We have seen that the following result holds true

Thm (Fermat)

Let $f: D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, differentiable on D, $\vec{x}_0$ be a local min/max s.t. $\vec{x}_0 \in \mathrm{Int}(D)$

$(\exists\, B(\vec{x}_0, r[\subset D)$

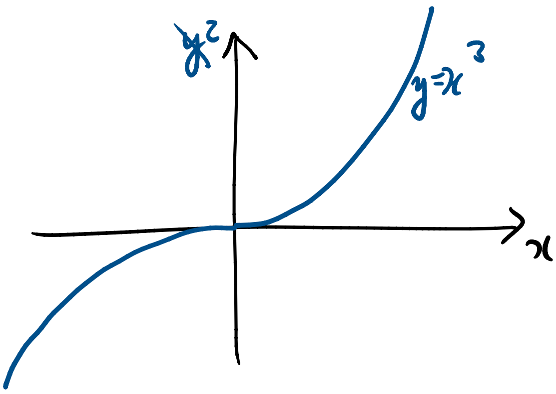
Then $\nabla f(\vec{x}_0) = \vec{0}$.

Rmk: The condition $\nabla f(\vec{x}_0) = \vec{0}$ does not mean that necessarily $\vec{x}_0$ is a local min/max.

This happens already with the case $f = f(x), x \in \mathbb{R}$

Take

$f = f(x) = x^3 \qquad f: \mathbb{R} \longrightarrow \mathbb{R}, \quad f \nearrow$ strictly

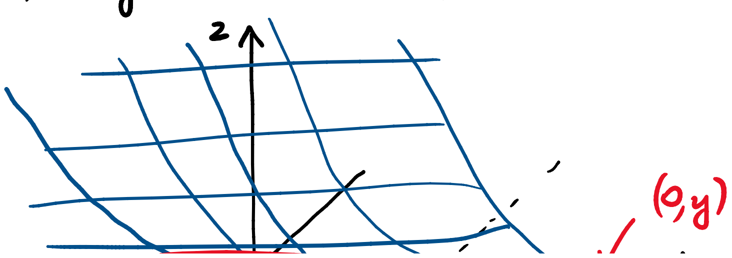


(if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$)

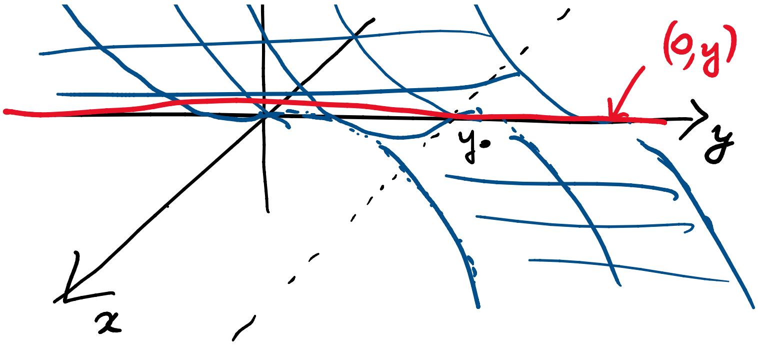
but $f'(x) = 3x^2$

$\Rightarrow f'(0) = 0$.

Take $f(x,y) = x^3$. $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$



$f(x,0) = x^3$



$f(x, y_0) = x^3$

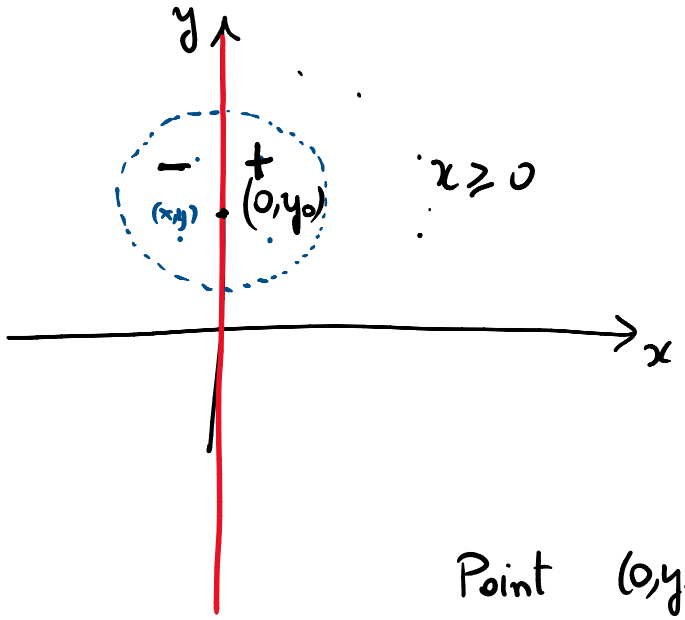
Now $\quad \nabla f(x,y) = (3x^2, 0)$, $\quad \nabla f(x,y) = \vec{0} \iff \begin{cases} 3x^2 = 0 \\ 0 = 0 \checkmark \end{cases}$

