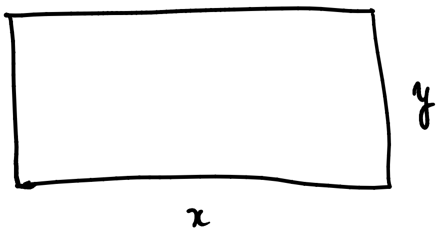
Constrained Optimization

Let's consider the

Pb: Find, among all the rectangles with fixed perimeter, that or those with max area.



$x, y > 0$

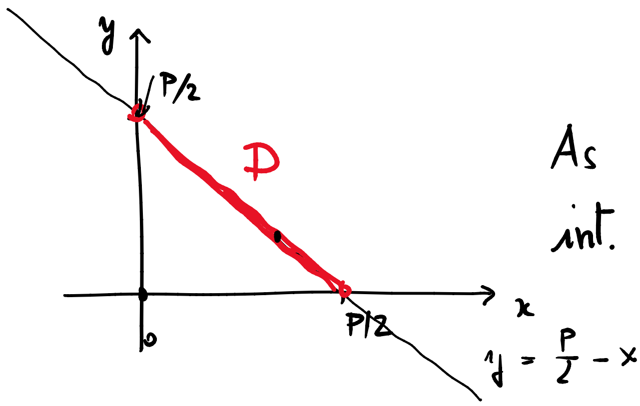
P = perimeter $= 2(x+y)$
↑
given and fixed

A = area $= xy$

The pb is to find

$$\max_{\{x,y:\ x,y>0,\ \underline{2(x+y)=P}\} =: D} \overset{f(x,y)}{\overset{/\!/}{xy}} = \max_D f(x,y)$$

$$D = \{(x,y) : x+y = P/2\} = \{(x,y) : y = P/2 - x\}$$
$x>0,\ y>0$



As we see D has not int. pts, so the search of st. pts for f $(\nabla f = \vec{0})$ is useless.

$\Big($ btw: $\nabla f = (y, x) = \vec{0} \iff (x,y) = \vec{0} \notin D$ $\Big)$

Because $(x,y) \in D \iff y = \frac{P}{2} - x \quad x \in]0, P/2[$

Because $(x,y) \in D \Leftrightarrow y = \frac{P}{2} - x \quad x \in]0, P/2[$

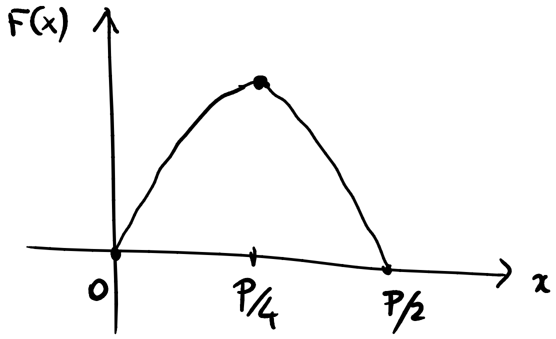
$$\Rightarrow f(x,y) = f\left(x, \frac{P}{2} - x\right) = x\left(\frac{P}{2} - x\right) =: F(x)$$

$$= \frac{P}{2}x - x^2$$

In other words

$$\max_{(x,y)\in D} f(x,y) = \max_{x \in]0, P/2[} F(x) \qquad F' = \frac{P}{2} - 2x$$

$$F' = 0 \Leftrightarrow x = P/4$$



$\Downarrow$

max F is achieved at $x = P/4$

max f is achieved at

$$(P/4, P/2 - P/4) = \left(\frac{P}{4}, \frac{P}{4}\right)$$

($\Rightarrow$ the rect with max A and fixed perimeter P is a <u>square</u> with side $P/4$) $\blacksquare$

What can we learn from this example?

