Ex 3.4.8

$f(x,y) := x^2(1-y)$ on $D = \{(x,y) : x^2 + |y| \leq 4\}$

i) Study sgn f

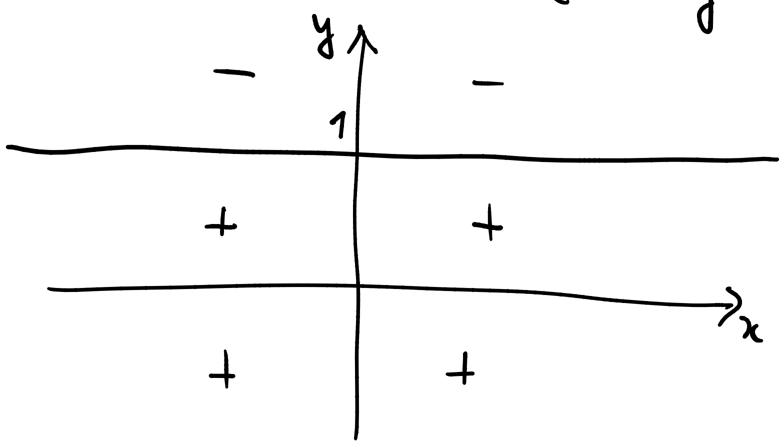
ii) Find st pts for f

iii) min / max f on D

Sol: i) $f(x,y) \geq 0 \iff x^2(1-y) \geq 0$

$$\iff 1-y \geq 0$$

$$\iff y \leq 1$$



ii) (x,y) is a st pt for $f \iff \nabla f(x,y) = \vec{0}$

We have

$$\nabla f = (\partial_x f, \partial_y f) = (2x(1-y), -x^2)$$

(by the way: $\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f$ is diff on $\mathbb{R}^2$)

Therefore

$$\nabla f(x,y) = \vec{0} \iff \begin{cases} 2x(1-y) = 0 & 0=0 \\ -x^2 = 0 & \iff x = 0 \end{cases}$$

$$\iff (0,y) \quad \forall y \in \mathbb{R}$$

iii) min/max f on D.

