venerdì 6 marzo 2020 08:24

Ex 4.5.1

i) Det min/max $f(x,y) = x^2 + 5y^2 - \frac{xy}{2}$

on $D = \{(x,y) \in \mathbb{R}^2 : x^2 + 4y^2 = 4\}$

Sol: Clearly $f \in \mathcal{C}(D)$, D is closed and bdd ($x^2 \leq 4$, $4y^2 \leq 4$ $\Rightarrow$ $|x| \leq 2$, $|y| \leq 1$)

$\Rightarrow$ D is compact $\Rightarrow$ (W. thm) f has min/max on D.

$\cancel{x^2+4y=4 \not\Downarrow x^2\leq 4,\ y\leq 4}$

Because $D = \{g(x,y)=0\}$ with

$$g(x,y) = x^2 + 4y^2 - 4$$

noticed that

$$\nabla g = (2x, 8y), \quad \nabla g = \vec{0} \iff (x,y) = \vec{0} \notin D$$

$\Rightarrow$ $\nabla g \neq \vec{0}$ on D, therefore by Lagr. multipliers thm., if (x,y) is min/max

$$\nabla f = \lambda \nabla g \iff \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = 0$$

$$\det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 2x - y/2 & 10y - x/2 \\ 2x & 8y \end{bmatrix}$$

$$= 8y\left(2x - \frac{y}{2}\right) - 2x\left(10y - \frac{x}{2}\right)$$

$$= 16xy - 4y^2 - 20xy + x^2$$

$$= x^2 - 4y^2 - 4xy$$

$$= (x - 2y)^2 - 8y^2$$

$$= (x - 2y - \sqrt{8}\,y)(x - 2y + \sqrt{8}\,y) = 0$$

$$\Updownarrow$$

$$x = (2 + 2\sqrt{2})y \quad \vee \quad x = (2 - 2\sqrt{2})y$$

$$\Updownarrow$$

$$(2(1+\sqrt{2})y, y) \qquad (2(1-\sqrt{2})y, y)$$

$$(2(1+\sqrt{2})y, y) \in D \iff \cancel{4}(1+\sqrt{2})^2 y^2 + \cancel{4}y^2 = \cancel{4}$$

$$\iff y^2 \underbrace{\left[1 + 1 + 2 + 2\sqrt{2}\right]}_{4+2\sqrt{2}} = 1$$

$$\iff y^2 = \frac{1}{4 + 2\sqrt{2}} = \frac{1}{2(2+\sqrt{2})}$$

$$\iff y = \pm \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}$$

$$\Rightarrow \left(\pm 2(1+\sqrt{2})\frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}, \pm \frac{1}{\sqrt{2}\sqrt{2+\sqrt{2}}}\right)$$

$+ \qquad\qquad + \qquad$ 2 pts.

$- \qquad\qquad -$

$$= \pm \frac{1}{\sqrt{4+2\sqrt{2}}} \left(\cancel{\pm}(2+2\sqrt{2}), \cancel{\pm}1 \right)$$

(same sgn)

$$(2(1-\sqrt{2})y, y) \in D \iff \cancel{4}(1-\sqrt{2})^2 y^2 + \cancel{4}y^2 = \cancel{4}1$$

$$\iff y^2 (1 + 1 + 2 - 2\sqrt{2}) = 1$$

$$\iff y^2 = \frac{1}{4-2\sqrt{2}}$$

$$\iff y = \pm \frac{1}{\sqrt{4-2\sqrt{2}}}$$

$$\Rightarrow \left(\pm(2-2\sqrt{2}) \frac{1}{\sqrt{4-2\sqrt{2}}}, \pm \frac{1}{\sqrt{4-2\sqrt{2}}} \right) =$$

$$\pm \frac{1}{\sqrt{4-2\sqrt{2}}} (2-2\sqrt{2}, 1)$$

$$f(x,y) = x^2 + 5y^2 - \frac{xy}{2}$$

$$f\left(\pm \frac{1}{\sqrt{4+2\sqrt{2}}} (2+2\sqrt{2}, 1) \right)$$

$$= \frac{4+8+8\sqrt{2}}{4+2\sqrt{2}} + \cancel{5} \frac{\cancel{1}5}{4+2\sqrt{2}} - \frac{1}{\cancel{2}} \frac{\cancel{2}+\cancel{2}\sqrt{2}}{4+2\sqrt{2}}$$