Consider a pb like

$$\min/\max \ f(x,y)$$

$$\underbrace{\{(x,y) \in \mathbb{R}^2 : g(x,y) = 0\}}_{D}$$

In general, a set like $\{g(x,y)=0\} \equiv \{g=0\}$

- can be <u>empty</u> $\{x^2+y^2 = -1\}$

$$\{\underbrace{x^2+y^2+1}_{g(x,y)} = 0\}$$

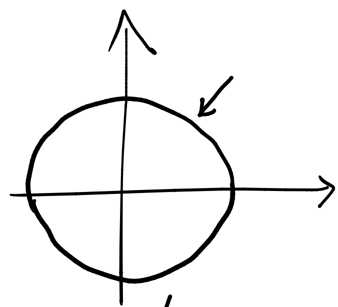
- can be a <u>singleton</u> $\{x^2+y^2=0\} = \{(0,0)\}$

- can be $\mathbb{R}^2$ $\{\underbrace{0}_{g(x,y)} = 0\}$

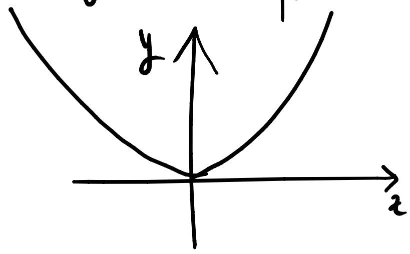
- but in general, avoiding pathologies,

$$\{x^2+y^2=1\}$$

$\{g(x,y)=0\}$ will be a line in the plane xy

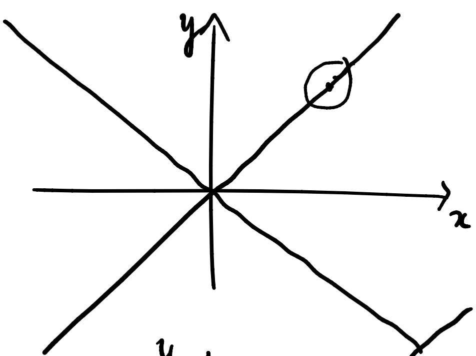


<u>Examples:</u> $\{y-x^2=0\}$

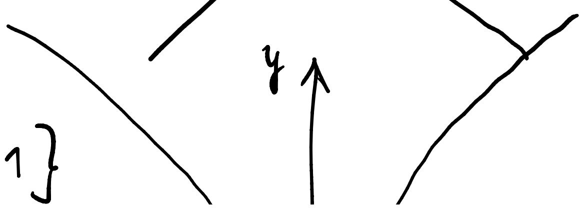


$$\{x^2-y^2=0\}$$

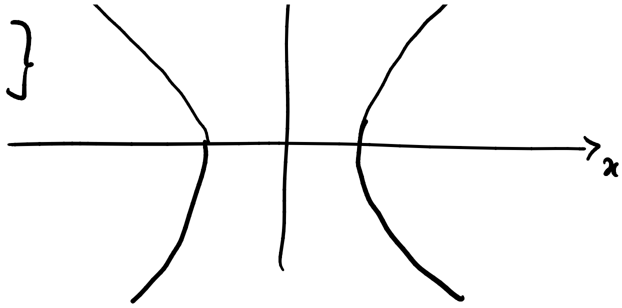
$$(x-y)(x+y)=0$$



$$\{x^2-y^2=1\}$$



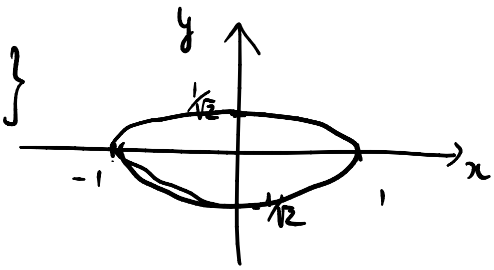
$$\{x^2 - y^2 = 1\}$$



$$\{x^2 + 2y^2 = 1\}$$

$$x^2 + \underbrace{(\sqrt{2}\,y)}^2 = 1$$

$$u^2 + v^2 = 1$$



So in general a set like

$$\{(x,y) : g(x,y) = 0\}$$

if non empty it will be a "line" in plane xy without interior pts. Therefore, st. pts for f are not necessarily interesting to solve

