Ex 3.4.2

#1 Let $f(x,y) := \begin{cases} \dfrac{x^3}{x^2+y^2} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$

i) is f cont at $(0,0)$?

ii) $\exists\ \partial_x f(0,0)$, $\partial_y f(0,0)$?

iii) is f differentiable at $(0,0)$?

Here $f = f(x,y) : D = \mathbb{R}^2 \longrightarrow \mathbb{R}$

i) f is cont at $(0,0)$ $\Leftrightarrow$

(*) $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

We may notice that

$\underset{(x,0)}{f(x,0) =} \dfrac{x^3}{x^2} = x \underset{|x|\to 0}{\longrightarrow} 0$

To check (*), we write f in polar coords:

$f(x,y) = f(\rho\cos\theta, \rho\sin\theta) = \dfrac{\rho^3(\cos\theta)^3}{\rho^2} = \rho(\cos\theta)^3$

$\Rightarrow |f(x,y) - 0| = |f(x,y)| = \rho \underbrace{|\cos\theta|^3}_{\leq 1} \leq \rho = \|(x,y)\| \underset{(x,y)\to\vec{0}}{\to} 0$

$\Rightarrow$ (*) true.

ii) Notice that for $(x,y) \neq \vec{0}$

$\partial_x f(x,y) = \partial\left(\dfrac{x^3}{\dots}\right) = \dfrac{3x^2(x^2+y^2) - x^3\cdot 2x}{\dots}$

$$\partial_x f(x,y) = \partial_x\left(\frac{x^3}{x^2+y^2}\right) = \frac{3x^2(x^2+y^2) - x^3\cdot 2x}{(x^2+y^2)^2}$$

(but unfortunately we cannot use this to compute $\partial_x f(0,0)$). By using the def

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - \overset{0}{\overset{\parallel}{f(0,0)}}}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{\frac{t^3}{t^2+0^2}}{t} = \lim_{t\to 0} \frac{t^3}{t^3} = 1$$

$\Rightarrow \quad \partial_x f(0,0) = 1.$

Similarly

$$\partial_y f(0,0) = \partial_{(0,1)} f(0,0) = \lim_{t\to 0} \frac{f(\overbrace{(0,0)+t(0,1)}^{(0,t)}) - \overset{0}{\overset{\parallel}{f(0,0)}}}{t}$$

$$= \lim_{t\to 0} \frac{f(0,t) \equiv 0}{t} = 0 \qquad \Rightarrow \partial_y f(0,0) = 0.$$

iii) (Rmk: the simple $\exists$ of $\nabla f(0,0) = (\partial_x f(0,0), \partial_y f(0,0))$
$= (1,0)$
does not mean that f is necessarily differentiable)

To be diff we have to prove that

$$f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(0,0)\cdot\vec{h} = o(\|\vec{h}\|)$$

$$\Leftrightarrow \quad \frac{f(\vec{0}+\vec{h}) - \overset{=0}{f(\vec{0})} - \overset{=(1,0)}{\nabla f(\vec{0})}\cdot\vec{h}}{} = 0.$$

$$\Leftrightarrow \lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h}) - \overset{=0}{f(\vec{0})} - \nabla f(\vec{0})\cdot\vec{h}}{\|\vec{h}\|} = 0.$$

$\vec{h} = (u,v)$

$$\lim_{(u,v)\to\vec{0}} \frac{f(u,v) - 0 - \overbrace{(1,0)\cdot(u,v)}^{u}}{\sqrt{u^2+v^2}} = 0 \;?$$

$$\Leftrightarrow \lim_{(u,v)\to\vec{0}} \frac{\frac{u^3}{u^2+v^2} - u}{\sqrt{u^2+v^2}} = 0\;?$$

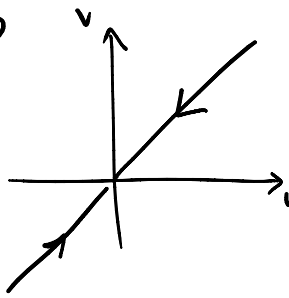
$$\Leftrightarrow \lim_{(u,v)\to\vec{0}} \frac{\overset{\cancel{u^3} - \cancel{u^3} - uv^2}{u^3 - u(u^2+v^2)}}{(u^2+v^2)^{3/2}} = 0\;?$$