$$= \frac{\text{[illegible]}}{4+2\sqrt{2}} + \cancel{5}\,\frac{4\,5}{4+2\sqrt{2}} - \frac{1}{\cancel{2}}\,\frac{2+8\sqrt{2}}{4+2\sqrt{2}}$$

$$= \frac{16+7\sqrt{2}}{4+2\sqrt{2}}$$

$$f\left(\pm\frac{1}{\sqrt{4-2\sqrt{2}}}\,(2-2\sqrt{2},1)\right) = \frac{4+8-8\sqrt{2}}{4-2\sqrt{2}} + \cancel{5}\,\frac{\cancel{4}5}{4-2\sqrt{2}}$$

$$- \frac{1}{\cancel{8}}\,\frac{\overset{1}{\cancel{8}}-8\sqrt{2}}{4-2\sqrt{2}}$$

$$= \frac{16-7\sqrt{2}}{4-2\sqrt{2}}$$

Now

$$\frac{16+7\sqrt{2}}{4+2\sqrt{2}} \overset{?}{>} \frac{16-7\sqrt{2}}{4-2\sqrt{2}}$$

$$<$$

$$(16+7\sqrt{2})(4-2\sqrt{2}) > (16-7\sqrt{2})(4+2\sqrt{2})$$

$$\cancel{64} - 32\sqrt{2} + 28\sqrt{2} - 28 \overset{?}{>} \cancel{64} + 32\sqrt{2} - 28\sqrt{2} - 28$$

$$\Updownarrow$$

$$56 \underset{?}{>} 64 \quad \text{No!}$$

$$\Rightarrow \quad \pm\frac{1}{\sqrt{4-2\sqrt{2}}}\,(2-2\sqrt{2},1) \quad \text{are max pts}$$

$$\pm\frac{1}{\sqrt{4+2\sqrt{2}}}\,(2+2\sqrt{2},1) \quad \text{are min pts.}$$

$$\pm \frac{1}{\sqrt{4+2\sqrt{2}}}\,(2+2\sqrt{2},\,1) \quad \text{are min pts.}$$

$\boxtimes$

Def: Given $g:\mathbb{R}^d \longrightarrow \mathbb{R}$ we say that g is a submersion on D if

$$\nabla g(\vec{x}) \neq \vec{0} \qquad \forall\, \vec{x} \in D.$$

(or if g has not st. pts on D.).

So the Lagrange multipliers thm may be restated as:

Thm: Let g be a submersion on $\{g=0\}$.
Then, if (x,y) is a min/max pt for f on $\{g=0\}$ we have

$$\boxed{\nabla f(x,y) = \lambda\, \underset{\neq 0}{\nabla g(x,y)}}$$

or, equivalently

$$\operatorname{rank}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2$$

It may be proved that the prev. result extends to the case

$$\min/\max\ f(x,y,z)$$

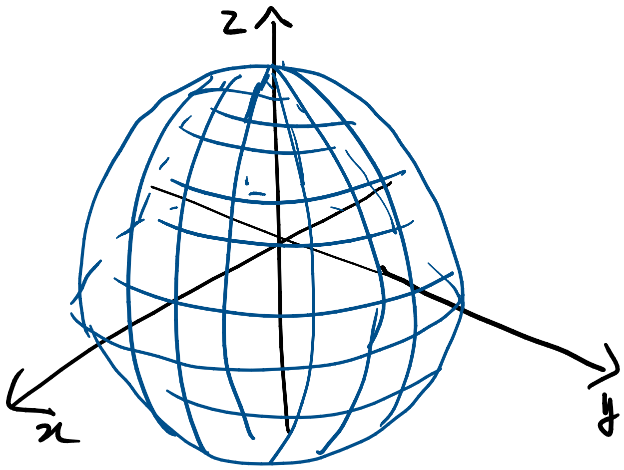
$$\min/\max \ f(x,y,z)$$
$$\{(x,y,z) : g(x,y,z) = 0\}$$

In this case, a set like $\{g(x,y,z)=0\}$ is a surface in $\mathbb{R}^3$.