$\iff x = 0$

$\Rightarrow \forall y, \; (0,y)$ is s.t. $\nabla f(0,y) = \vec{0}$

but non of these points is local min/max

Why? Because



domain of $f(x,y) = x^3$

Point $(0,y_0)$ is <u>not</u> a local min/max because

- to be a local min we need

$\exists B((0,y_0), r[\; : \; 0 = f(0,y_0) \leq f(x,y) \quad \forall (x,y) \in B((0,y_0), r[$

$\|$

$\in B((0,y_0),r[$

$(\cap \mathbb{R}^2)$

$\Downarrow$

$f(x,y) \geq 0 \quad \forall (x,y) \in B((0,y_0),r[$

But $\quad f(x,y) = x^3 \geq 0 \iff x \geq 0$

$\Rightarrow \quad f \not\geq 0$ on $B((0,y_0),r[$. $\quad (\Rightarrow (0,y_0)$ is not local min$)$

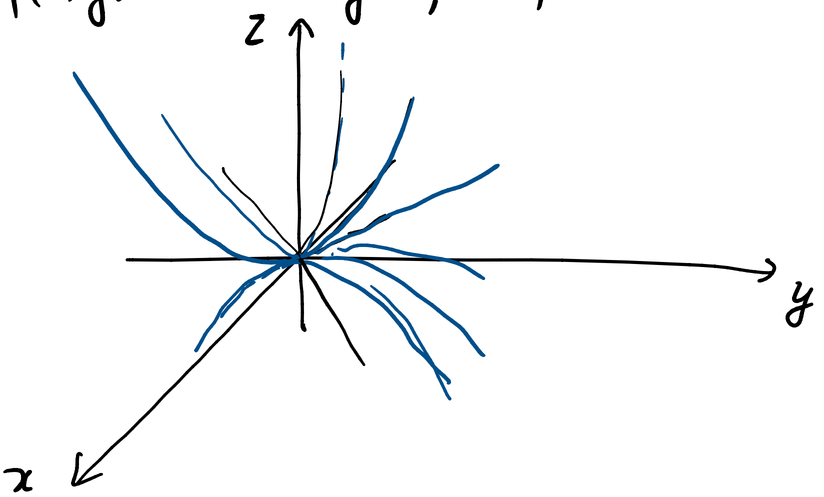
– to be a local max we need

$\exists B((0,y_0),r[\; : \quad f(x,y) \leq f(0,y_0) = 0$
$\forall (x,y) \in B((0,y_0),r[$

but because $f(x,y) > 0$ for $x > 0$
it cannot be true $f \leq 0 \;\; \forall (x,y) \in B((0,y_0),r[$

$\Rightarrow (0,y_0)$ cannot be a loc max.

Take $\quad f(x,y) = x^2 - y^2, \quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$



$f(x,0) = x^2$

$f(0,y) = -y^2$

$f(x,x) = 0$

Here $\quad \nabla f(x,y) = (\partial_x f, \partial_y f) = (2x, -2y)$

(notice that $\quad \partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f$ is diff on $\mathbb{R}^2$ because of total diff thm) on $\mathbb{R}^2$ because of total diff thm)

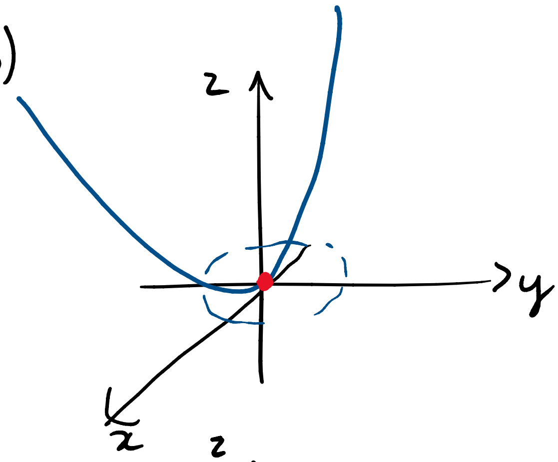
Now

$$\nabla f(x,y) = \vec{0} \iff \begin{cases} 2x = 0 \\ -2y = 0 \end{cases} \iff (x,y) = (0,0)$$

Is $(0,0)$ a local min or max? No:

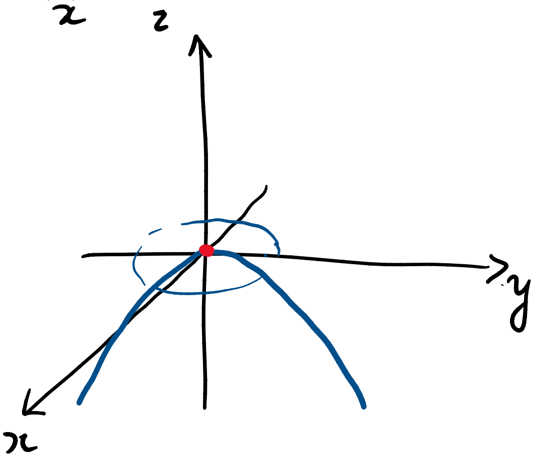
$f(x,0) = x^2 \geq 0 = f(0,0)$

$\Rightarrow (0,0)$ is min for section $y=0$



$f(0,y) = -y^2 \leq 0 = f(0,0)$

$\Rightarrow (0,0)$ is max for sect. $x = 0$



$\Rightarrow (0,0)$ cannot be a local min/max.