Let's consider

$$\lim_{(u,v)\to(0,0)} \underbrace{\frac{uv^2}{(u^2+v^2)^{3/2}}}_{g(u,v)} \qquad (u^2)^{1/2} = |u|$$

$$g(u,0) \equiv 0 \xrightarrow[(u,0)\to\vec{0}]{} 0 \qquad g(u,u) = \frac{u^3}{(2u^2)^{3/2}} = \frac{1}{2^{3/2}}\frac{u^3}{|u|^3}$$

$$g(0,v) \equiv 0 \longrightarrow 0 \qquad = \frac{1}{2^{3/2}}\left(\frac{u}{|u|}\right)^3$$



$$= \frac{1}{2^{3/2}} \operatorname{sgn} u \quad \xrightarrow{u\to0^+} +\frac{1}{2^{3/2}} \;\neq\; \xrightarrow{u\to0^-} -\frac{1}{2^{3/2}}$$

$\nexists$ lim ... f is not differentiable

$\Rightarrow \nexists \lim_{(u,v)\to(0,0)} g \Rightarrow f$ is not differentiable ■

4:

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} + x - y & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

Same questions.

Sol: Is f cont at $\vec{0}$? We need to check if

$$\lim_{(x,y)\to\vec{0}} f(x,y) = f(0,0) = 0$$

$$\lim_{(x,y)\to\vec{0}} \left(\frac{x^2y^2}{x^2+y^2} + \underbrace{x-y}_{\downarrow \, 0} \right) \stackrel{?}{=} 0$$

Let's check if $\displaystyle\lim_{(x,y)\to\vec{0}} \frac{x^2y^2}{x^2+y^2} = 0$ (*)

$f(x,0) \equiv 0 \xrightarrow[(x,0)\to(0,0)]{} 0$.

$$|f(x,y) - 0| = |f(x,y)| = f(x,y) = f(\rho\cos\theta, \rho\sin\theta)$$

$$= \frac{\rho^4 (\cos\theta)^2(\sin\theta)^2}{\rho^2} = \rho^2 \underbrace{C(\theta)}_{\leq 1} \leq \rho^2 \xrightarrow{\rho\to 0} 0$$

(*) holds true $\Rightarrow f$ cont at $(0,0)$.

(*) holds true $\Rightarrow$ f cont at $(0,0)$.

ii) $$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f(\overbrace{(0,0)+t(1,0)}^{=(t,0)}) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{t - \cancel{0}}{t} = 1$$

$$\partial_y f(0,0) = \partial_{(0,1)} f(0,0) = \lim_{t\to 0} \frac{\overset{=-t}{f(0,t)} - \overset{=0}{f(0,0)}}{t} = -1$$

iii) By ii) $\nabla f(0,0) = (1,-1)$. To be diff at $\vec{0}$ we need

$$\frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\cdot\vec{h}}{\|\vec{h}\|} \to 0$$

$\vec{h} = (u,v)$

$$\frac{f(u,v) - (1,-1)\cdot(u,v)}{\sqrt{u^2+v^2}} \xrightarrow[(u,v)\to\vec{0}]{} 0 \quad ?$$

$$\left(\left(\quad = \frac{\overbrace{\frac{u^2v^2}{u^2+v^2} + \cancel{u-v}}^{f(u,v)} - \cancel{(u-v)}}{(u^2+v^2)^{1/2}}\right.\right.$$

$$= \frac{u^2v^2}{(u^2+v^2)^{3/2}}.$$

so we've to check if $\displaystyle\lim_{(u,v)\to\vec{0}_2} \underbrace{\frac{u^2v^2}{(u^2+v^2)^{3/2}}}_{g(u,v)} = 0$

$$\underbrace{\qquad}_{g(u,v)}$$

$$|g(u,v) - 0| = g(u,v) = g(\rho\cos\theta, \rho\sin\theta)$$
$$= \frac{\rho^4 (\cos\theta)^2 (\sin\theta)^2}{\underbrace{(\rho^2)^{3/2}}_{=\rho^3}} = \rho(\cos\theta)^2(\sin\theta)^2$$
$$\leq \rho \longrightarrow 0 \quad \rho \to 0$$

true.

