'(\vec{x})$ is called jacobian matrix

Now we have this def, the first pb is:

how do we compute the jacobian matrix?

Prop: If f is differentiable at $\vec{x}$ $(\exists f'(\vec{x}))$

$$\Rightarrow \quad \exists\, \partial_{\vec{v}} f(\vec{x}) = f'(\vec{x})\vec{v}.$$

Proof: f differentiable means

$$(*) \quad \boxed{\underbrace{f(\vec{x}+\vec{h}) - f(\vec{x})} - \underbrace{f'(\vec{x})\vec{h}} = o(\|\vec{h}\|).}$$

Now, if $\vec{v} \neq 0$ $\quad (\partial_{\vec{0}} f(\vec{x}) = 0)$

$$\partial_{\vec{v}} f(\vec{x}) \underset{\text{by def}}{=} \lim_{t\to 0} \frac{f(\vec{x}+t\vec{v}) - f(\vec{x})}{t}$$

by $(*)$ $\quad f(\vec{x}+\vec{h}) - f(\vec{x}) = f'(\vec{x})\vec{h} + o(\|\vec{h}\|)$

$\vec{h} = t\vec{v}$

$$= \lim_{t\to 0} \frac{f'(\vec{x})(t\vec{v}) + o(\|t\vec{v}\|)}{t}$$

$$= \lim_{t\to 0} \frac{t\left(f'(\vec{x})\vec{v} + o(|t|\,\|\vec{v}\|)\right)}{t}$$

$$= \lim_{t\to 0} f'(\vec{x})\vec{v} + \frac{o(|t|)}{|t|} \nearrow^{0} \operatorname{sgn} t$$

$$= f'(\vec{x})\vec{v}. \quad \blacksquare$$

So in particular

f differentiable $\Rightarrow$ $\exists$ all directional deriv.

$\not\Leftarrow$

Let's use

$$\boxed{\partial_{\vec{v}} f(\vec{x}) = f'(x)\vec{v}}$$

to determine the entries of jac. matr.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2j} & \cdots & a_{2d} \\ \vdots & & & & a_{ij} & & \\ a_{m1} & a_{m2} & \cdots & \cdots & & \cdots & a_{md} \end{bmatrix} \cdot \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_j} \; j$$

$$\underbrace{\qquad\qquad a_{ij} \qquad\qquad}_{A} \quad \underbrace{\qquad}_{\vec{e}_j}$$

$$= \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ a_{mj} \end{pmatrix}$$

If now $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{pmatrix} \cdot \vec{e}_i \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_i = a_{ij}.$

$$\Rightarrow \quad a_{ij} = (A\vec{e}_j) \cdot \vec{e}_i$$

$\Rightarrow$ so in the case of jacobian matrix

$$a_{ij} = \underbrace{f'(\vec{x})\, \vec{e}_j}_{\partial_{\vec{e}_j} f(\vec{x})} \cdot \vec{e}_i = \partial_{\vec{e}_j} f(\vec{x}) \cdot \vec{e}_i$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \| $$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \partial_j f(\vec{x}) \cdot \vec{e}_i$$

$f = \vec{f}(\vec{x}) \in \mathbb{R}^m$ $\qquad f = (f_1, f_2, \dots, f_m)$

$$\partial_j f = (\partial_j f_1, \partial_j f_2, \dots \partial_j f_m)$$

so $$\partial_j f \cdot \vec{e_i} = \begin{pmatrix} \partial_j f_1 \\ \partial_j f_2 \\ \vdots \\ \partial_j f_m \end{pmatrix} \cdot \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array}$$

$$= \partial_j f_i (\vec{x})$$

$$f'(x) = [a_{ij}] \quad \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, d \end{matrix} \qquad a_{ij} = \partial_j f_i (\vec{x})$$

or

$$f'(x) = \begin{bmatrix} \partial_1 f_1 & \partial_2 f_1 & \partial_3 f_1 & \cdots & \partial_d f_1 \\ \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2 & & \partial_d f_2 \\ \partial_1 f_3 & \partial_2 f_3 & \partial_3 f_3 & \cdots & \partial_d f_3 \\ \vdots & & & & \\ \partial_1 f_m & \partial_2 f_m & \partial_3 f_m & \cdots & \partial_d f_m \end{bmatrix}$$

$$f = (f_1 \cdots , f_m)$$

$$f(x_1 \cdots x_d) = (f_1(x_1 \cdots x_d), f_2(x_1 \cdots x_d), \dots , f_m(x_1 \cdots x_d))$$

An important and special case is that of

$$f = f(x_1, \dots, x_d) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^1 \equiv \mathbb{R}$$

In this case the jac matrix is an $1 \times d$ matrix

$$\underbrace{[\partial_1 f \ \ \partial_2 f \ \ \partial_3 f \ \cdots \ \partial_d f]}_{\nabla f \ \text{(gradient of } f)} \in \mathbb{R}^d$$