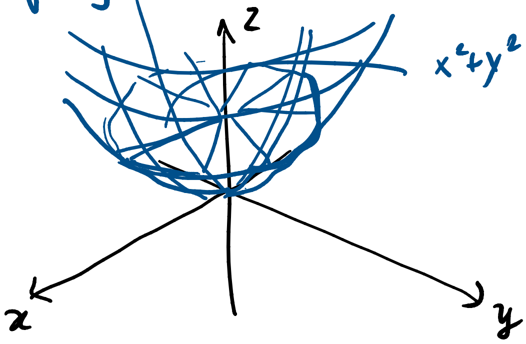
Examples:

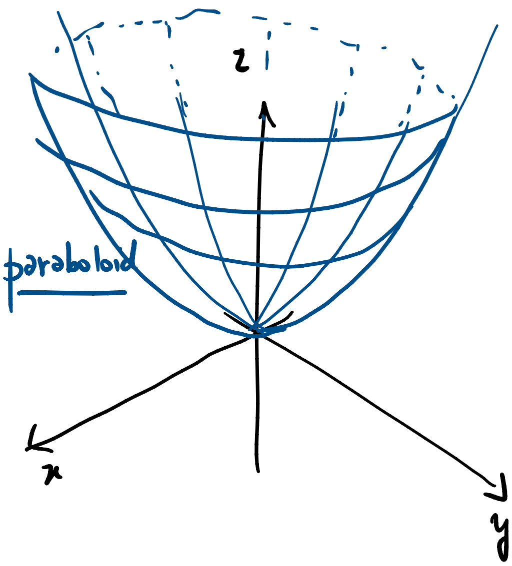
$$\{x^2+y^2+z^2=1\} = \{x^2+y^2+z^2-1=0\}$$

(sphere centered at $(0,0,0)$, radius 1)



$$\{z = x^2+y^2\} = \{x^2+y^2-z=0\}$$

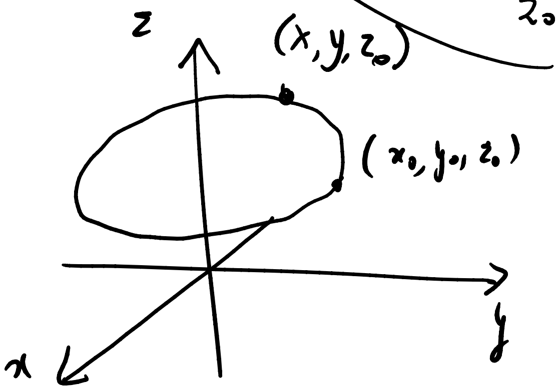




$$\{ z^2 = x^2 + y^2 \} \ni S$$

If $(x_0, y_0, z_0) \in S$ then any other pt. (x, y, z_0) s.t. $x^2+y^2 = x_0^2 + y_0^2$ $\in S$,

$$z_0^2 \overset{?}{=} x^2+y^2 = \underbrace{x_0^2+y_0^2}_{=}$$



More in gen $S = \{ g(x,y,z) = 0 \} = \{ h(x^2+y^2, z) = 0 \}$

is invariant by rots around the z-axis.

Indeed: $(x_0, y_0, z_0) \in S \Rightarrow h(x_0^2 + y_0^2, z_0) = 0$

$$\Downarrow$$

(x, y, z_0) is s.t. $x^2+y^2 = x_0^2+y_0^2$

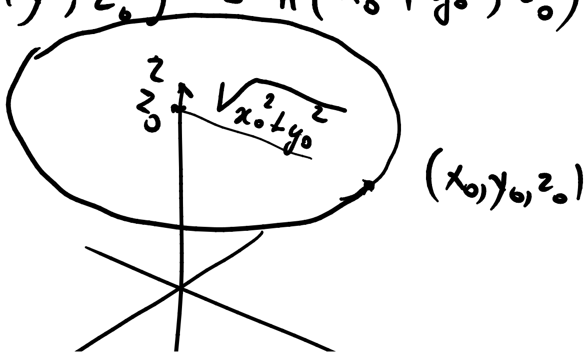
$$\Downarrow$$

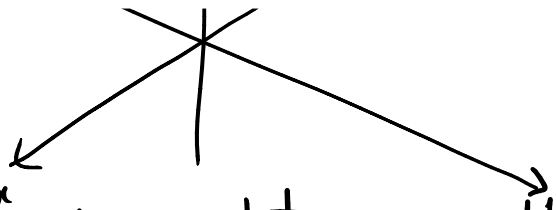
$$h(x^2+y^2, z_0) = h(x_0^2+y_0^2, z_0)$$