$$\min/\max f(x,y)$$
$$g(x,y) = 0$$

Now suppose we may express pts $(x,y) : g(x,y) = 0$ as

$$\boxed{y = \varphi(x)} \quad \text{or} \quad x = \psi(y).$$

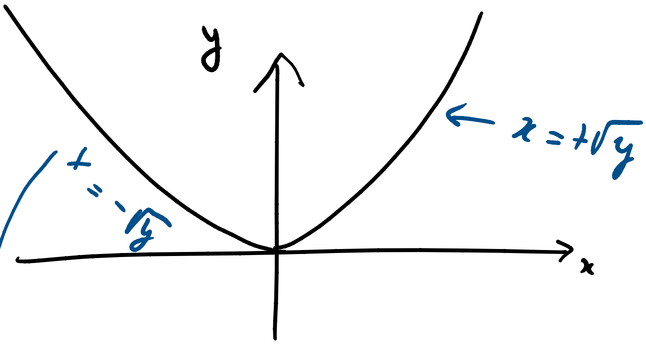
Then

$$\min/\max f(x,y) = \min/\max f(x, \varphi(x))$$

$$\min_{g(x,y)=0}/\max f(x,y) = \min_{x \in I \subset \mathbb{R}}/\max f(x,\varphi(x))$$

Examples:

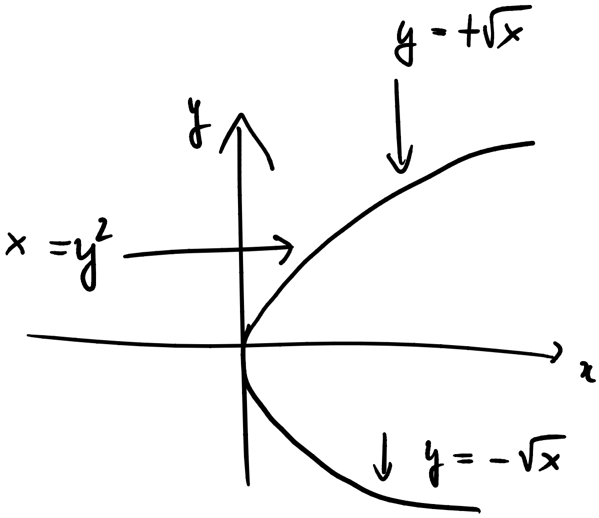
1. $\{ y - x^2 = 0 \}$



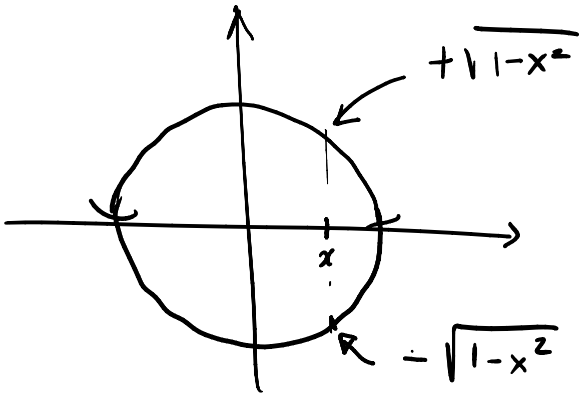
$$y = x^2 = \varphi(x)$$

$$x = \pm\sqrt{y}$$

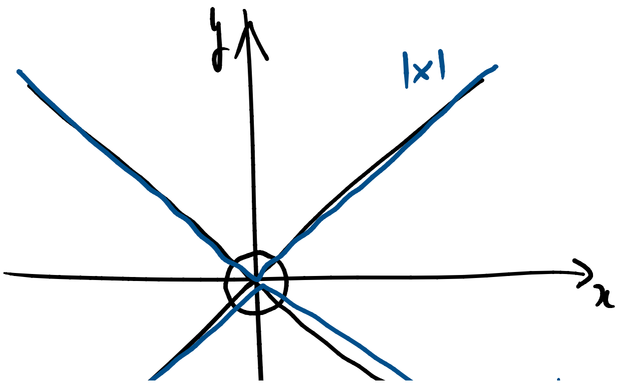
$$\{ x - y^2 = 0 \}$$

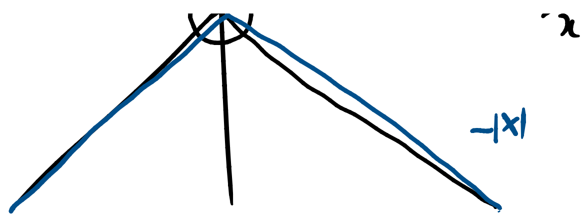


2. $\{ x^2 + y^2 = 1 \}$



3. $\{ x^2 - y^2 = 0 \}$





4. What is $\{ x^3 + y^2 x - x^2 + xy + y^3 = 0 \}$

Now we show a method that finally will not require the explicit knowledge of functs

$$\boxed{y = \varphi(x)} \quad \text{or} \quad \boxed{x = \psi(y)}$$

as conds for pts s.t

$$g(x, y) = 0.$$

Suppose we found $\boxed{y = \varphi(x)}$ s.t.

$$\boxed{g(x, \varphi(x)) = 0.}$$

Consider

$$\max_{g(x,y)=0} / \min f(x,y) \overset{y=\varphi(x)}{=} \max_{g(x,\varphi(x))=0} / \min \boxed{f(x, \varphi(x))}$$

$$F(x) := f(x, \varphi(x))$$

at min/max pts.

$$\frac{d}{dx} F(x) = 0$$

$$\Updownarrow$$

$$\frac{d}{dx}\left[f(x, \varphi(x)) \right] = 0$$

$$\|$$

$$\boxed{\partial_x f(x, \varphi(x)) \cdot 1 + \partial_y f(x, \varphi(x))\, \varphi'(x) = 0}$$

Total Derivative

$$\frac{d}{dt} f(x_1(t), x_2(t), \ldots, x_d(t))$$

$$= \partial_{x_1} f(\quad) x_1'(t) +$$

$$\partial_{x_2} f(\quad) x_2'(t) +$$

$$\vdots$$

$$\partial_{x_d} f(\quad) x_d'(t)$$

$$\frac{d}{dx} f(\varphi_1(x), \varphi_2(x)) \qquad f(x,y)$$

$$= \partial_x f(\quad) \varphi_1'(x) + \partial_y f(\quad) \varphi_2'(x)$$

Let's det $\varphi'(x)$: because

$$g(x, \varphi(x)) \equiv 0$$

$$\Downarrow$$

$$\frac{d}{dx}\left[g(x, \varphi(x)) \right] = 0$$

$$\|$$

$$\partial_x g(x, \varphi(x))\, 1 + \partial_y g(x, \varphi(x))\, \varphi'(x) = 0$$

$$\Rightarrow \quad \boxed{\varphi'(x) = - \frac{\partial_x g\,(x, \varphi(x))}{\partial_y g(x, \varphi(x))}}$$

if $\boxed{\partial_y g(x, \varphi(x)) \neq 0}$

Recalling that $\qquad \partial_x f + \partial_y f\, \varphi' = 0$

$$\Updownarrow$$

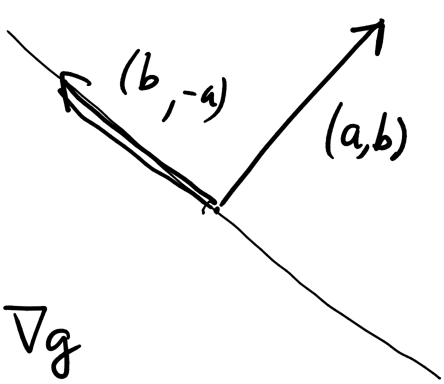
$$\partial_x f + \partial_y f \cdot \left(- \frac{\partial_x g}{\partial_y g} \right) = 0$$