Conclusion: f is diff at $(0,0)$ ∎

Total Differential Thm:

Let $\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{f} = (f_1, \cdots, f_m)$

(here $f_j = f_j(x_1, \cdots, x_d) : D \subset \mathbb{R}^d \to \mathbb{R}$)

If

$$\boxed{\exists\ \partial_j f_i \in \underline{\mathcal{C}(D)} \quad \forall i = 1, \dots, m \quad \forall j = 1, \dots, d}$$

$\Rightarrow \vec{f}$ is diff at every pt $\vec{x} \in D$.

Ex. 3.4.3 Show that $f(x,y) = x\sqrt{x^2+y^2}$ $(x,y) \in \mathbb{R}^2$ is diff on $\mathbb{R}^2$ $(\forall (x,y) \in \mathbb{R}^2)$.

Sol: Here $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. and

$$\partial_x f(x,y) = \partial_x \left(x\sqrt{x^2+y^2} \right) = \sqrt{x^2+y^2} + x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\partial_x f(x,y) = \partial_x\left(x\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2} + x\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}}. \quad \forall (x,y)\neq\vec{0}$$

$$\Rightarrow \partial_x f \in \mathcal{C}(\mathbb{R}^2\setminus\{\vec{0}\})$$

$$\partial_y f(x,y) = \partial_y\left(x\sqrt{x^2+y^2}\right) = x\,\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$(x,y)\neq(0,0)$$

$$\Rightarrow \partial_y f \in \mathcal{C}(\mathbb{R}^2\setminus\{\vec{0}\})$$

so on $\mathbb{R}^2\setminus\{\vec{0}\}$ f is diff. (by tot. diff thm)

What about $\vec{0}$?

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0}\frac{\overset{t\sqrt{t^2}}{f(t,0)} - \overset{0}{\overset{\parallel}{f(0,0)}}}{t}$$

$$= \lim_{t\to 0}\frac{\cancel{t}|t|}{\cancel{t}} = 0 \Rightarrow \exists\, \partial_x f(0,0) = 0$$

$$\partial_y f(0,0) = \partial_{(0,1)} f(0,0) = \lim_{t\to 0}\frac{\overset{0}{\overset{\parallel}{f(0,t)}} - \overset{0}{\overset{\parallel}{f(0,0)}}}{t} = 0$$

$$\Rightarrow \nabla f(0,0) = (0,0).$$

How do we check that f is diff?

Two ways:

- use the def
- use the total diff thm (but warning! if this fails f could be still diff)

$$\partial_y f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq \vec{0} \\ 0 & (x,y) = \vec{0} \end{cases}$$

is $\partial_y f$ cont at $(0,0)$?

$$\Updownarrow$$

$$\lim_{(x,y)\to\vec{0}} \underbrace{\frac{xy}{\sqrt{x^2+y^2}}}_{g(x,y)} = 0 \quad ?$$

$$g(x,0) \equiv 0 \longrightarrow 0$$

$$|g(\rho\cos\theta, \rho\sin\theta) - 0| = |g(\rho\cos\theta, \rho\sin\theta)|$$

$$= \frac{\rho^2 |\cos\theta||\sin\theta|}{\rho} = \rho|\cos\theta||\sin\theta|$$