$$\Downarrow$$

$(x, y, z_0) \in S$

$\|$





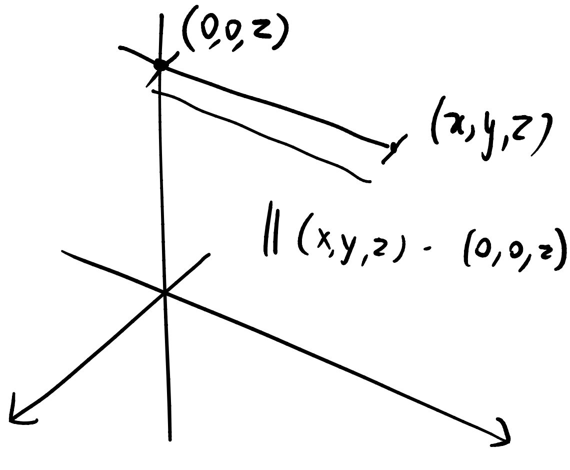
$\Downarrow$

S is invariant by rotations around z-axis.

So $\{(x^2+y^2)^2 + z^2 + x^2+y^2 = 10\}$

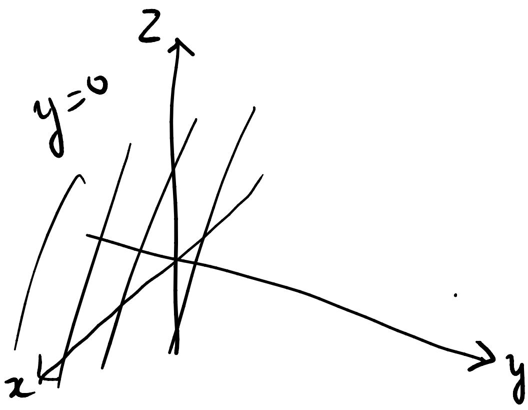
rotation invariant resp. to z-axis.