$$\Updownarrow$$

$$\boxed{\partial_x f\, \partial_y g - \partial_y f\, \partial_x g = 0}$$

$$\underbrace{(\partial_x f, \partial_y f)}_{\nabla f} \cdot (\partial_y g, -\partial_x g) = 0$$

$$\Rightarrow \quad \nabla f \perp (\partial_y g, - \partial_x g)$$



$$\Leftrightarrow \quad \nabla f \parallel (\partial_x g, \partial_y g) = \nabla g$$

$$\Leftrightarrow \quad \boxed{\nabla f = \lambda \nabla g}$$

So : if $g(x,y)=0$ is such that $y = \varphi(x)$

at min/max pts for f on $g(x,y)=0$

$$\nabla f = \lambda \nabla g$$

(provided $\partial_y g \neq 0$)

Similarly: if $g(x,y)=0$ is s.t. $x = \psi(y)$, at min/max

pt. for f on $g(x,y)=0$

$$\nabla f = \lambda \nabla g$$

(provided $\partial_x g \neq 0$)

<u>Conclusion</u>: If $\nabla g \neq 0$, at <u>min/max</u> pt for f on

$g(x,y)$ it must be

$$\nabla f = \lambda \nabla g$$

(Lagrange multipliers thm)

Lagrange multipliers thm gives a method to search

for min/max $f(x,y)$. If (x,y) : $g(x,y)=0$

$g(x,y)=0$

is a min/max for f and

$\nabla g(x,y) \neq 0 \implies \nabla f = \lambda \nabla g$

So candidates to be min/max are pts (x,y) s.t.

$$\begin{cases} g(x,y)=0 \\ \nabla g(x,y)\neq 0 \\ \nabla f(x,y) = \lambda\, \nabla g(x,y) \quad \text{for some } \lambda \end{cases}$$

$$\Updownarrow$$

$$\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{bmatrix} \quad \text{has det} = 0$$

$$\Updownarrow$$

$$\boxed{\text{rk} < 2}$$

<u>Example:</u>

$$\max_{\substack{x+y = P/2 \\ x,y>0}} xy$$

Here $f(x,y) = xy$

$g(x,y) = x+y - P/2$

so our pb becomes

$$\max_{g(x,y)=0} f(x,y)$$

$\nabla g = (\partial_x g, \partial_y g) = (1,1) \neq \vec{0} \implies \nabla g \neq \vec{0} \quad \forall (x,y) \in \{g(x,y)=0\}$

$$\begin{cases} g(x,y)=0 \\ (\nabla g \neq 0) \\ \nabla f = \lambda \nabla g \end{cases} \iff \begin{cases} x+y = P/2 \\ \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \end{cases}$$

$$\nabla f = (\partial_x f, \partial_y f) = (y,x) \qquad \Rightarrow \quad y=x$$

$$\iff \begin{cases} y = x \\ 2x = P/2 \end{cases} \iff \begin{matrix} y = P/4 \\ x = P/4 \end{matrix}$$

In alt to check $\nabla f = \lambda \nabla g$, we may check

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix} = y-x$$

$$\Rightarrow \boxed{y = x}. \qquad \boxtimes$$

Ex. 4.5.1

Det $\min_D / \max f$ where

i) $f(x,y) = x+y \qquad D = \{ \overbrace{x^2+y^2-1}^{g(x,y)} = 0 \} = \{x^2+y^2 = 1\}$

Here $f \in \mathcal{C}(\mathbb{R}^2)$, D closed and bded

$(x^2+y^2 = 1 \Rightarrow x^2 \leq 1, y^2 \leq 1 \Rightarrow |x|, |y| \leq 1)$

$\Rightarrow D$ is compact $\Rightarrow (W)$ f has min/max

$\Rightarrow D$ is compact $\Rightarrow$ (W) f has min/max on D.