$$\leq \rho \to 0 \quad \rho \to 0$$

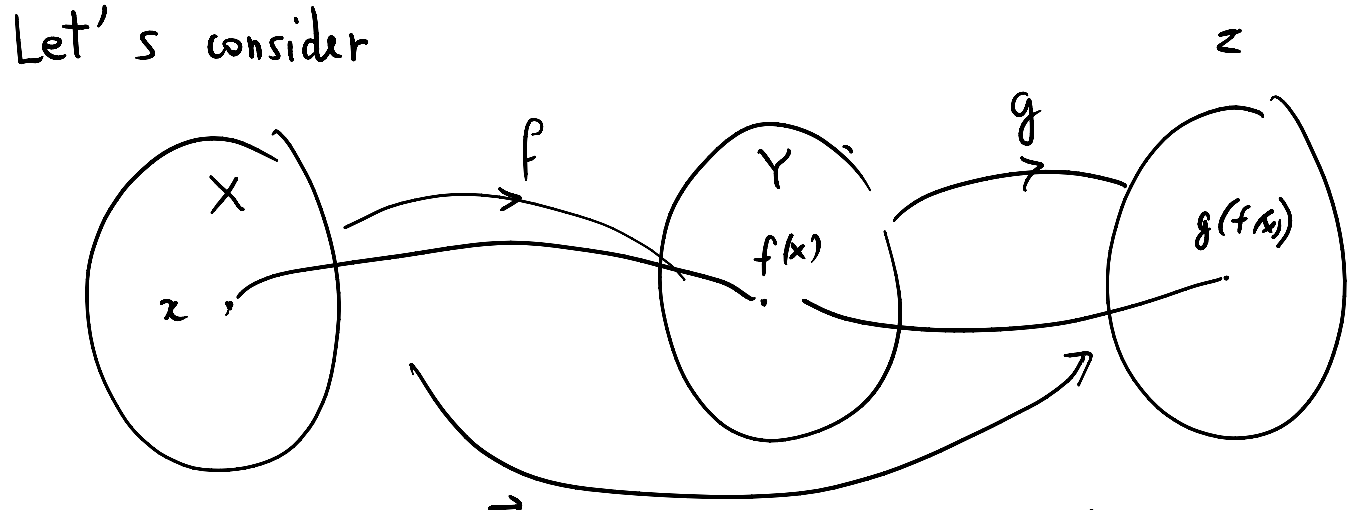
$\Rightarrow$ $\partial_y f$ cont at $(0,0)$

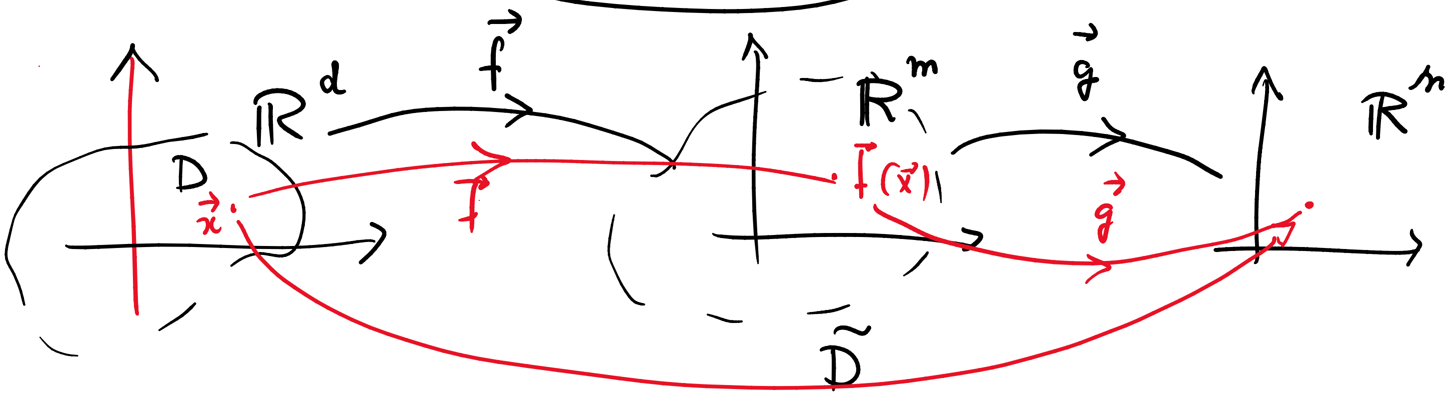
Similarly $\lim_{(x,y)\to\vec{0}} \partial_x f(x,y) = \partial_x f(0,0) = 0$

$\Rightarrow \ \partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow$ (tot diff thm) f is diff on $\mathbb{R}^2$ $\blacksquare$

Rules of Calculus

Let's consider





$$\vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$$
$$\vec{g} : \tilde{D} \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
s.t. $\vec{f}(\vec{x}) \in \tilde{D} \quad \forall \vec{x} \in D$