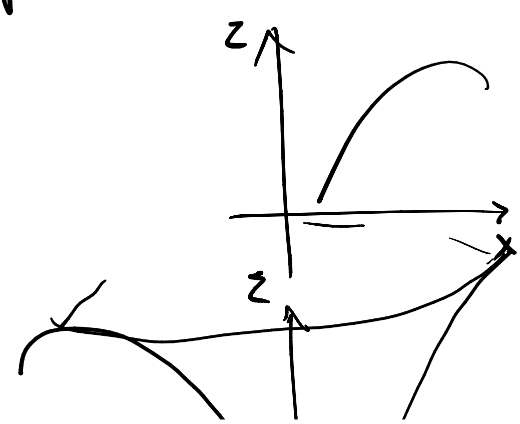
$\{x^2+y^2+z+xy=10\}$

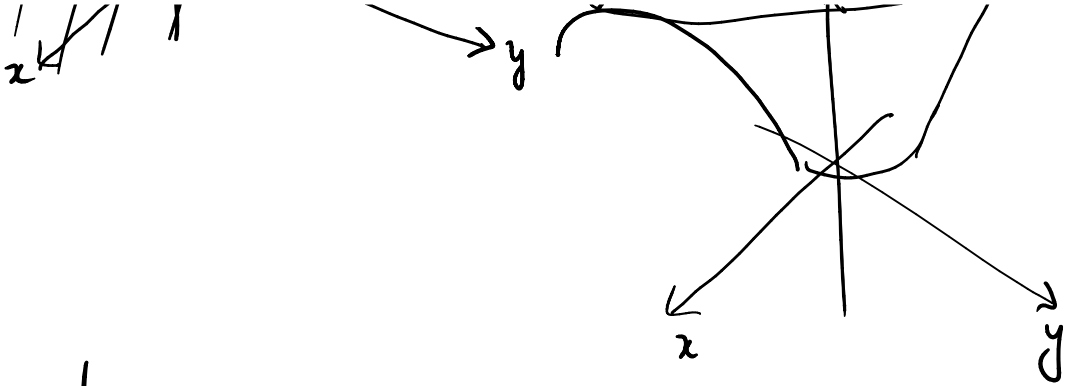


$\|(x,y,z)-(0,0,z)\| = \|(x,y,0)\| = \sqrt{x^2+y^2}$

To plot S we may take a section of S on a plane $y=0$



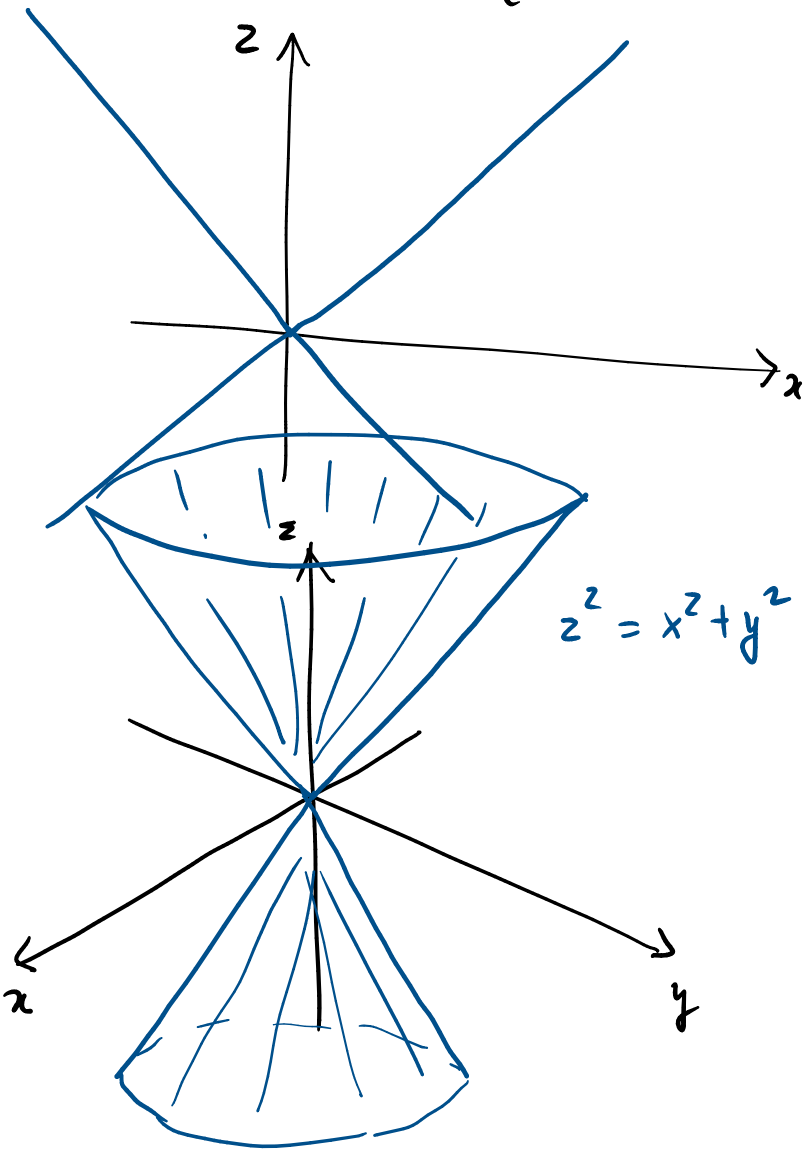




Returning to

$$\{ z^2 = x^2 + y^2 \}.$$

$$y = 0 \implies \{ z^2 = x^2 \} = \{ z = \pm |x| \}$$



It might be proved that

Thm: Assume $g = g(x,y,z)$ be a submersion on $\{g=0\}$ (that is $\nabla g \neq \vec{0}$ $\forall (x,y,z) \in \{g=0\}$)

Then, if (x,y,z) is min / max for f on $\{g=0\}$

we have

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$

or, equivalently

$$\text{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2$$

(2x3 matrix)

Rmk: How do we check $\text{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2$.

Notice that

$$\text{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} = 2 \iff$$

$\left[\boxed{1}\ \ \boxed{2}\ \ \boxed{3}\right]$ at least one between

$\det\left[\boxed{1}\ \boxed{2}\right]$

$\det\left[\boxed{1}\ \boxed{3}\right]$ be $\neq 0$

$\det\left[\boxed{2}\ \boxed{3}\right]$

so $\text{rk}\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix} < 2 \iff$ all 2x2 subdeterminants of $\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix}$ are simultaneously $=0$

$\left[\nabla g\right]$ ... all 2×2 ...