$D = \{(x,y) : x^2 + |y| \leq 4\}$. Notice that D is closed and bded

$$\Downarrow$$

$\left(x^2 \leq 4, \quad |y| \leq 4 \right)$

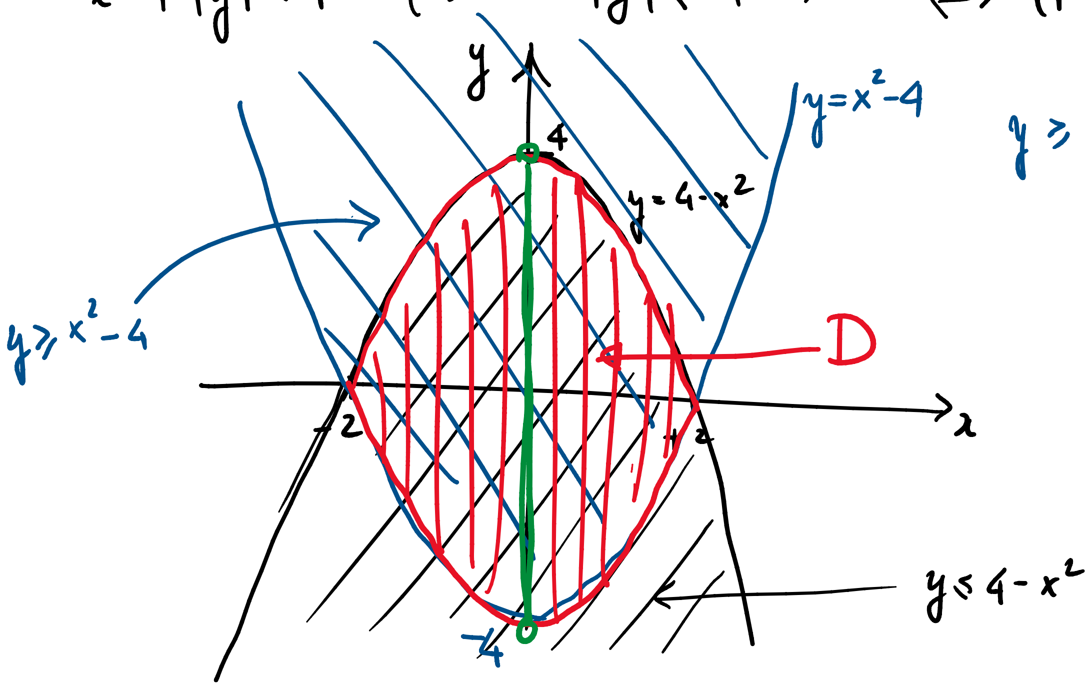
$\Downarrow$

$|x| \leq 2$

$\Downarrow$

D is compact

$$x^2 + |y| \leq 4 \iff |y| \leq 4 - x^2 \iff -(4-x^2) \leq y \leq 4 - x^2$$



$y \geqslant -(4-x^2) = x^2 - 4$

Because D is compact and $f \in \mathscr{C}(D) \Rightarrow$ (W thm)

Because D is compact and $f \in \mathscr{C}(D)$ $\Rightarrow$ (W thm)
$\exists$ min/max pts for f on D.

Let (x,y) be a min/max

$\Downarrow$

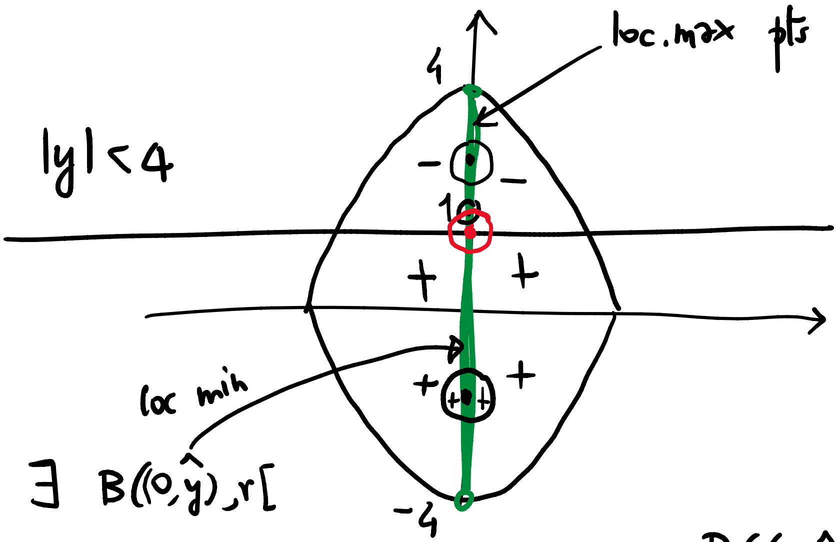
<u>if</u> $\underline{(x,y) \in \text{Int } D}$, being f diff

$\Downarrow$ F. thm

$\nabla f(x,y) = \vec{0}$

$\Downarrow$

$(0,y)$ $\quad |y|<4$



Recalling i),

$f(0,y) = 0$

$(0,\hat{y})$ $\quad 1 < \hat{y} < 4$, $\quad \exists\ B((0,\hat{y}), r[$

s.t. $\quad f(x,y) \leq 0 = f(0,\hat{y}) \quad \forall (x,y) \in B((0,\hat{y}), r[$

$\Downarrow$

$(0,\hat{y})$ is local max pt

similarly $(0,\hat{y})$ $\ -4 < \hat{y} < 1$, $\ \exists\ B((0,\hat{y}), r[)$:

$f(x,y) \geq 0 = f(0,\hat{y}) \quad \forall (x,y) \in B((0,\hat{y}), r[$

$\Downarrow$

$(0,\hat{y})$ is loc. min pt

for pt $(0,1)$ however, $\forall B((0,1), r[$

$\exists\ (x,y) \in B((0,1), r[$: $\quad f(x,y) > 0 > f(\tilde{x},\tilde{y})$

$\exists\ (x,y) \in B((0,1),r[\quad : \quad f(x,y) > 0 > f(\tilde{x},\tilde{y})$
$(\tilde{x},\tilde{y}) \in$