∇f (gradient of f)

Examples:

1. $f(x,y) = x + y^2x^2$

In this case $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$.

$$\nabla f = (\partial_x f, \partial_y f)$$

$$= (1 + 2xy^2, 2yx^2)$$

2. $f(x,y,z) = e^{xy} + \sin(xz)$

Here $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f)$$

$$= (ye^{xy} + \cos(xz)\cdot z, \ xe^{xy}, \ \cos(xz)\, x)$$

3. $f(x,y) = (\overset{f_1}{x+y}, \overset{f_2}{xy})$ Here $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

The jac matrix

$$f'(x,y) = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ y & x \end{bmatrix}$$

4. $f(x,y) = (\overset{f_1}{x + y^2x}, \overset{f_2}{x}, \overset{f_3}{y})$ $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

and

$$f'(x,y) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \cdots \end{bmatrix} \quad \begin{bmatrix} 1 + y^2 & 2yx \\ \cdots \end{bmatrix}$$

and

$$f'(x,y) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{bmatrix} = \begin{bmatrix} 1+y^2 & 2yx \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. $f(x,y,z) = (\overset{f_1}{xyz}, \overset{f_2}{x+y+z})$ Here $f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

and

$$f'(x,y,z) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \end{bmatrix} = \begin{bmatrix} yz & xz & xy \\ 1 & 1 & 1 \end{bmatrix}$$

$$f'(\vec{x}) = \begin{bmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{bmatrix} \quad \text{if } f = (f_1, \dots, f_m)$$

Rmk: Does the $\exists$ of Jac. matrix $\Rightarrow$ f is diff.?

(or, in other words, once I computed the Jac matrix or the ∇, is this sufficient to say f is diff.?)

NO: In the example

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq \vec{0} \\ 0 & (x,y) = 0 \end{cases}$$

one can easily check that $(\partial_x f(0,0), \partial_y f(0,0)) = (0,0)$

but f is not diff at $(0,0)$

but f is not diff at $(0,0)$

This is because f is not even cont and there's a general property that says

Thm: If f is differentiable at $\vec{x}$ $\Rightarrow$ f is continuous at $\vec{x}$.

Ex 3.2.4

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

Compute $\nabla f(0,0)$ and say if f is diff at $(0,0)$.

<u>Sol:</u> $\nabla f(0,0) = \left(\partial_x f(0,0), \partial_y f(0,0)\right)$.

How can we compute $\partial_x f(0,0)$, $\partial_y f(0,0)$?

$$\partial_x f = \partial_x \left(\frac{x^2 y^2}{x^2+y^2} \right) = y^2 \, \frac{2x(x^2+y^2) - x^2 2x}{(x^2+y^2)^2}$$

but $\partial_x f(0,0)$ does not make sense with this.

To compute $\partial_x f(0,0)$ we use the def

$$\partial_x f(0,0) = \lim_{t \to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t}$$

$$\overset{\parallel}{\partial_{\vec{e}_1}} = \lim_{t\to 0} \frac{\overset{0}{f(t,0)} - \overset{0}{f(0,0)}}{t} = \lim_{t\to 0} \frac{0}{t} = 0$$

Similarly $\partial_y f(0,0) = 0$

$\Rightarrow \nabla f(0,0) = (0,0)$

To check that f is diff at $(0,0)$ we need to prove

$$\underbrace{f((0,0)+\vec{h})}_{= f(\vec{h})} - \underbrace{f(0,0)}_{=0} - \underbrace{f'(0,0)\vec{h}}_{=[0\ 0]\cdot\vec{h}=0} = o(\|\vec{h}\|)$$

$$\Leftrightarrow \quad \frac{f(\vec{h})}{\|\vec{h}\|} \to 0 \quad \vec{h} \to 0$$

$\Leftrightarrow$
$\vec{h} = (u,v)$

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}}$$

$$= \lim_{(u,v)\to(0,0)} \frac{\frac{u^2v^2}{u^2+v^2}}{(u^2+v^2)^{1/2}}$$

$$= \lim_{(u,v)\to(0,0)} \frac{u^2v^2}{(u^2+v^2)^{3/2}} \overset{(*)}{=}$$

$$0 \leq \frac{u^2v^2}{(u^2+v^2)^{3/2}} \underset{v=\rho\sin\theta}{\overset{u=\rho\cos\theta}{=}} \frac{\rho^4(\cos\theta)^2(\sin\theta)^2}{\underbrace{(\rho^2)^{3/2}}_{=\rho^3}}$$

$$= \rho(\cos\theta)^2(\sin\theta)^2 \leq \rho \xrightarrow[\rho\to 0]{} 0$$

$\Rightarrow \overset{(*)}{=} 0. \quad \Rightarrow f$ is diff at $(0,0)$ ☒

$\Rightarrow \overset{(*)}{=} 0. \quad \Rightarrow f$ is diff at $(0,0)$ $\boxtimes$

Do 3.4.2