In this case the composition

$$(\vec{g} \circ \vec{f})(\vec{x}) := \vec{g}\left(\vec{f}(\vec{x})\right) \qquad x \in D$$

is well defined

$$\vec{g} \circ \vec{f} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^n$$

Recall that if

$$f: D\subset\mathbb{R}\to\mathbb{R}, \quad g:\tilde{D}\subset\mathbb{R}\to\mathbb{R}, \ \exists f', g'$$

$$\Rightarrow \quad (g\circ f)'(x) = g'(f(x))\,f'(x).$$

Thm: If

- $\vec{f}: D\subset\mathbb{R}^d\to\mathbb{R}^m$ is diff on D
- $\vec{g}:\tilde{D}\subset\mathbb{R}^m\to\mathbb{R}^n$ is diff on $\tilde{D}$
- $\vec{f}(\vec{x})\in\tilde{D}$ $\forall \vec{x}\in D$

$\Rightarrow$ $\vec{g}\circ\vec{f}$ is diff on D and

$$(\vec{g}\circ\vec{f})'(\vec{x}) = \overset{n\times m}{g'(\vec{f}(\vec{x}))}\cdot \overset{m\times d}{f'(\vec{x})}$$

jacobian matrix of f/g.

Rmk: A particular case is the so called total derivative

Imagine we have to compute

$$\frac{d}{dt}\left[g\left(x_1(t), x_2(t), \dots, x_m(t)\right)\right] \qquad t\in\mathbb{R}$$

Here $g:\tilde{D}\subset\mathbb{R}^m\longrightarrow\mathbb{R}^1$ $(n=1)$

$f: D\subset\mathbb{R}^1\longrightarrow\mathbb{R}^m$

$$f(t) = \left(x_1(t), x_2(t), \dots, x_m(t)\right)$$

$$\Rightarrow \frac{d}{dt}\left[g(x_1(t), \dots, x_m(t)) \right] = \frac{d}{dt}\left[g(f(t)) \right]$$

$$= g'(f(t))\, f'(t)$$

$$f'(t) \hookrightarrow \begin{bmatrix} \partial_t x_1 \\ \partial_t x_2 \\ \vdots \\ \partial_t x_m \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_m'(t) \end{bmatrix}$$

g' : jacobian matrix $\quad \left[\partial_1 g \quad \partial_2 g \quad \cdots \quad \partial_m g \right] = \nabla g$

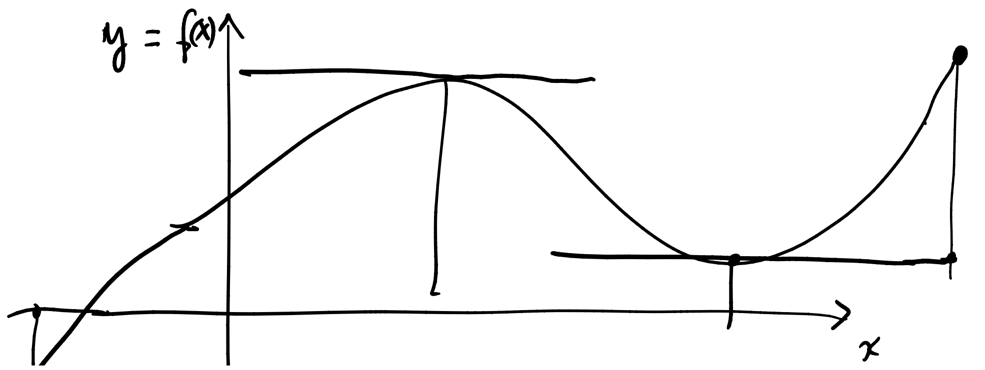
$$= \nabla g(x_1(t), \dots, x_m(t)) \cdot \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_m'(t) \end{bmatrix}$$

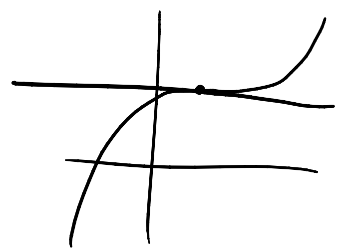
$$= \sum_{j=1}^{m} \partial_j g(x_1(t), \dots, x_m(t))\, x_j'(t)$$