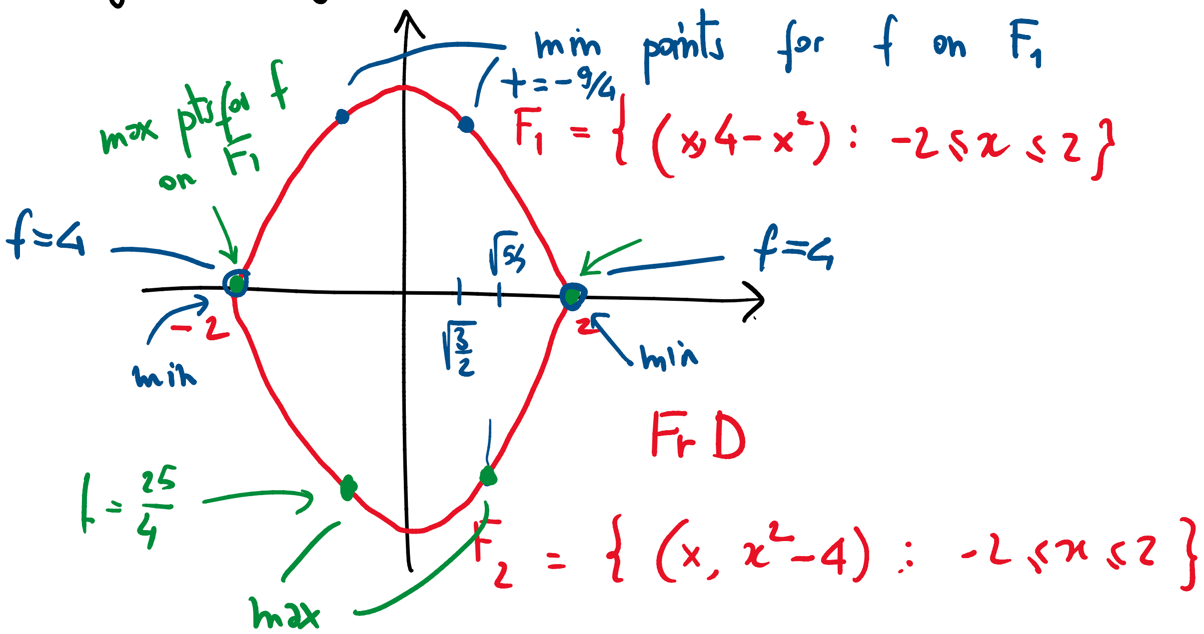
$$0 = f(0,1)$$

$\Rightarrow (0,1)$ is not local min/max.

(Rmk: none of $(0,\hat{y})$ $1<\hat{y}<4$ is global max
$-4<\hat{y}<1$ is global min.

otherwise $f=0$.)

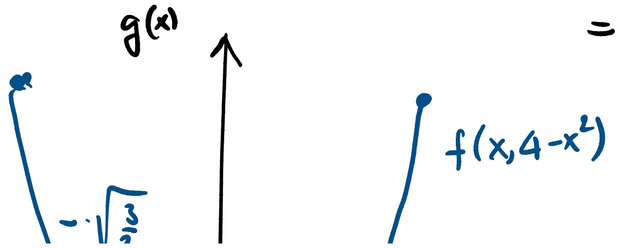
If (x,y) (glo. min/max of f) $\in$ Fr. D $\Rightarrow$



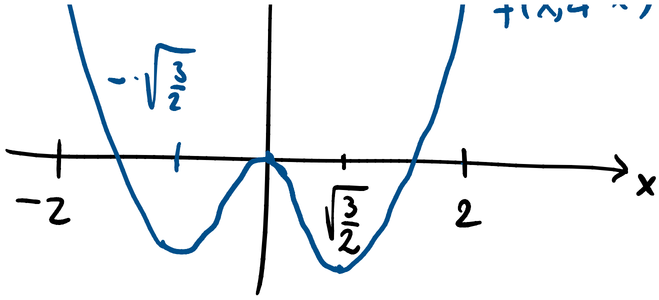
$\Rightarrow (x,y) \in F_1 \cup F_2$.