In alt., because $f(0,0)=0$, and being

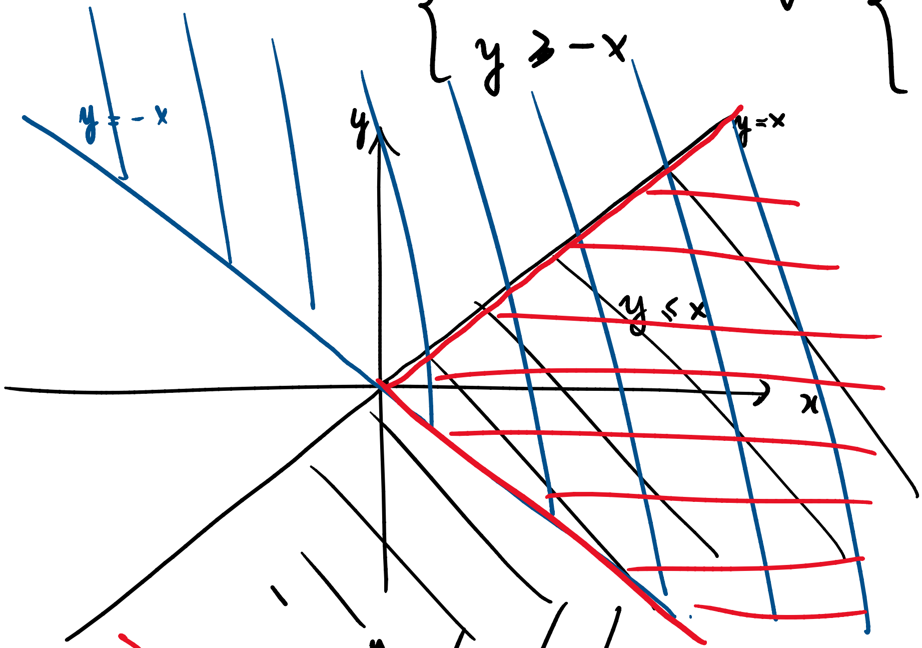
$$\boxed{f(x,y) \geq 0} \iff x^2 - y^2 \geq 0 \iff (x-y)(x+y) \geq 0$$

$$\iff \begin{cases} x - y \geq 0 \\ x + y \geq 0 \end{cases} \vee \begin{cases} x - y \leq 0 \\ x + y \leq 0 \end{cases}$$

$\Uparrow$

$\Updownarrow$

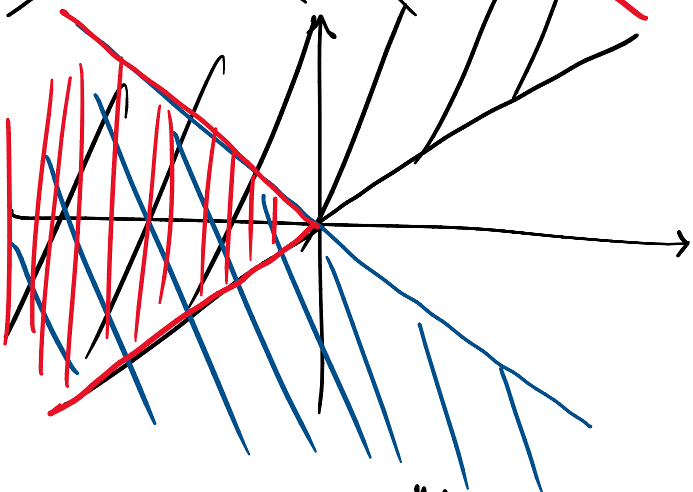
$$\begin{cases} y \le x \\ y \ge -x \end{cases} \vee \begin{cases} y \ge x \\ y \le -x \end{cases}$$



black = $y \le x$

blue = $y \ge -x$

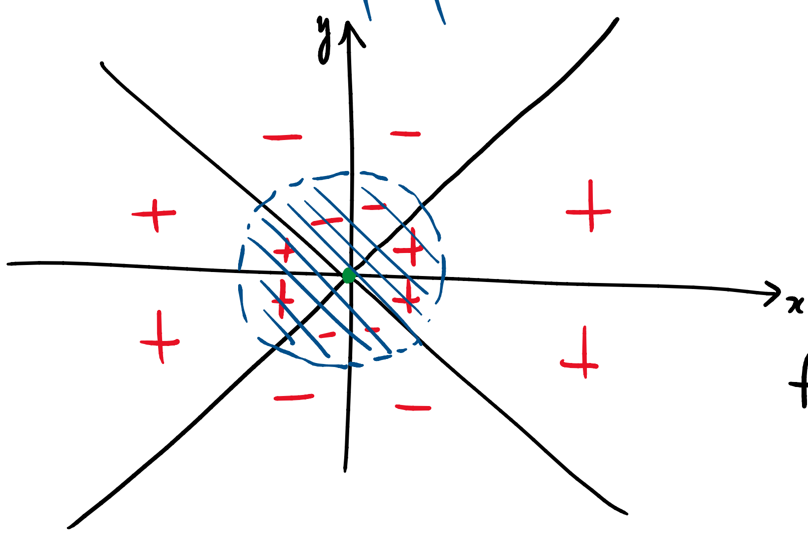
red $\begin{cases} y \le x \\ y \ge -x \end{cases}$