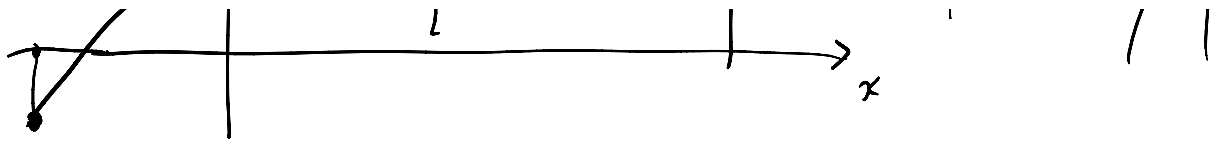
$$\Rightarrow \frac{d}{dt}\left[g(x_1(t), \dots, x_m(t)) \right] = \sum_{j=1}^{m} \partial_j g(x_1(t) \dots x_m(t))\, x_j'(t)$$

Min/Max pts

Let's return to the case $f = f(x)$ $x \in \mathbb{R}$.







<u>Thm</u> (Fermat)
If x_0 is a local min/max and x_0 is in the <u>interior</u> of domain of f $\Rightarrow$ $f'(x_0)=0$.

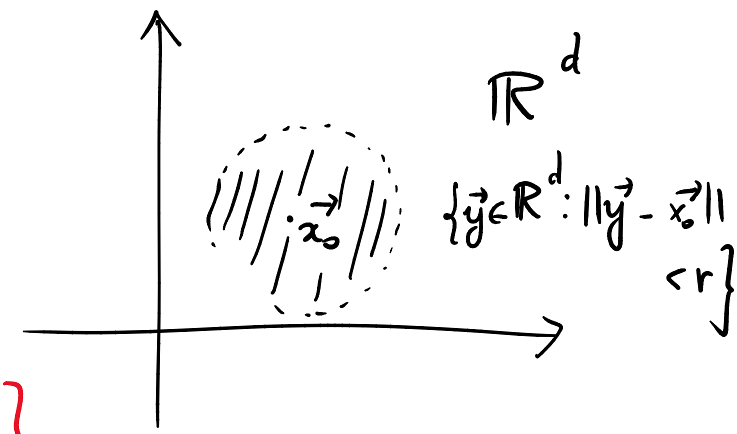
We now consider the case

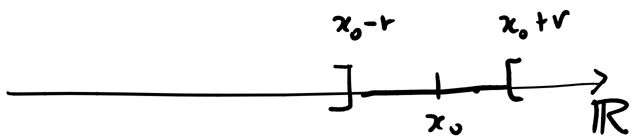
$$\boxed{f = f(x_1, \dots, x_d) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}}$$

To extend the Fermat thm we need few defs:

<u>Def</u>: We call <u>neighbourhood</u> of $\vec{x}_0$ any set of type

$$B(\vec{x}_0, r[:= \left\{ \vec{y} \in \mathbb{R}^d : \|\vec{y} - \vec{x}_0\| < r \right\}$$



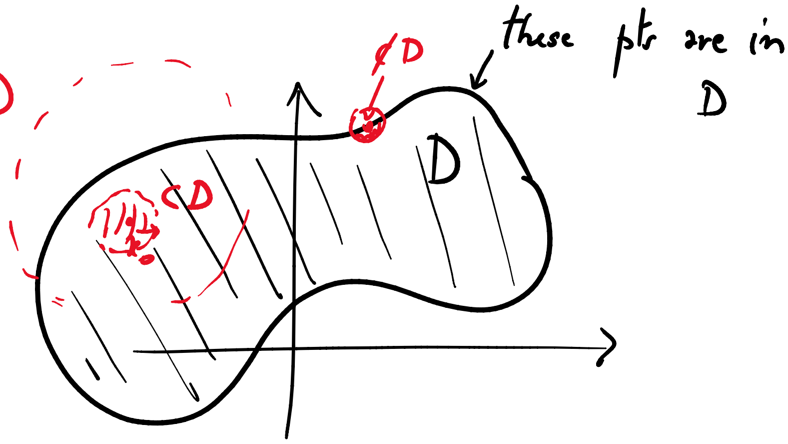


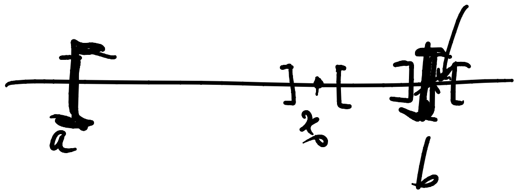
$$]x_0 - r, x_0 + r[= \{ y \in \mathbb{R} : |y - x_0| < r \}$$