Now, if $(x,y) \in D$ is a min/max pt. such that $\nabla g(x,y) \neq \vec{0}$

$$\Rightarrow \nabla f = \lambda \nabla g$$

Notice that $g(x,y) = x^2 + y^2 - 1$

$$\nabla g = (2x, 2y) = \vec{0} \iff (x,y) = \vec{0} \notin D$$

$$\nabla g \neq 0 \quad \forall (x,y) \in D \quad \Leftarrow$$

therefore we have to seek at (x,y)

$$\begin{cases} g(x,y) = 0 \\ \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0 \end{cases} \iff \begin{cases} x^2 + y^2 = 1 \\ \det \begin{bmatrix} 1 & 1 \\ 2x & 2y \end{bmatrix} = 0 \end{cases}$$

$$\iff \begin{cases} x^2 + y^2 = 1 \\ 2y - 2x = 0 \end{cases} \iff \begin{cases} y = x \\ 2x^2 = 1 \end{cases} \iff \begin{cases} x = \pm \frac{1}{\sqrt{2}} \\ y = x \end{cases}$$

$$\Downarrow$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

and because $f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$

$f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

$\Rightarrow \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ is max, $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ is min. $\blacksquare$

Do ii)

iii) $f(x,y) = xy$ $\quad D = \{\overbrace{x^2+y^2+xy-1}^{g} = 0\}$.

Here $f \in \mathcal{C}(D)$, D closed. Is D bounded?

$x = \rho\cos\theta$
$y = \rho\sin\theta$

$$\rho^2 + \rho^2\cos\theta\sin\theta = 1$$

$$\rho^2\underbrace{\left(1+\tfrac{1}{2}\sin(2\theta)\right)}_{\neq 0} = 1$$

$$x^2+y^2 = \rho^2 = \frac{1}{1+\frac{1}{2}\sin(2\theta)} \leq 2$$

$$\Downarrow \qquad \Downarrow$$

$|x| \leq \sqrt{2}$, $|y| \leq \sqrt{2}$ $\Rightarrow$ D bded

$\Rightarrow$ D is compact $\Rightarrow$ (W) f has min/max on D.

Let's det. these pts. We apply Lagr. multipliers

Let's det. these pts. We apply Lagr. multipliers thm: Let's see first if $\nabla g \neq \vec{0}$ on D

$$\nabla g = (2x+y,\ 2y+x) = \vec{0} \iff \begin{cases} 2x+y=0 \\ 2y+x=0 \end{cases} \iff \begin{cases} x=0 \\ y=0 \end{cases}$$

$$\Rightarrow (x,y) = \vec{0} \notin D \Rightarrow \nabla g \neq \vec{0} \text{ on } D$$

$\Rightarrow$ if (x,y) is min/max for f on D

$$\begin{cases} g(x,y) = 0 \\ \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0 \end{cases} \iff \det \begin{bmatrix} y & x \\ 2x+y & 2y+x \end{bmatrix} = 0$$

$$\iff \begin{cases} \boxed{x^2+y^2+xy-1=0} \\ y(2y+x) - x(2x+y) = 0 \end{cases}$$

$$\cancel{2}y^2 + \cancel{yx} - \cancel{2}x^2 - \cancel{xy} = 0$$

$$\iff \begin{cases} \overline{} \\ y^2 - x^2 = 0 \end{cases} \iff \begin{cases} y = x \\ 3x^2-1=0 \end{cases} \lor \begin{cases} y=-x \\ x^2-1=0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} x^2 = 1/3 \\ y = x \end{cases} \lor \begin{cases} x^2 = 1 \\ y = -x \end{cases}$$

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \Bigg| \ (1,-1), \ (-1,1)$$

$$\begin{array}{cccc} \downarrow f = xy & \downarrow & \downarrow & \downarrow \\ 1/3 & 1/3 & -1 & -1 \\ \text{max} & & \text{min.} & \end{array}$$

$\blacksquare$

Do iv).