$\begin{bmatrix}\nabla f\\ \nabla g\end{bmatrix}$ are simultaneously $=0$

$$\iff \begin{cases} \det\begin{bmatrix} \boxed{1} & \boxed{2}\end{bmatrix} = 0 \\ \det\begin{bmatrix} \boxed{1} & \boxed{3}\end{bmatrix} = 0 \\ \det\begin{bmatrix} \boxed{2} & \boxed{3}\end{bmatrix} = 0 \end{cases}$$

Ex. 4.5.1.

vi) Find min/max f where $f(x,y,z) = z^2 e^{xy}$
$\quad\quad\quad\quad\quad\quad D$ $\quad\quad\quad\quad\quad D = \{x^2+y^2+z^2 = 1\}$

Sol: $f \in \mathscr{C}(D)$, D is closed and bdd (evident)

$\Rightarrow D$ is compact $\Rightarrow$ (W thm) f has min/max on D.

To det these pts, let's apply the Lagr. multipliers thm. on $D = \{g=0\}$ where

$$g(x,y,z) = x^2+y^2+z^2-1$$

Let's first check if g is submersive on D.

$\nabla g = (2x, 2y, 2z)$, $\Rightarrow$ $\nabla g = \vec{0} \iff (x,y,z) = \underset{\pi}{\vec{0}}$

D

$\Rightarrow \nabla g \neq \vec{0}$ on D $\Rightarrow$ g is submersive on D.

According to L. mult. thm, if $(x,y,z) \in D$ is min/max pt for f on D we have

$$\nabla f = \lambda \nabla g \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2$$

Because

$$\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \begin{bmatrix} y z^2 e^{xy} & x z^2 e^{xy} & 2z e^{xy} \\ 2x & 2y & 2z \end{bmatrix}$$

we have

$$\operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2 \iff \begin{cases} \det\begin{bmatrix} y z^2 e^{xy} & x z^2 e^{xy} \\ 2x & 2y \end{bmatrix} = 0 \\ \det\begin{bmatrix} y z^2 e^{xy} & 2z e^{xy} \\ 2x & 2z \end{bmatrix} = 0 \\ \det\begin{bmatrix} x z^2 e^{xy} & 2z e^{xy} \\ 2y & 2z \end{bmatrix} = 0 \end{cases}$$

$$\iff \begin{cases} \cancel{2e^{xy}}\left[y^2 z^2 - x^2 z^2 \right] = 0 \\ \cancel{2e^{xy}}\left[y z^3 - 2zx \right] = 0 \\ \cancel{2e^{xy}}\left[x z^3 - 2zy \right] = 0 \end{cases} \iff \begin{cases} \boxed{z^2 (y-x)(y+x) = 0} \\ z(\overset{-x}{y} z^2 - 2x) = 0 \\ z(x z^2 - 2y) = 0 \end{cases}$$

$$\iff \begin{cases} z = 0 \\ 0 = 0 \end{cases} \vee \begin{cases} y = x,\ z \neq 0 \\ \cancel{z}x(z^2 - 2) = 0 \end{cases} \vee \begin{cases} y = -x,\ z \neq 0 \\ -\cancel{z}y(z^2 + 2) = 0 \end{cases}$$

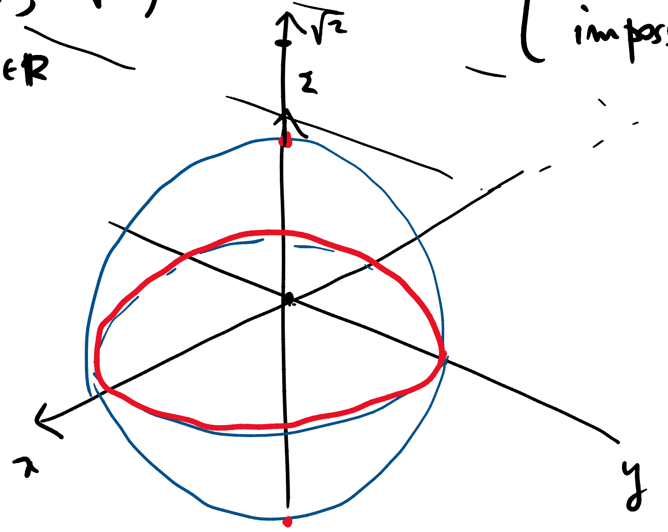
$$\Longleftrightarrow \begin{cases} 0=0 \\ 0=0 \end{cases} \vee \begin{cases} \cancel{z}y(z^2-2)=0 \\ \cancel{zy(z^2-2)=0} \end{cases} \vee \begin{cases} -\cancel{z}y(z^2+2)=0 \\ -\cancel{z}y(z^2+2)=0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$(x,y,0) \qquad \begin{cases} y=x, \; z\neq 0 \\ y=0 \\ (0,0,z) \end{cases} \vee \begin{cases} y=x, \; z\neq 0 \\ z^2=2 \end{cases} \vee \begin{cases} y=-x, \; z\neq 0 \\ y=0 \\ (0,0,z) \end{cases} \vee \begin{cases} y=-x \\ z\neq 0 \\ z^2+2=0 \\ \text{imposs} \end{cases}$$

$$z^2=2 \Updownarrow z=\pm\sqrt{2}$$

$$(x,x,\pm\sqrt{2}) \quad x\in\mathbb{R}$$



$\Rightarrow$ we found pts.