$y \ge x$

$y \le -x$

$\begin{cases} y \ge x \\ y \le -x \end{cases}$



$(0,0)$ is loc. min?

$\Updownarrow$

$\exists B((0,0),r[$:

$f(x,y) \ge f(0,0) = 0$

false

and similarly $(0,0)$ cannot be a local

cannot be a local max (same argument)

Def: We say that $\vec{x}_0 \in \mathbb{R}^d$ is <u>stationary point</u> for f if $\nabla f(\vec{x}_0) = \vec{0}$

We could rephrase Fermat in the following way

$$\boxed{\text{if } \vec{x}_0 \text{ is loc min/max}, \ \vec{x}_0 \in \text{Int } D \Rightarrow \vec{x}_0 \text{ st. point}}$$

$$\not\Leftarrow$$

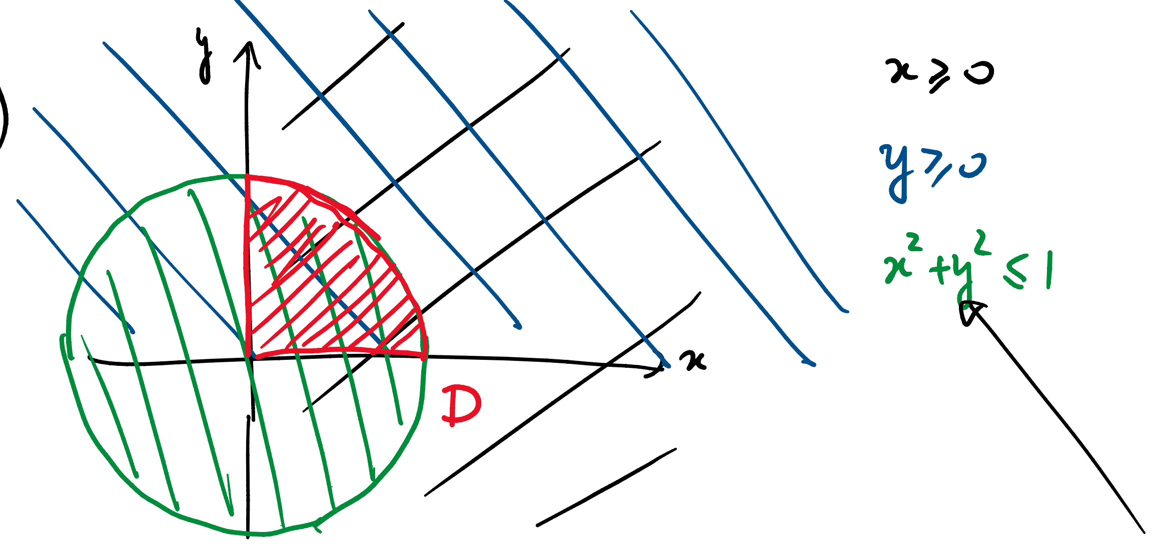
<u>Example 3.3.6</u>

$$f(x,y) = xy e^{x^2+y^2} \quad \text{on } D = \left\{ (x,y) \in \mathbb{R}^2 : \begin{array}{l} x \geq 0, y \geq 0 \\ x^2+y^2 \leq 1 \end{array} \right\}$$

i) Draw D. Is D open? closed? compact?

ii) Det min / max f on D.

<u>Sol:</u> i)



D is not open. it is closed, compact (closed & bded)

D is not open, it is closed, compact (closed & bded)

ii) Because D is compact, $f \in \mathcal{C}(D) \overset{\text{(Weierstrass)}}{\Longrightarrow}$

$\exists \min\limits_{D} f \;/\; \max\limits_{D} f$.