$(x,y) \in F_1 \quad : \quad f(x, 4-x^2) = x^2\left(1-(4-x^2)\right)$

$$= x^2(x^2-3) = g(x)$$



g even
$g(0)=0$



$g(0) = 0$

$g(2) = 4$

$g' = 2x(x^2 - 3) + x^2(2x)$

$g\left(\pm\sqrt{\frac{3}{2}}\right) = \frac{3}{2}\left(\frac{3}{2} - 3\right) = -\frac{9}{4}$ $\quad = 2x\left[2x^2 - 3\right]$

$0 \leq x \leq 2 \qquad g' \geq 0 \iff 2x^2 - 3 \geq 0 \iff x^2 \geq \frac{3}{2}$

$\overset{x \geq 0}{\iff} x \geq \sqrt{\frac{3}{2}}$

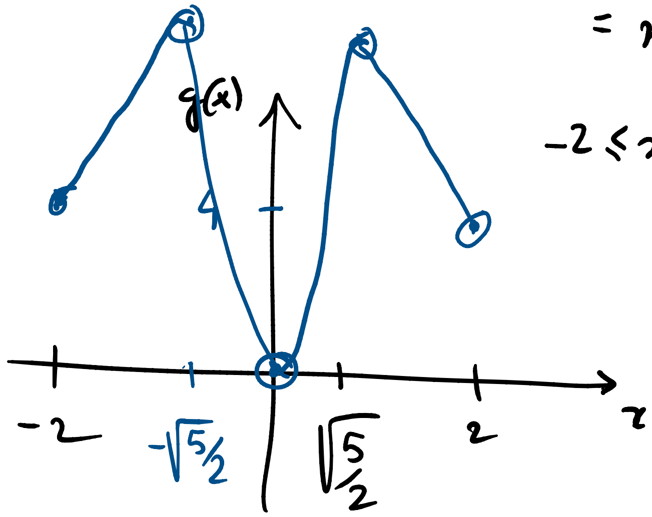
we see that

$f(x, 4-x^2)$ is max when $x = \pm 2$

min where $x = \pm\sqrt{\frac{3}{2}}$.

On F_2 : $f(x, x^2 - 4) = x^2(1 - (x^2 - 4))$

$= x^2(5 - x^2) = g(x)$



$-2 \leq x \leq 2$

$g(\pm 2) = 4 \qquad g$ even

$g(0) = 0$

$g' = 2x(5 - x^2) + x^2(-2x)$

$= 2x\left[5 - 2x^2\right]$

for $x \geq 0$, $g' \geq 0 \iff 5 - 2x^2 \geq 0 \qquad x^2 \leq \frac{5}{2}$

$\Updownarrow x \geq 0$

$x \leq \sqrt{5/2}$

$$x \leq \sqrt{\tfrac{5}{2}}$$

$$g\left(\sqrt{\tfrac{5}{2}}\right) = \frac{5}{2}\Big(\underbrace{5-\frac{5}{2}}_{=\frac{5}{2}}\Big) = \frac{25}{4}$$

so $f(x, x^2-4)$ is min at $x = 0$

max at $x = \pm\sqrt{\tfrac{5}{2}}$

Conclusion: $\left(\pm\sqrt{\tfrac{3}{2}},\ 4-\tfrac{3}{2}\right) = \left(\pm\sqrt{\tfrac{3}{2}},\ \tfrac{5}{2}\right)$ are min points for f on D

$\left(\pm\sqrt{\tfrac{5}{2}},\ \tfrac{5}{2}-4\right) = \left(\pm\sqrt{\tfrac{5}{2}},\ -\tfrac{3}{2}\right)$ are max pts for f on D. $\blacksquare$