$(x,y,0) \qquad x,y\in\mathbb{R}$

$(0,0,z) \qquad z\in\mathbb{R}$

$(x,x,\pm\sqrt{2}) \qquad x\in\mathbb{R}$

Let's see which of these are in domain:

- $(x,y,0)\in D \;\Longleftrightarrow\; x^2+y^2=1$
- $(0,0,z)\in D \;\Longleftrightarrow\; z^2=1 \Longleftrightarrow z=\pm 1 \Rightarrow (0,0,\pm 1)$
- $\underline{(x,x,\pm\sqrt{2})\in D} \;\Longleftrightarrow\; x^2+x^2+2=1$

  $2x^2=-1$ imposs.

$$\downarrow$$

$$f(x,y,0) = 0^2 e^{xy} = 0$$

$$f(x,y,0) = 0^2 e^{xy} = 0$$
on $x^2+y^2=1$

$$f(0,0,\pm 1) = (\pm 1)^2 \cdot e^{0\cdot 0} = 1 \cdot e^0 = 1$$

$\Rightarrow$ $(0,0,\pm 1)$ are max pts
$(x,y,0)$: $x^2+y^2=1$ are min pts. $\blacksquare$

Do v)

Do 4.5.3/4/7/8/9/14

$\boxed{4.5.3}$ $M = \{ (x,y,z) \in \mathbb{R}^3 : z^2 = xy+1 \}$

i) $M \neq \emptyset$ and it is a diff manifold of dim ...

$\Updownarrow$

$M = \{g=0\}$ g submersion on M

ii) Say if M is compact or not

iii) Determine pts of M at min/max dist to $\vec{0}$

Sol: $M = \{ z^2 = xy+1 \}$

$x=y=0$
$z^2=1$

$M \neq \emptyset$ $(0,0,\pm 1) \in M$

$\mathcal{M} \neq \emptyset \qquad (0,0,\pm 1) \in \mathcal{M} \qquad z^2 = 1$

Let $g(x,y,z) = z^2 - xy - 1$. We check that g is a submersion on $\{g = 0\} = \mathcal{M}$

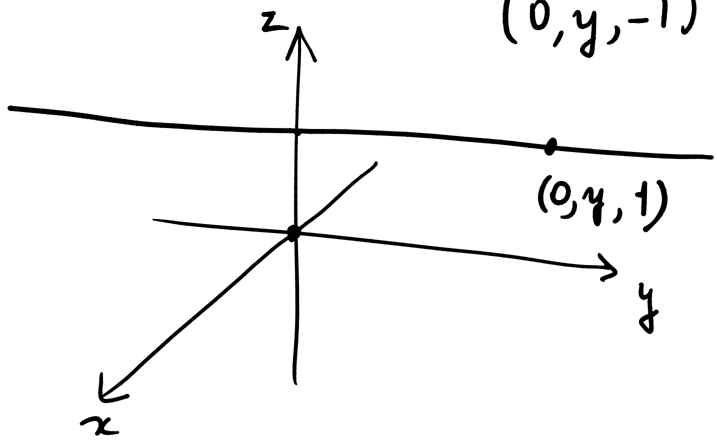
$$\nabla g = \vec{0} \iff (-y, -x, 2z) = (0,0,0) \iff$$

$$\iff (x,y,z) = \vec{0} \notin \mathcal{M}.$$

ii) $\mathcal{M} = \{g = 0\} \quad g \in \mathcal{C} \implies \mathcal{M}$ is closed

Is $\mathcal{M}$ also bded? $\quad z^2 = xy + 1.$

$$\left.\begin{array}{l} x = 0 \\ z^2 = 1 \end{array}\right. \implies \mathcal{M} \ni \begin{array}{l} (0,y,1) \\ (0,y,-1) \end{array} \quad \forall y \in \mathbb{R}$$



$\implies \mathcal{M}$ is unbded!