Let (x,y) be a min/max for f on D

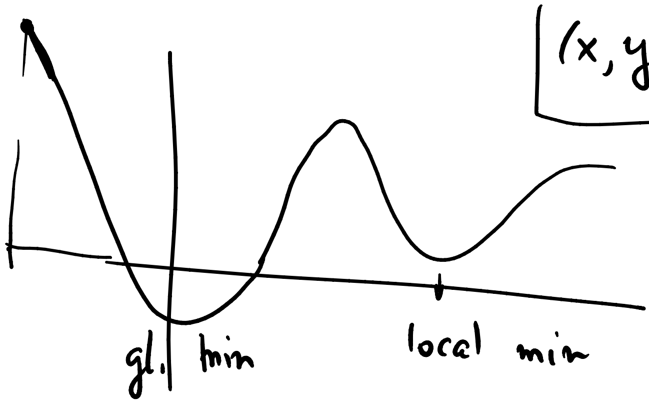
$$\Downarrow$$

$$\boxed{(x,y) \text{ is a loc. min/max}}$$

$$\Downarrow \text{ Fermat}$$

$$\nabla f(x,y) = \vec{0} \;\; ?$$

this is not correct

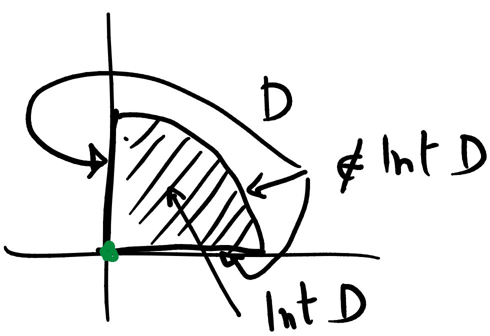


(x,y) is loc min/max

$(x,y) \in \text{Int } D$ $\qquad\qquad$ $(x,y) \notin \text{Int } D$

$$\Downarrow \text{ Fermat}$$

$$\nabla f(x,y) = \vec{0}$$



$$\nabla f(x,y) = \left(e^{x^2+y^2}(y + xy\cdot 2x),\ e^{x^2+y^2}(x + 2xy^2)\right)$$

$$f(x,y) = xy\,e^{x^2+y^2}$$

$$= e^{x^2+y^2}\left(y(1+2x^2),\ x(1+2y^2)\right)$$

$$\Rightarrow \nabla f(x,y) = \vec{0} \iff \begin{cases} y(\underbrace{1+2x^2}_{\geq 1}) = 0 \\ x(\underbrace{1+2y^2}_{\geq 1}) = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

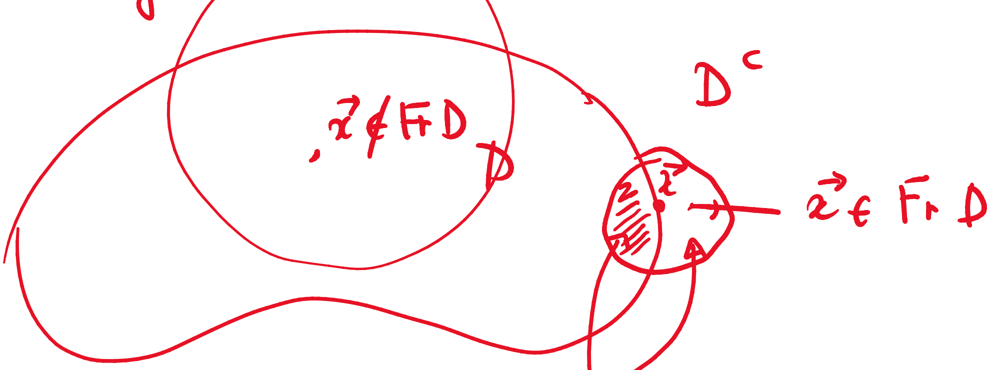
$\Rightarrow$ there's a unique st. point for f, it is $(0,0) \notin \operatorname{Int} D$

$\Rightarrow$ There aren't st points in $\operatorname{Int}(D)$

$\Rightarrow$ min / max pt for $f \notin \operatorname{Int} D$

$\Rightarrow$ " " " " $\in$ boundary of $D = \operatorname{Fr} D$.

Def: We say that $\vec{x} \in \operatorname{Fr}(D)$ (frontier of D)

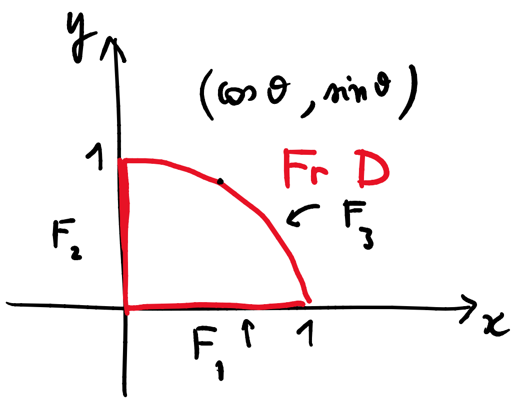


if $\forall B(\vec{x}, r]$ we have $B(\vec{x}, r] \cap D \neq \emptyset$

$B(\vec{x}, r] \cap D^c \neq \emptyset$

$B(\bar{x}, r) \cap D \neq \phi$

In the case of current pb we have



$F_1 = \{(x,0) : 0 \leq x \leq 1\}$

$F_2 = \{(0,y) : 0 \leq y \leq 1\}$

$F_3 = \{(x,y) : x^2+y^2=1, x \geq 0, y \geq 0\}$

$= \{(\cos\theta, \sin\theta) : 0 \leq \theta \leq \pi/2\}$

On F_1:

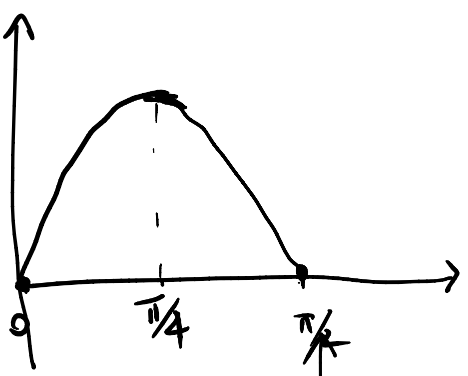
$f(x,0) \equiv 0$

On F_2

$f(0,y) \equiv 0$

On F_3

$$f(\cos\theta, \sin\theta) = \cos\theta \sin\theta \, e^1 = \frac{e}{2} \sin(2\theta) \qquad \theta \in [0, \pi/2]$$

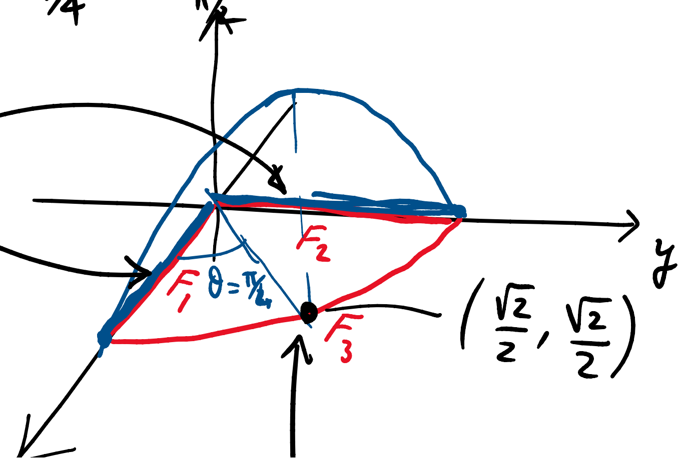


$0 \leq 2\theta \leq \pi$

$\cancel{2}\theta = \pi/4$

$f = f(x,y)$

all these are min pts for f on D



for f on D

x

this is max for f on D ■

Do Ex 3.4.4

#2. Det st pts of

$$f(x,y) = x^2 + y^2 + xy.$$

Sol: Here $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\nabla f(x,y) = (\partial_x f, \partial_y f) = (2x + y, 2y + x)$$

therefore, (x,y) is st pt for f $\Leftrightarrow$

$$\nabla f(x,y) = \vec{0} \iff \begin{cases} 2x + y = 0 \\ 2y + x = 0 \end{cases}$$

$$\iff \begin{cases} y = -2x \\ -4x + x = 0 \end{cases} \Rightarrow x = 0 \Rightarrow y = 0$$

$\Rightarrow$ unique st pt $(0,0)$.

Rmk: because $\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f$ is diff on $\mathbb{R}^2$. ■

#5 $f(x,y,z) = (x^3 - 3x - y^2)z^2 + z^3$.

Sol: Here $f \in \mathcal{C}(\mathbb{R}^3)$,

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f) = \left((3x^2 - 3)z^2, \; -2yz^2, \; 2z(x^3 - 3x - y^2) + 3z^2 \right)$$

thus (x,y,z) is st pt $\Leftrightarrow$

$$\nabla f(x,y,z) = \vec{0} \iff \begin{cases} \cancel{3}(x^2-1)z^2 = 0 & \leftarrow \\ \cancel{-2}yz^2 = 0 & \leftarrow \\ 2z(x^3 - 3x - y^2) + 3z^2 = 0 \end{cases}$$

$$\iff \begin{cases} y = 0 \\ (x^2-1)z^2 = 0 \\ 2z(x^3-3x) + 3z^2 = 0 \end{cases} \vee \begin{cases} z = 0 \\ 0 = 0 \\ 0 = 0 \end{cases}$$

$\Updownarrow$ $\qquad\qquad (x,y,0) \quad x,y \in \mathbb{R}$

$$\begin{cases} z = 0 \\ y = 0 \\ 0 = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = 1 \\ 3z^2 - 4z = 0 \end{cases} \vee \begin{cases} y = 0 \\ x = -1 \\ 3z^2 + 4z = 0 \end{cases}$$

already included

$\Updownarrow$

$$\begin{cases} x = 1 \\ y = 0 \\ z(3z-4) = 0 \end{cases} \vee \begin{cases} x = -1 \\ y = 0 \\ z(3z+4) = 0 \end{cases}$$

$$\begin{cases} y=0 \\ z(3z-4)=0 \end{cases} \quad \vee \quad \begin{cases} y=0 \\ z(3z+4)=0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x=1 \\ y=0 \\ z=0 \end{cases} \vee \begin{cases} x=1 \\ y=0 \\ z=4/3 \end{cases} \quad \vee \quad \begin{cases} x=-1 \\ y=0 \\ z=0 \end{cases} \vee \begin{cases} x=-1 \\ y=0 \\ z=-4/3 \end{cases}$$

$(1,0,4/3)$ $\qquad$ $(-1,0,-4/3)$

<u>Conclusion:</u> St pts $(x,y,0)$, $x,y\in\mathbb{R}$

$(1,0,4/3)$, $(-1,0,-4/3)$.