What happens if we have to search for $\min/\max_D f$ when D is not compact. precisely when

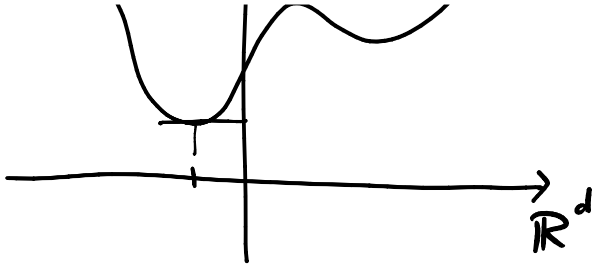
D is closed but unbounded

(for example: $D = \mathbb{R}^2, \mathbb{R}^3$).

Thm: Suppose that $f \in \mathcal{C}(D)$ D closed and unbounded such that

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty.$$

$$\vec{x} \to \infty_d$$



Then $\exists\ \vec{x}_{min} \in D$:

$$f(\vec{x}_{min}) \leq f(\vec{x}) \quad \forall\ \vec{x} \in D.$$

Similarly, if

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = -\infty \implies \exists \text{ max pt for } f \text{ on } D.$$

__EX 3.4.6__

Let $f(x,y) = (x^2+y^2)^2 - 3x^2y \qquad (x,y) \in \mathbb{R}^2$

i) $\displaystyle\lim_{(x,y)\to\infty_2} f$

ii) Find st. pts for f

iii) Det min/max f on $\mathbb{R}^2$

__Sol:__ i) We have

$$f(x,0) = x^4 \xrightarrow[(x,0)\to\infty_2]{} +\infty$$

$$\Updownarrow$$

$$|x| \to +\infty$$

Now let's check that $\displaystyle\lim_{(x,y)\to\infty_2} f = +\infty.$

We have

$$f(\rho\cos\theta, \rho\sin\theta) = (\rho^2)^2 - 3(\rho\cos\theta)^2 \rho\sin\theta$$

$$f(\rho\cos\theta, \rho\sin\theta) = (\rho^2)^2 - 3(\rho\cos\theta)^2 \rho\sin\theta$$

$$= \rho^4 - 3\rho^3 \underbrace{(\cos\theta)^2 \sin\theta}_{\leq 1}$$

$$\geq \rho^4 - 3\rho^3 = \lambda(\rho) \xrightarrow[\rho \to +\infty]{} +\infty$$

ii) (x,y) is st pt for $f \iff \nabla f(x,y) = \vec{0}$.

$$\nabla f(x,y) = \left(2(x^2+y^2)2x - 6xy,\ 2(x^2+y^2)2y - 3x^2\right)$$

Btw $\partial_x f, \partial_y f \in \mathcal{C}(\mathbb{R}^2) \Rightarrow f$ is diff.

$$\nabla f = \vec{0} \iff \begin{cases} 2x\left(2(x^2+y^2) - 3y\right) = 0 \\ 4y(x^2+y^2) - 3x^2 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ 4y^3 = 0 \end{cases} \quad \vee \quad \begin{cases} 2(x^2+y^2) - 3y = 0 \\ 4y(x^2+y^2) - 3x^2 = 0 \end{cases}$$

$2(x^2+y^2) - 3y = 0 \iff x^2+y^2 = \frac{3}{2}y$

$$\begin{cases} x = 0 \\ 4y^3 = 0 \end{cases} \iff (0,0)$$

$$\begin{cases} 2(x^2+y^2) - 3y = 0 \\ 4y(x^2+y^2) - 3x^2 = 0 \end{cases} \iff \begin{cases} x^2+y^2 = \frac{3}{2}y \\ 4y^2 \cdot \frac{\cancel{3}}{\cancel{2}\cancel{y}} - 2\cancel{3}x^2 = 0 \end{cases}$$