<u>Def</u>: Given a set $D \subset \mathbb{R}^d$ we say that $\vec{x}_0$ is in the <u>interior</u> of D (notation: $\vec{x}_0 \in \text{Int}(D)$)

if

$\exists\, B(x_0, r[\ \subset D$





Def: Given $f: D \subset \mathbb{R}^d \to \mathbb{R}$ we say that $\vec{x}_0$ is a local min for f if

$$\exists\, B(\vec{x}_0, r[\ :\quad f(\vec{x}_0) \leq f(\vec{x}),\quad \forall\, \vec{x} \in B(\vec{x}_0, r[\ \cap D.$$

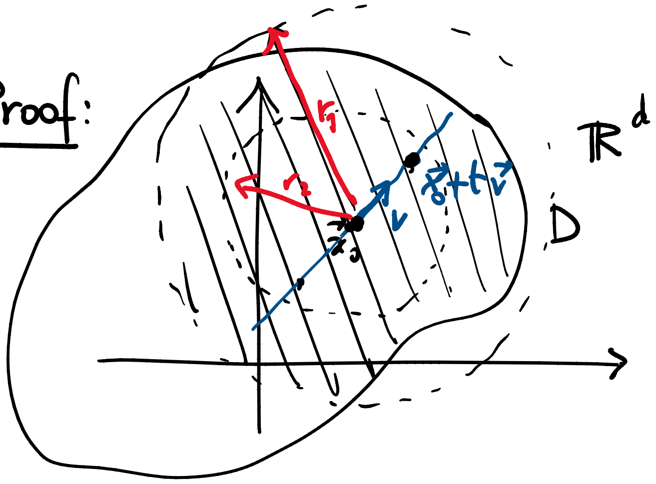
Similarly the concept of local max is def'd.

With these defs we can finally state and prove the

Thm (Fermat)
Let $f: D \subset \mathbb{R}^d \to \mathbb{R}$, differentiable, $\vec{x}_0$ be a local min/max for f s.t. $\vec{x}_0 \in \operatorname{Int} D$. Then

$$\nabla f(\vec{x}_0) = \vec{0}.$$

<u>Proof:</u>



Assume $\vec{x}_0$ be a local min: $\exists\, B(x_0, r_1[$:

$$f(\vec{x}_0) \le f(\vec{x})$$

$$\forall\, \vec{x} \in B(\vec{x}_0, r_1[\cap D$$

Because $\vec{x}_0 \in \operatorname{Int} D \Rightarrow \exists\, B(x_0, r_2[\subset D$

Taking $r = \min\{r_1, r_2\}$ we may assume

$$\underline{f(\vec{x}_0) \le f(\vec{x})} \qquad \forall\, \vec{x} \in B(x_0, r[\subset D$$

Define

$$g(t) := f(\vec{x}_0 + t\vec{v}) \in \mathbb{R} \qquad t \in \mathbb{R} \qquad g: \mathbb{R} \to \mathbb{R}$$

$$g(0) = f(\vec{x}_0)$$

$$\Rightarrow \quad g(t) \geq g(0)$$

$\Rightarrow$ g has min at $t=0$

$\Downarrow$ Fermat for real var functs

$$g'(0) = 0$$

Now

$$g'(t) = \frac{d}{dt}\left[f(x_{0,1}+tv_1, x_{0,2}+tv_2 \dots x_{0,d}+tv_d) \right]$$

$$= \sum_{j=1}^{d} \partial_j f(\vec{x}_0 + t\vec{v}) \cdot v_j = \nabla f(x_0 + t\vec{v}) \cdot \vec{v}$$

$$\Rightarrow \quad 0 = g'(0) = \nabla f(x_0) \cdot \vec{v} \qquad \forall \vec{v} \in \mathbb{R}^d$$

$$\nabla f(x_0) \cdot \vec{v} = 0 \iff \nabla f \perp \vec{v} \quad \forall \vec{v}$$

$$\Rightarrow \nabla f(\vec{x_0}) = 0 \quad \blacksquare$$