<u>Rmk:</u> Because $\partial_x f, \partial_y f, \partial_z f \in \mathcal{C}(\mathbb{R}^3) \Rightarrow f$ is diff. on $\mathbb{R}^3$ $\blacksquare$

Do #1, #3, #4.

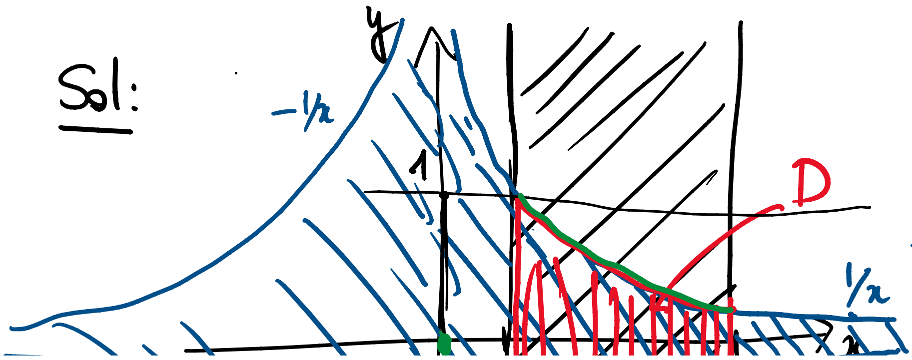
Do Ex 3.4.8 / 3.4.9 / 3.4.10 (*) / 3.4.13.

<u>Ex 3.4.9</u> $f(x,y) = xy e^{-xy}$ on $D = \{(x,y)\in\mathbb{R}^2 : 1\le x \le 4,\ |xy|\le 1\}$

i) Draw D. Is D compact.

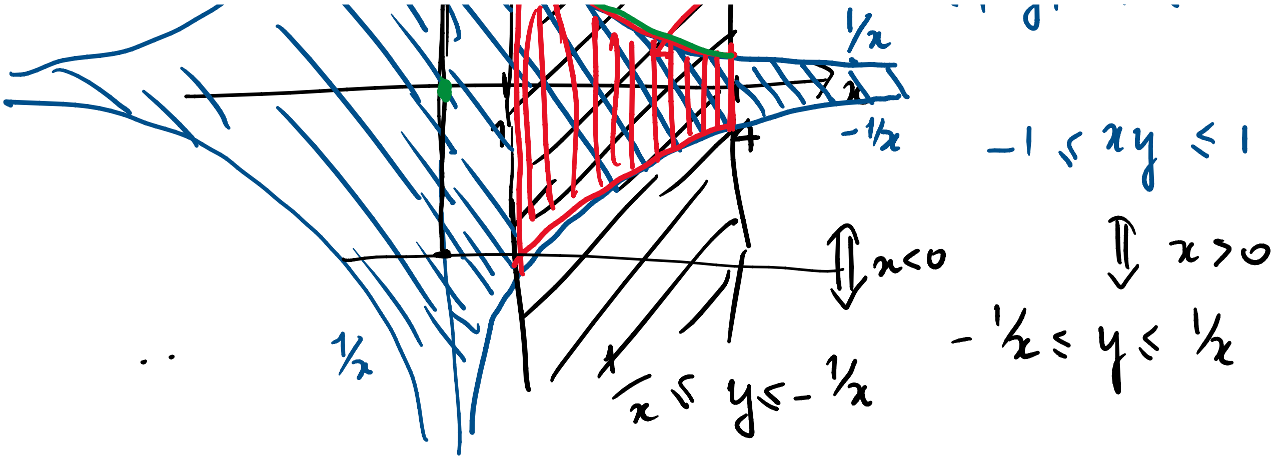
ii) Det min/max f on D if any.

<u>Sol:</u>



$1 \le x \le 4$

$\rightarrow |xy| \le 1 \Leftrightarrow$



D is compact because

- D is <u>closed</u> (it is def'd by large inequalities involving cont functs)
- D is <u>bded</u> (indeed, $1 \leq x \leq 4 \Rightarrow x$ bded, $-1 \leq y \leq 1$)

ii) Because D is compact, $f \in \mathcal{C}(D) \Rightarrow$ W. thm ensures the $\exists$ of min / max for f on D.

Let's det min/max pts.

Let (x,y) be a pt of min/max.