$\Updownarrow$

$$2y^2 - x^2 = 0$$

$$\begin{cases} x^2+y^2 = \frac{3}{2}y \\ (\sqrt{2}y - x)(\sqrt{2}y + x) = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x = \sqrt{2}y \\ 2y^2+y^2 = \frac{3}{2}y \end{cases} \quad \vee \quad \begin{cases} x = -\sqrt{2}y \\ 2y^2+y^2 = \frac{3}{2}y \end{cases}$$

$$\cancel{3}y^2 = \frac{\cancel{3}}{2}y \qquad\qquad \cancel{3}y^2 = \frac{\cancel{3}}{2}y$$

$$\begin{cases} y=0 \\ x=0 \end{cases} \qquad \begin{cases} y=\frac{1}{2} \\ x=\frac{\sqrt{2}}{2} \end{cases} \qquad \begin{cases} y=0 \\ x=0 \end{cases} \vee \begin{cases} y=\frac{1}{2} \\ x=-\frac{\sqrt{2}}{2} \end{cases}$$

$$(0,0) \qquad \left(\frac{\sqrt{2}}{2}, \frac{1}{2}\right) \qquad (0,0) \qquad \left(-\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$$

Conclusion: st pts are $(0,0)$, $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

(3 pts)

iii) We've to det min/max for f on $\mathbb{R}^2$ (closed but unbded). Because of i)

$\Rightarrow$ f has not any max pt but $\exists$ min pt for f on $\mathbb{R}^2$.

Let (x,y) be a min pt $\Rightarrow (x,y) \in \operatorname{Int}\mathbb{R}^2$

$$\Downarrow \text{ Fermat}$$

$$\nabla f = \vec{0}$$

$$\Downarrow$$

(x,y) is one between $(0,0)$, $\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$

$$f(0,0) = 0$$

$$f\left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right) = \left(\frac{1}{2} + \frac{1}{4}\right)^2 - 3 \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\frac{9}{16} - \frac{3}{4} \overset{\frac{12}{16}}{} = -\frac{3}{16} < 0$$

$\Rightarrow \left(\pm\frac{\sqrt{2}}{2}, \frac{1}{2}\right)$ are the min pts for f on $\mathbb{R}^2$ $\boxtimes$

Ex 3.4.7 $\quad f(x,y,z) = (x^2+y^2+z^2)^2 - xyz$

i) $\lim\limits_{(x,y,z)\to\infty_3} f$

ii) Find st points for f

iii) $\min/\max\limits_{\mathbb{R}^3} f$.

Sol: i) We prove that $\lim\limits_{(x,y,z)\to\infty_3} f = +\infty$

by using spherical coords

$$f(x,y,z) = (\rho^2)^2 - \rho^3 \overbrace{\left((\sin\varphi)^2 \cos\theta \sin\theta \cos\varphi\right)}^{\leq 1}$$

$$f(x,y,z) = (\rho^2)^2 - \rho^3 \left((\sin\varphi)^2 \cos\theta \sin\theta \cos\varphi \right)$$

$$= \rho^4 - \rho^3 C(\theta,\varphi) \geq \rho^4 - \rho^3 \to +\infty$$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho\cos\varphi \end{cases}$$

$\rho \to +\infty$

$\Downarrow$

$\rho \geq 0, \quad \theta \in [0, 2\pi], \quad \varphi \in [0,\pi]$

$$\lim_{(x,y,z)\to\infty_3} f = +\infty.$$

ii) St pts: (x,y,z) is st pt for $f \iff$

$$\nabla f(x,y,z) = \vec{0}$$

$$\nabla f(x,y,z) = (\partial_x f, \partial_y f, \partial_z f)$$

$$= \left(2(x^2+y^2+z^2)2x - yz,\ 2(x^2+y^2+z^2)2y - xz,\ 2(x^2+y^2+z^2)2z - xy \right)$$

$\Rightarrow \partial_x f, \partial_y f, \partial_z f \in \mathcal{C}(\mathbb{R}^3) \Rightarrow f$ is diff on $\mathbb{R}^3$.