$\Downarrow$

$\mathcal{M}$ is not compact.

Other examples:
$$y = x$$
$$\Downarrow$$
$$z^2 = x^2 + 1 \iff z = \pm\sqrt{x^2+1}$$
$$\Downarrow$$
$$(x, x, \pm\sqrt{x^2+1}) \in \mathcal{M} \quad \forall x \in \mathbb{R}$$

$$y = -x$$

$$z^2 = -x^2 + 1 = 1 - x^2 \qquad z = \pm\sqrt{1-x^2}$$

$$(x, -x, \pm\sqrt{1-x^2}) \qquad |x| \leq 1$$

not good pts to prove $\mathcal{M}$ unbdd.

iii) We have to det min/max $f(x,y,z)$

$$\boxed{f(x,y,z) = \sqrt{x^2+y^2+z^2}}$$

So we have to discuss $\min/\max f(x,y,z)$
$\mathcal{M} = \{g=0\}$

Because f is a bit heavy (∇f)

we may notice that

$\sqrt{x^2+y^2+z^2}$ is min/max for pts (x,y,z)

for which $x^2+y^2+z^2$ is min/max

so, to find min/max pts for f we may

use

$$f(x,y,z) = x^2+y^2+z^2$$

So we study $\min/\max_{\mathcal{M}} (x^2+y^2+z^2)$.

Existence: because $\mathcal{M}$ is closed and unbdd and $$\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$$

$$\Rightarrow \quad \nexists \max_{M} f \quad \text{but} \quad \exists \min_{M} f.$$

To det min pt for f we apply Lagr. mult. thm. We already checked that g is a submersion on M (i). So, if $(x,y,z) \in M$ is min for f on M, we must have

$$\nabla f = \lambda \nabla g$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} < 2$$

$$\Leftrightarrow \quad \operatorname{rk}\begin{bmatrix} 2x & 2y & 2z \\ -y & -x & 2z \end{bmatrix} < 2$$

$$\Leftrightarrow \quad \begin{cases} \det\begin{bmatrix} 2x & 2y \\ -y & -x \end{bmatrix} = 0 \\ \det\begin{bmatrix} 2x & 2z \\ -y & 2z \end{bmatrix} = 0 \\ \det\begin{bmatrix} 2y & 2z \\ -x & 2z \end{bmatrix} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} -\cancel{2}x^2 + \cancel{2}y^2 = 0 \\ 2\,\cancel{4}xz + \cancel{2}yz = 0 \\ \cancel{2}\cancel{4}yz + \cancel{2}xz = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \boxed{(y-x)(y+x) = 0} \\ z(2x+y) = 0 \\ z(2y+x) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ \cancel{3}zx = 0 \\ \cancel{3zx = 0} \end{cases} \quad \vee \quad \begin{cases} y = -x \\ zx = 0 \\ \cancel{zx = 0} \end{cases}$$

$$\Updownarrow$$

$$\cdot \begin{cases} y = x \\ z = 0 \end{cases} \vee \begin{cases} y = x = 0 \\ x = 0 \end{cases} \quad \vee \quad \begin{cases} y = -x \\ z = 0 \end{cases} \vee \begin{cases} y = -x = 0 \\ x = 0 \end{cases}$$

$$\cdot \underset{x \in \mathbb{R}}{(x,x,0)} \vee \underset{z \in \mathbb{R}}{(0,0,z)} \quad \vee \quad \underset{x \in \mathbb{R}}{(x,-x,0)} , \vee \underset{z \in \mathbb{R}}{(0,0,z)}$$

To finish, let's see which are on m:

- $(x,x,0) \in m \Leftrightarrow 0 = x^2+1$ impos.
- $(0,0,z) \in m \Leftrightarrow z^2 = 0+1 \Leftrightarrow z = \pm 1$

$$\boxed{(0,0,\pm 1)}$$

- $(x,-x,0) \in m \Leftrightarrow 0 = -x^2+1 \Leftrightarrow x^2 = 1 \Leftrightarrow x = \pm 1$

$$\boxed{(1,-1,0) , (-1,1,0)}$$

Finally

$$f(0,0,\pm 1) = 1, \qquad f(1,-1,0) = f(-1,1,0) = 2$$

$\Rightarrow (0,0,\pm 1)$ are min pts, that is pts of m at min dist to $\vec{0}$. ∎