Because f is <u>diff</u> ($\nabla f = (\partial_x f, \partial_y f)$

$$= \Big(\underbrace{e^{-xy}(y + xy(-y))}_{\in \mathcal{C}},\ \underbrace{e^{-xy}(x + yx(-x))}_{\in \mathcal{C}} \Big) \Big)$$

according to Fermat thm we have:

if $(x,y) \in \text{Int } D \Rightarrow \nabla f(x,y) = \vec{0}$

$$\Updownarrow$$

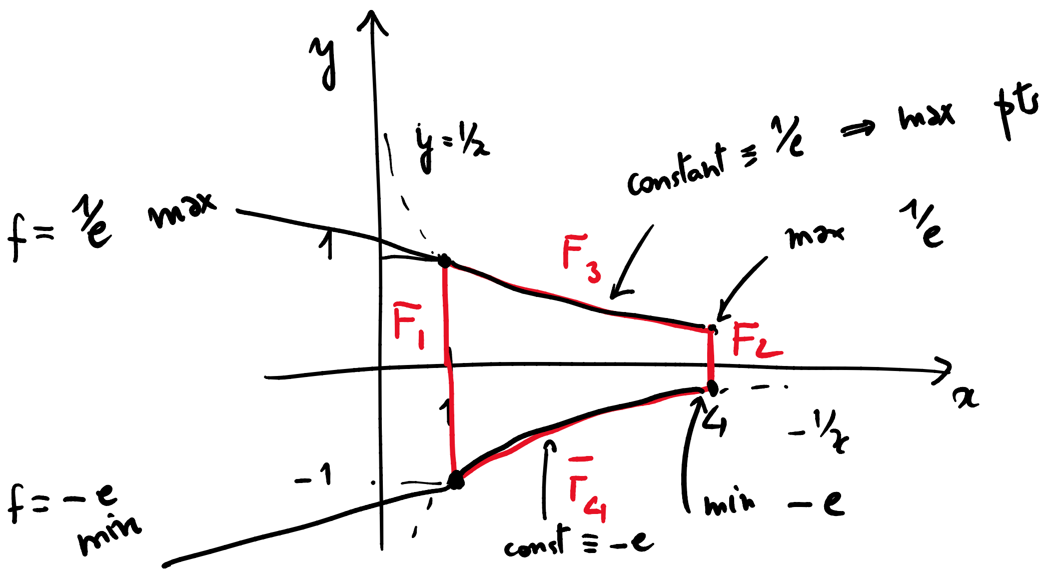
$$\begin{cases} y(1-xy)=0 \\ x(1-xy)=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y=0 \\ x=0 \end{cases} \quad \vee \quad \begin{cases} xy=1 \\ 0=0 \end{cases}$$

$$\Rightarrow \underset{D}{\cancel{\in}}(0,0) \qquad \underset{\mathrm{Int}\,D}{\cancel{\in}}\ xy=1 \quad \left(x, \tfrac{1}{x}\right)$$

$$\Downarrow$$

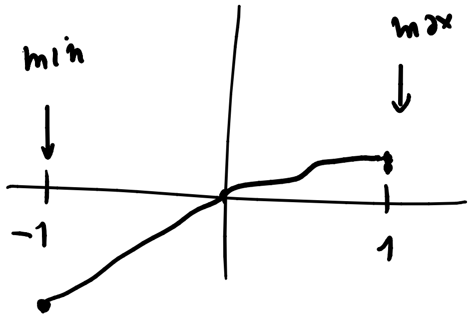
$$(x,y) \notin \mathrm{Int}\, D \implies (x,y) \in \mathrm{Fr}\, D$$



$$F_1 = \{(1,y) : -1 \leq y \leq 1\} \qquad F_3 = \{(x, 1/x) : 1 \leq x \leq 4\}$$

$$F_2 = \{(4,y) : -1/4 \leq y \leq 1/4\} \qquad F_4 = \{(x, -1/x) : 1 \leq x \leq 4\}$$

On F_1: $f(1,y) = \underbrace{ye^{-y}}_{= g(y)}$ $\quad y \in [-1,1]$

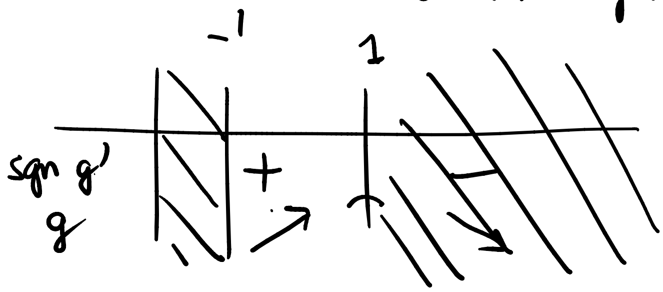


$g(-1) = -e$

$g(1) = e^{-1}$

$g'(y) = e^{-y}(1 + y(-1))$

$= \underbrace{e^{-y}}_{>0} \underbrace{(1-y)}_{\geq 0} \iff y \leq 1$



On F_2: $f(4,y) = 4ye^{-4y} =: g(y) \quad y \in [-1/4, 1/4]$

same funct

On F_3 $\quad f(x, 1/x) = 1 \cdot e^{-1} = 1/e$

On F_4 $\quad f(x, -1/x) = -1\, e = -e$