Now

$$\nabla f = \vec{0} \iff \begin{cases} 4x(x^2+y^2+z^2) - yz = 0 \\ 4y(x^2+y^2+z^2) - xz = 0 \\ 4z(x^2+y^2+z^2) - xy = 0 \end{cases}$$

$$4x(x^2+y^2+z^2) - yz = 0 \overset{?}{\iff} 4(x^2+y^2+z^2) = \frac{yz}{x}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ \boxed{yz = 0} \\ 4y(y^2+z^2) = 0 \\ 4z(y^2+z^2) = 0 \end{cases} \quad \vee \quad \begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ y\,\frac{yz}{x} - xz = 0 \\ z\,\frac{yz}{x} - xy = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x = 0 \\ y = 0 \\ 0 = 0 \\ 4z^3 = 0 \end{cases} \vee \begin{cases} x = 0 \\ z = 0 \\ 4y^3 = 0 \\ 0 = 0 \end{cases}$$

$$\Updownarrow$$

$$(0,0,0)$$

$$\Updownarrow$$

$$\begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ y^2z - x^2z = 0 \Leftrightarrow \boxed{z(y-x)(y+x)=0} \\ z^2y - x^2y = 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ z = 0 \\ \cancel{x^2}y = 0 \end{cases} \vee \begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ y = x \\ \cancel{x}(z^2-x^2) = 0 \end{cases} \vee \begin{cases} x \neq 0 \\ 4(x^2+y^2+z^2) = \frac{yz}{x} \\ y = -x \\ -\cancel{x}(z^2-x^2) = 0 \end{cases}$$

$$\Updownarrow \qquad 12x^2 = \pm x \quad \Updownarrow$$

$$\begin{cases} y = 0 \\ z = 0 \\ 4\cancel{x}^2 = 0 \\ x \neq 0 \end{cases}$$

NO SOL

$$\begin{cases} 4 \cdot 3x^2 = \pm \frac{x^{\cancel{2}}}{\cancel{x}} = \pm x \\ y = x \\ z = \pm x \\ x \neq 0 \end{cases} \vee \begin{cases} 4 \cdot 3x^2 = \mp \frac{x^{\cancel{2}}}{\cancel{x}} = \pm x \\ y = -x \\ z = \pm x \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} 12x = \pm 1 \\ y = x \\ \ldots \end{cases} \qquad \begin{cases} 12x = \pm 1 \\ \ldots \end{cases}$$

$$\begin{cases} y = x \\ z = \pm x \end{cases} \qquad \begin{cases} 12x = \pm 1 \\ y = -x \\ z = \pm x \end{cases}$$

$$\left(+\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) \left(\frac{1}{12}, \frac{1}{12}, -\frac{1}{12}\right) \quad \left(\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}\right) \left(\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right)$$

$$\left(-\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}\right) \left(-\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}\right) \quad \left(-\frac{1}{12}, +\frac{1}{12}, \frac{1}{12}\right) \left(-\frac{1}{12}, +\frac{1}{12}, -\frac{1}{12}\right)$$

$+ \ (0,0,0)$ are the st pts for f.

iii) $f \in \mathcal{C}(\mathbb{R}^3)$, $\mathbb{R}^3$ closed but unbded, by i)

$$\lim_{(x,y,z) \to \infty_3} f = +\infty \implies f \text{ has not max but has min pt.}$$

Let (x,y,z) be a min pt. Clearly $(x,y,z) \in \operatorname{Int} \mathbb{R}^3$

$\Rightarrow$ (F.) $\quad \nabla f(x,y,z) = \vec{0} \implies (x,y,z)$ is one of the st pts.

$f(0,0,0) = 0$

$f(x,y,z) = (x^2+y^2+z^2)^2 - xyz$

Because of the part nature of st pts

$(x^2+y^2+z^2)^2$ is the same for all pts $\neq (0,0,0)$

so to search for min f we have to find for which pts $-xyz$ is min $\Leftrightarrow$ xyz is max

We easily see that min pts for f are

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\boxtimes$

Do 3.4.12/11